ADVANCED CALCULUS

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by

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WITH 166 DIAGRAMS

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PREFACE

Although this book is intended for students who have already acquired some knowledge of the elements of the calculus, it must be regarded merely as an introduction to the more advanced parts of the subject. The scope of modern analysis is so wide and its ramifications are so numerous that it is not possible within the limits of one volume to give even a short account of the more important developments that lie beyond the elementary stage. I have therefore confined myself to that part of the calculus that does not involve the theory of differential equations or of the functions that arise directly from these equations. My aim has been not only to cover a fairly wide field, keeping in view the possible applications of mathematical ideas, but it has also been to give a reasonably rigorous account of the principal limiting processes that are characteristic of modern analytical methods. But in view of the different requirements of students of varied interests, it is not easy either to make an appropriate choice of subject matter or to decide how deeply to probe the foundations of analysis. The introduction that the normal student receives to the calculus is usually one in which many results may legitimately be regarded as intuitively obvious, an appeal being made to geometry or physics; and such an introduction is good, up to a point, if the subject stimulates interest not only for its intrinsic merits but also for its practical value. It is essential, however, at some stage which should not be too long deferred, to re-examine the concepts that have been introduced and to place them on a more satisfactory basis; for otherwise it is not possible to appreciate the more advanced parts of the subject, especially in those problems where intuition fails to give correct results. Any attempt, however, to give an exhaustive treatment of the foundations of analysis is beyond the scope of a work, whose object is to cover the field that is chosen here; and I have therefore confined myself in general to those fundamental theorems that prove useful in subsequent developments or are likely to arise in applications. Nevertheless, I have taken the opportunity, when it occurred, of introducing such important topics as Theory of Sets, Lebesgue

Integration, Algebraic Functions, Finite Differences, Tensors, One-one Correspondence, Analytic Continuation, &c., even although it has been possible to deal only with the simplest theorems that relate to these topics or in some cases to give only such a descriptive account as to indicate their importance in analysis.

The subject has been developed from the beginning in order that the reader may revise his previous knowledge from a more mature standpoint and at the same time have a surer foundation on which to base the subsequent development. With few exceptions, the examples may be solved directly from the text; and in any case sufficient indications have been given to enable the reader to make satisfactory progress without supervision.

Some importance has been attached to approximations, not only to the approximate forms of functions but also to approximate integration and summation of series, although lack of space has prevented the inclusion of other than the more outstanding results. It is suggested by this that the theory of finite differences, interpolation formulae and approximate integration might well replace much of the elaborate detail that is often associated with indefinite integration.

It is hoped that the book will prove helpful not only to those whose interests are purely mathematical but also to those whose main interests lie in some related field. All the topics introduced are appropriate to the curriculum of an Honours Mathematics course; at the same time, much of the book is suitable for a student taking a General Honours course that includes Mathematics and also to a student in any other Honours course, like Physics, in which Mathematics is indispensable. Whilst it may be too much to expect that a physicist should be conversant with the rigorous formal proofs of general theorems in Pure Mathematics, it is important that he should know sufficient conditions for the validity of those limiting processes he may encounter in his investigations.

Where there is so much variety in the nature of the subjects discussed and where most of the work must now be regarded as well-known, if not available in a single text-book, I have not attempted to give detailed references, except in those special cases that appeared to warrant them. Of the many works I have consulted in the preparation of the book, I must, however, acknowledge the help I have obtained from the following :

Bromwich, *Infinite Series*; Hardy, *Pure Mathematics*; Goursat, *Cours d'Analyse Mathématique, I–III*; Hobson, *Functions of a Real Variable, I*; Knopp, *Theorie und Anwendung der Unendlichen Reihen*; De la Vallée-Poussin, *Cours d'Analyse Infinitésimale*; Whittaker and Watson, *Modern Analysis*; Titchmarsh, *Theory of Functions*; Bieberbach, *Lehrbuch der Funktionentheorie I*; Osgood, *Differential and Integral Calculus*; Weatherburn, *Vector Analysis, I, II*.

I wish to acknowledge my indebtedness to Professor P. J. Daniell, who has read the manuscript and given me much valuable advice and helpful criticism. I wish also to thank Mr. F. Bowman for the assistance he has given me by his criticism of the earlier chapters.

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CHAPTER I

REAL VARIABLES. SEQUENCES. LIMITS. RATIONAL FUNCTIONS

1. Functions of One Variable. When two variables x, y are so related that y is determined when x is given, y (the *dependent* variable) is called a *function* of x (the *independent* variable). The set of values that x may take is called the *domain* of x; and the functional relationship then determines a domain for y. Functions may be expressed in various ways, some of which are illustrated in the following examples:

(i) $y = x/(x^2+1)$, (ii) $y = \sqrt{(x-1)}$, (iii) $y = \sqrt{x^3}$

(iv) $y = 4x^2$, $(0 \leqslant x \leqslant 2)$; $y = 16(3x-4)$, $(2 \leqslant x \leqslant 3)$;
$y = 2x^3 - (2x^2+9)\sqrt{(x^2-9)}$, $(x \geqslant 3)$

Here $\pi y/3$ is the volume of overflow when a sphere of radius x is lowered into a cylindrical vessel full of liquid, the height of the cylinder being 4 and its radius 3.

(v) y is the sum of x terms of the series $8 - 4 + 2 - 1 + \frac{1}{2} + \ldots$

(vi) $y = -(0{\cdot}16)x^2(1-x)^3$. Here y is the deflection from the horizontal of a certain beam of unit length, x being the distance from one end.

(vii) $y = 0$, (x rational). $y = 1$, (x irrational).

(viii) $y = f(x) + f(90 - x)$, where $f(x) = \sin x°$, $(0 < x < 180)$, and $f(x) = 0$ for other values.

(ix) y is given approximately by the following table:

x	0	1	2	3	4	5	6	7	8	9	10
y	0	100	380	400	510	350	210	240	410	200	0

and $5y$ is the height in feet of an aeroplane x minutes after the commencement of a ten-minute flight.

(x) y is the barometric height at a certain place x hours after noon on a certain date, the function being given approximately by a chart which is automatically registered by an instrument.

The usual graphical representation of such functions as shown in *Fig. 1* is, of course, approximate and may, as in *Example* (vii), be deceptive. The domain of x, if not stated nor implied, consists of those values for which the function has a meaning. This domain may, however, be explicitly restricted, or its limitations may be determined by the context in which the function occurs. Thus in *Examples* (i), (iii), x may be any real number, but in *Example* (ii) x cannot be less than 1. In applications, y may be given only for a certain domain of x, whilst the formula for y, if it exists, may have significance outside that domain. For example, the deflection of the unit beam in *Example* (vi) applies only to the range $0 \leqslant x \leqslant 1$. A function may be known only approximately as in *Examples* (ix), (x), but it may be subject to an appropriate analysis from which approximate or probable properties may be established.

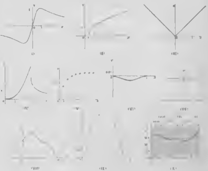


FIG. 1

1.01. *Implicit Functions.* In the examples chosen above, y is *explicitly* given, but there are many ways in which a function can be *implicitly* expressed. It will often be found that implicit relations determine *classes* of functions rather than particular functions and that the domains of both x and y are restricted.

Examples. 1. If y is real and is given by the relation

$$xy^2 - 2(2x - 1)y + x = 0$$

we have, on solving,

$$xy = 2x - 1 \pm \sqrt{(x-1)(3x-1)}, \quad (x \neq 0), \quad y = 0, \quad (x = 0).$$

Thus y does not exist if $\frac{1}{3} < x < 1$, and, otherwise, except for $x = 0, 1, \frac{1}{3}$, has *two* values. When the quadratic occurs in a problem, it is often possible to reject one of the solutions when the problem is set in such a way as to admit of one solution only.

2. If y is given by the relation $y'' = 0$, for all values of x, we obtain for y the class of linear functions $ax + b$, where a, b are arbitrary constants. A particular function may be specified by supplying additional data that determine a and b.

1.1. The Real Variable. An essential basis for the development of analysis is a correct definition of real number. It will be assumed here that *rational* numbers have already been defined and their properties established, although for completeness these numbers also require definitions in terms of more elementary ideas. In modern analysis we continually meet with limiting processes and a correct interpretation of these processes demands, as a foundation, a knowledge of the characteristic properties of a properly defined real number system.

The inadequacy of rational numbers for analytic processes is most simply illustrated by the equation $x^2 = 2$, for this equation is easily shown to have no rational solution. Although it has no rational solution, we can, however, find rational approximations (e.g. decimal approximations) to a solution. Thus, by a definite process, we can form the sequence of numbers:

$$1,\ 1.4,\ 1.41,\ 1.414,\ \ldots$$

whose squares differ from 2 by smaller and smaller numbers as the sequence proceeds.

This suggests that the definition and the properties of irrational numbers can be based upon the consideration of such sequences.

1.11. Simple Sequences. A set of numbers (of any kind) written in order:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

is called a simple sequence. It may be denoted by $\{a_n\}$ or simply by a_n.

1.12. Convergence of a Sequence. If by taking n large enough, the difference between every two of the terms $a_{n+1}, a_{n+2}, \ldots$ can be made as small as we please, the sequence is said to be *convergent*.

More precisely. The sequence a_n is said to be convergent if, given ϵ, any positive number, however small, we can find n_0 such that

$$|a_p - a_q| < \epsilon$$

for all integers $p, q > n_0$.

For simplicity we shall say that for convergence the difference between every two of the terms is '*ultimately small*', this phrase being used in the strict sense of the above definition.

Example. The sequence $5/2, 7/3, 9/4, \ldots, (2n+3)/(n+1), \ldots$ is convergent because

$$|a_p - a_q| = \left|\frac{1}{p+1} - \frac{1}{q+1}\right| \text{ and } \frac{1}{p+1}, \frac{1}{q+1}$$

can both be made as small as we please when $p, q > n_0$ and n_0 is taken large enough.

1.13. Limit of a Sequence. A sequence a_n is said to tend to a limit l when n tends to infinity, if $|l - a_n|$ is ultimately small.

By 'ultimately small' we mean here that given ϵ, any positive number, however small, we can find n_0 such that $|l - a_n| < \epsilon$ for all $n > n_0$.

We then write $\lim\limits_{n\to\infty} a_n = l$ or simply $\lim a_n = l$.

Note. The symbol ϵ, unless otherwise specified, will be used to denote an arbitrary positive number.

Example. Since $\left|\dfrac{2n+3}{n+1} - 2\right| = \dfrac{1}{n+1}$, the limit of $\dfrac{2n+3}{n+1}$ is equal to 2.

1.14. Null Sequence. If $\lim a_n = 0$, a_n is called a *null sequence*.

It follows from the definition of limit that if a_n is not null, a positive number K (independent of n) can be found such that $|a_n| > K$ for all $n > n_0$.

1.15. The Fundamental Theorems on Convergence. If a_n, b_n converge, then the sequences $a_n + b_n$, $a_n - b_n$, a_nb_n, a_n/b_n all converge (except the last when $\lim b_n = 0$).

It follows from the definition of convergence that, given ϵ (> 0), we can find n_0 such that both inequalities

$$|a_p - a_q| < \epsilon, \ |b_p - b_q| < \epsilon$$

are satisfied for all $n \geq n_0$.

Therefore (i) $|(a_p + b_p) - (a_q + b_q)| \leq |a_p - a_q| + |b_p - b_q| < 2\epsilon$, i.e. the sequence $a_n + b_n$ converges.

(ii) The convergence of b_n implies that of $-b_n$ and therefore by (i) $a_n - b_n$ converges.

(iii) $|a_pb_p - a_qb_q| = |(\lambda - \lambda')b_{n_0} + (\mu - \mu')a_{n_0} + \lambda\mu - \lambda'\mu'|$, (where $\lambda = a_p - a_{n_0}$, $\lambda' = a_q - a_{n_0}$, $\mu = b_p - b_{n_0}$, $\mu' = b_q - b_{n_0}$)

$$< 2\epsilon\{|b_{n_0}| + |a_{n_0}| + \epsilon\}$$

i.e. a_nb_n converges.

(iv) If b_n is not a null sequence, a positive constant K can be found such that $|b_n| > K$ for all $n \geq n_0$, n_0 being sufficiently large. Thus n_0 can be determined so that all the inequalities

$$|b_n| > K, \ |a_p - a_q| < \epsilon, \ |b_p - b_q| < \epsilon$$

can be simultaneously satisfied for all $n \geq n_0$.

Then
$$\left|\frac{a_p}{b_p} - \frac{a_q}{b_q}\right| = \left|\frac{(\lambda - \lambda')b_{n_0} + (\mu' - \mu)a_{n_0} + \lambda\mu' - \lambda'\mu}{b_pb_q}\right|$$
$$< \frac{2\epsilon}{K^2}\{|b_{n_0}| + |a_{n_0}| + \epsilon\}$$

i.e. a_n/b_n converges.

1.16. Definition of Real Number (Cantor). A real number may be defined as a convergent sequence $\{a_n\}$ of *rational* numbers.

(i) Two numbers $\{a_n\}$, $\{b_n\}$ are said to be *equal* if $\{a_n - b_n\}$ is a null sequence. This criterion is necessary since a given number may be defined in an infinite number of ways as a sequence.

(ii) The number $\{a_n\}$ is said to be *greater* (*less*) than the number $\{b_n\}$ if $(a_n - b_n)$ is *ultimately* positive (negative).

(iii) The *sum*, *product*, and *quotient* of the numbers $\{a_n\}$, $\{b_n\}$ are defined to be the convergent sequences $\{a_n + b_n\}$, $\{a_nb_n\}$, $\{a_n/b_n\}$ (where, in the quotient, b_n is not null).

The number $-\{a_n\}$ is defined to be $\{-a_n\}$ and the difference $\{b_n\} - \{a_n\}$ to be $\{b_n\} + [-\{a_n\}]$ i.e. $\{b_n - a_n\}$.

Note. It is necessary, for completeness, to show that the sum, product, and quotient is independent of the sequences that define $\{a_n\}$ and $\{b_n\}$.

Thus if $\{a_n'\} = \{a_n\}$, $\{b_n'\} = \{b_n\}$, we must show that $\{a_n' + b_n'\} = \{a_n + b_n\}$, $\{a_n'b_n'\} = \{a_nb_n\}$, $\{a_n'/b_n'\} = \{a_n/b_n\}$.

Now since $\{a_n'\} = \{a_n\}$, $\{b_n'\} = \{b_n\}$, a suffix n_0 exists such that both of the inequalities

$$|a_n' - a_n| < \epsilon, \ |b_n' - b_n| < \epsilon \text{ are true for } n \geq n_0$$

Therefore $\quad |a_n' + b_n' - a_n - b_n| < |a_n' - a_n| + |b_n' - b_n| < 2\epsilon$

i.e. $\quad \{a_n' + b_n'\} = \{a_n + b_n\}$

The proofs of the other equalities $\{a_n'b_n'\} = \{a_nb_n\}$; $\{a_n'/b_n'\} = \{a_n/b_n\}$ are left to the reader.

(iv) If $\{a_n\}$ has a *rational* limit l, then $\{a_n\} = \{l\}$ by the criterion of equality. The number $\{a_n\}$ is in this case called *rational*. Strictly, it should be called rational real, but without ambiguity it may be denoted by the symbol l (of the purely rational domain) and called rational, although it is not logically identical with it. The properties of the rational real numbers are inclusive of those of the real numbers, since the definitions of the operations on real numbers are obviously consistent with those on the purely rational numbers. For example, if the sequence of purely rational numbers $\{a_n\}$ is convergent, it is obvious that the sequence of rational real numbers $\{a_n\}$ is also convergent.

If $\{a_n\}$, a sequence of rational numbers does not possess a rational limit, the real number that it defines is called irrational.

(v) A rational number (and consequently an infinite number of rational numbers) can be determined lying between two unequal real numbers α, β.

Let $\alpha = \{a_n\}$, $\beta = \{b_n\}$, where a_n, b_n are rational and (for definiteness) $\alpha > \beta$.

Since $a_n - b_n$ is > 0 ultimately and does not tend to zero, and since $|\alpha - a_n|$, $|\beta - b_n|$ are ultimately small, a positive constant K exists and a corresponding index n_0 such that all the inequalities $a_n - b_n > K$, $|\alpha - a_n| < \frac{1}{4}K$, $|\beta - b_n| < \frac{1}{4}K$ are simultaneously satisfied for all $n > n_0$.

Thus $\alpha - \frac{1}{2}(a_n + b_n) = \frac{1}{2}(a_n - b_n) + (\alpha - a_n) > \frac{1}{2}(a_n - b_n) - |\alpha - a_n| > \frac{1}{4}K$ and $\frac{1}{2}(a_n + b_n) - \beta = \frac{1}{2}(a_n - b_n) + (b_n - \beta) > \frac{1}{2}(a_n - b_n) - |b_n - \beta| > \frac{1}{4}K$

i.e. $$\beta < \tfrac{1}{2}(a_n + b_n) < \alpha.$$

(vi) *Every Convergent Sequence of Real Numbers tends to a Limit, which is a Real Number.* Let $\{x_n\}$ be a convergent sequence of real numbers. Then, given ε, we can find n_0 such that

$$|x_q - x_p| < \varepsilon \text{ for all } n > n_0.$$

(a) Suppose that no term of $\{x_n\}$ is equal to the preceding term. Then by (v) above, a rational number a_n can always be determined lying between x_n and x_{n+1}.

Now
$$\begin{aligned} |a_m - a_n| &\leq |a_m - x_m| + |x_m - x_n| + |x_n - a_n| \\ &\leq |x_{m+1} - x_m| + |x_m - x_n| + |x_n - x_{n+1}| \\ &< 3\varepsilon \text{ if } m, n > n_0. \end{aligned}$$

Therefore $\{a_n\}$ tends to a limit l, viz. the real number defined by the convergent sequence $\{a_n\}$. But an index n_0 can be found such that both of the inequalities $|l - a_n| < \varepsilon$, $|x_{n+1} - x_n| < \varepsilon$ are simultaneously satisfied for $n > n_0$

i.e. $$|l - x_n| < |l - a_n| + |a_n - x_n| < |l - a_n| + |x_{n+1} - x_n| < 2\varepsilon$$

or $\{x_n\}$ tends also to the limit l.

(b) Suppose that from and after some fixed term x_p, all the terms are equal to x_p.

Then $\{x_n\} \longrightarrow x_p$, since $|x_m - x_p| = 0$, for $m > p$.

(c) If (a) or (b) is not true, we can form an *infinite* sequence $\{\beta_n\}$ by omitting every term from $\{x_n\}$ that is equal to the preceding term.

This sequence $\{\beta_n\}$ is convergent, since $\beta_n = x_m$ for some value of $m \geq n$

i.e. $|\beta_q - \beta_p| = |x_r - x_s| < \varepsilon$ for all $p, q > n_0$, since $r > q$, $s > p$.

By (a), the sequence $\{\beta_n\}$ tends to a limit l: and if ε is given, an index n_1 can be found such that $|l - \beta_n| < \varepsilon$ for all $n > n_1$.

But if n_1 is given, there exists an index $n_2 (> n_1)$ such that $\beta_{n_1} = x_{n_2}$ and therefore $|l - x_n| < \varepsilon$ for all $n > n_2$: i.e. $\{x_n\}$ tends to the limit l.

1.17. The Principle of Convergence. It follows from the previous paragraphs that the necessary and sufficient condition that the sequence $\{x_n\}$ of real numbers should possess a limit is that, given $\varepsilon (> 0)$, it should be possible to find an index n_0 such that

$$|x_p - x_q| < \varepsilon \text{ for all } n > n_0$$

This is known as the '*Principle of Convergence*'. That the condition is *necessary* follows from the definition of a limit; for if the numbers $|l - x_p|$, $|l - x_q|$ are ultimately small, so also is $|x_p - x_q|$. The proof

of its sufficiency, established in the previous paragraph, depends on a correct definition of real number.

Note. Other definitions of real number have been given that are theoretically equivalent to that of Cantor. The best known of these is that given by Dedekind, who defines an irrational number as a *section* of the rational numbers. (See Hobson, *Functions of a Real Variable*, I, 1.)

1.18. The Fundamental Theorems on Limits of Sequences (Real). If $\lim \alpha_n = \alpha$, $\lim \beta_n = \beta$, we may prove by a method very similar to that used for the corresponding theorems on convergence that

$$\lim(\alpha_n \pm \beta_n) = \alpha \pm \beta; \quad \lim(\alpha_n\beta_n) = \alpha\beta; \quad \lim\frac{\alpha_n}{\beta_n} = \frac{\alpha}{\beta}(\beta \neq 0)$$

For example, if $\alpha - \alpha_n = \rho$, $\beta - \beta_n = \sigma$, we can find n_0 such that $|\rho|$, $|\sigma|$ are ultimately small and therefore

$$|\alpha\beta - \alpha_n\beta_n| = |\alpha_n\sigma + \beta_n\rho + \rho\sigma| \leq |\alpha_n|\,|\sigma| + |\beta_n|\,|\rho| + |\rho|\,|\sigma|$$

i.e. $|\alpha\beta - \alpha_n\beta_n|$ is ultimately small and $\alpha_n\beta_n \to \alpha\beta$. The reader should have no difficulty in establishing the other two limit theorems in a similar way.

FIG. 2

1.19. Geometrical Representation of Real Number. Points O, A are chosen on a straight line $X'X$ to represent the numbers 0, 1 respectively (*Fig.* 2). It is then assumed that to every real number x there corresponds a single point P and conversely; the order of the points corresponds to the order of the numbers so that if $x_1 > x_2$, P_1 is to the *right* of P_2. For practical purposes we usually take $x_1 - x_2$ as the magnitude of the displacement P_2P_1, although in special cases it may be more convenient to take some other function of x_1, x_2.

Sequences of numbers $x_1, x_2, \ldots, x_n, \ldots$ converging to α correspond to sequences of points $P_1, P_2, \ldots, P_n, \ldots$ converging to Q, the point that corresponds to α (*Fig.* 3). When x is a variable taking

FIG. 3

all real numbers as its values and any x is regarded as the common limit of the infinite number of equivalent convergent sequences that tend to x, we call x the *real continuous variable*.

The set of *all* real numbers that satisfy the relation $a \leq x \leq b$ is called a *continuum*, and the set of *all* real numbers is called the *Arithmetical continuum*.

In the continuum $a \le x \le b$, the limit of every convergent sequence belongs to the domain, which is therefore said to be *closed*. The intervals specified by $a < x \le b$, $a \le x < b$, $a < x < b$ are not closed.

1.2. Simple Sequences in General. We frequently meet with sequences that are not convergent, and it is convenient at this stage to specify the various possibilities that arise.

1.21. Bounded Sequences. A simple sequence is said to be *bounded* if a positive number K exists, independent of n, such that $|a_n| \le K$ for *all* values of n. Otherwise it is *unbounded*.

For example $\sin\frac{1}{2}n\pi$, $2 - 100/n$; $(\sin\frac{1}{2}n\pi)/n + \sqrt{2}\cos\frac{1}{2}n\pi$, are bounded sequences, suitable values of K being 1, 98, 2 respectively. But $n(n-1)$; $(-1)^{n-1}n$, $(-1)^{n-1}n$ are unbounded.

Note. It is the behaviour of a_n when n is *large* that is important. Thus if a_n were given by such a formula as $\{(n-5)(n-10)\}^{-1}$, a_n is bounded since we should omit $n = 5$, $n = 10$ or commence the sequence at $n = 11$.

1.22. A Convergent Sequence is Bounded. For if l is the limit, ultimately a_n lies between $l - \epsilon$ and $l + \epsilon$ where ϵ is any positive number.

1.23. Finite Oscillation. If a bounded sequence is not convergent, it is said to *oscillate finitely*.

Example. $\sqrt{2}(\sin\frac{1}{4}n\pi + \cos\frac{1}{4}n\pi)$ assumes each of the values $0, \sqrt{2}, -\sqrt{2}, 2, -2$ an infinite number of times. It oscillates finitely.

1.24. Sequences tending to Positive or Negative Infinity. If in an unbounded sequence a_n, the terms are ultimately large and positive, the sequence is said to *tend to positive infinity* and we write $\lim a_n = +\infty$. More precisely: if, given G, a positive number, however large, we can find n_0 such that $a_n > G$ for *all* $n \ge n_0$, then, $\lim a_n = +\infty$; and it is in this sense that the phrase 'ultimately large and positive' is used. If $-a_n \to +\infty$, then a_n is said to *tend to negative infinity* and we write $\lim a_n = -\infty$.

Examples. $n(n^2+1) \to +\infty$, $1 - n^2 \to -\infty$.

1.25. Infinite Oscillation. An unbounded sequence that does not tend to $+\infty$ nor to $-\infty$ is said to *oscillate* infinitely.

Examples. $n^2\cos n\pi$, $n + (-1)^n n^2$ oscillate infinitely.

1.26. Summary of Types of Simple Sequences. There are therefore five types of simple sequences:

(i) *Convergent*. *Ex.* $\lim\{3 + (-1)^n/n\} = 3$.
(ii) *Oscillating Finitely*. *Ex.* $(-1)^n$.
(iii) *Diverging to* $+\infty$. *Ex.* n^2.
(iv) *Diverging to* $-\infty$: *Ex.* $-n^4$.
(v) *Oscillating Infinitely*. *Ex.* $n\{1 + (-1)^n\}$.

1.3. Methods of establishing Convergence of Sequences. For establishing convergence at this stage it will be found sufficient to use—

(i) the characteristic property of a certain type of sequence called a *monotone*;

(ii) the fundamental theorems on limits.

1.31. Monotones. If $a_n < a_{n+1}$ (all n), a_n is called an *increasing monotone*; and if $a_n > a_{n+1}$ (all n), a_n is called a *decreasing monotone*.

Examples. (i) 1, 2, 2, 3, 2, 3, 4, 4, 4, 4, . . . is an increasing monotone.
(ii) 3, $2\frac{1}{2}$, $2\frac{1}{3}$, . . ., $2 + 1/n$, . . . is a decreasing monotone.

Notes. (i) The terms 'increasing' and 'decreasing' are used here in the *broad* sense. If $a_n < a_{n+1}$, (all n), a_n increases in the *narrow* sense.
(ii) It is usually sufficient for the application of the properties of monotones that the sequence should be *ultimately* monotonic.

1.32. A Bounded Monotone is Convergent. This is the characteristic property. If a monotone were *not* convergent it would be possible to find an unending set of increasing suffixes $n_1, n_2, \ldots, n_m, \ldots$ such that, given $\varepsilon\ (> 0)$,

$$|a_{n_2} - a_{n_1}| > \varepsilon,\ |a_{n_3} - a_{n_2}| > \varepsilon,\ \ldots$$

and so, if the monotone were increasing, $a_{n_m} > a_{n_1} + m\varepsilon$ and if the monotone were decreasing $a_{n_m} < a_{n_1} - m\varepsilon$. But $m\varepsilon \to +\infty$ when $m \to \infty$ and therefore an unbounded increasing monotone tends to $+\infty$ whilst an unbounded decreasing monotone tends to $-\infty$: and if the monotone is bounded it must converge.

Note. The monotone is important not only because of its frequent occurrence but also because it can be shown that *every* convergent sequence is equivalent to *two* monotones, one increasing up to the limit and the other decreasing down to the limit. (*Read* Bromwich, *Infinite Series*, I.)

1.33. Application of the Fundamental Limit-Theorems. This application may be summarized in the following form:

If $\lim a_n = a$, $\lim b_n = b$, $\lim c_n = c$, . . ., there being a finite number of sequences $a_n, b_n, c_n, \ldots$, then

$$\lim R(a_n, b_n, c_n, \ldots) = R(a, b, c, \ldots)$$

where R denotes a finite number of the fundamental operations (provided there is no division by zero).

This result follows by repeated applications of the fundamental limit-theorems.

1.34. The Sequence x^n. When $0 < x < 1$, $x^n > 0$ and is monotonic decreasing. Therefore x^n tends to a limit l; but $\lim x^{n+1} = x \lim x^n$, i.e. $l = xl$ and therefore $l = 0$ since $x \neq 1$. When $x = 1$, $x^n = 1$, $\lim x^n = 1$; when $x = 0$, $x^n = 0$, $\lim x^n = 0$. By writing $x = 1 + y$ when $x > 1$, and $x = -y$ when $x < 0$ we may immediately deduce the nature of the sequence x^n for values of x outside the interval $0 \leq x \leq 1$. Thus: $x^n \to 0$ when $|x| < 1$; $x^n \to 1$ when $x = 1$, $x^n \to +\infty$ when $x > 1$; x^n oscillates finitely between $+1$ and -1 when $x = -1$; x^n oscillates infinitely when $x < -1$.

Note. The reader may easily prove that if s_n is an increasing (decreasing) monotone which is such that $s_n < K\ (> K)$ for all n, then s_n tends to a limit $l \leq K\ (\geq K)$.

Examples. (i) The sequence $x^n/n! \to 0$ for all finite x.

Let $x > 0$: then $x^n/n!$ is ultimately monotonic decreasing, i.e. after $n > x - 1$. Therefore $x^n/n!$ tends to a limit $l\,(\geq 0)$.

But $\quad l = \lim \{x^{n+1}/(n+1)!\} = x \lim (x^n/n!) . \lim (1/(n+1)) = 0$

Also since $x^n/n! = (-1)^n(-x)^n/n!$, it follows that $\lim (x^n/n!)$ is zero for all finite x, positive or negative.

(ii) Find $\lim \,(2-3n^2)(1+4n^3)x^n/\{(n-1)^2(1-2n)^3\}$.

Here $a_n = x^n \dfrac{3}{2} \dfrac{(1-2/3n^2)(1+1/4n^3)}{(1-1/n)^2(1-1/2n)^3}$ which $\to (3/2) \lim x^n$ if $\lim x^n$ exists.

Thus the limit is 0 when $x = 0$ and $3/2$ when $x = 1$. For other values of x, the sequence has the character of x^n.

(iii) Prove that if $a_{n+1} = (3+2a_n)/(2+a_n)$ where $a_1 = 1$, then $\lim a_n = \sqrt{3}$.

In this sequence $(a_{n+1}{}^2 - 3)(a_n+2)^2 = a_n{}^2 - 3$; but $a_1{}^2 < 3$ and therefore $a_n{}^2 < 3$, all n.

Also $(a_{n+1} - a_n)(a_n + 2) = (3 - a_n{}^2)$, i.e. a_n is an increasing monotone; and it is bounded since $a_n{}^2 < 3$. Therefore a_n tends to a positive limit l, where

$$l = (3+2l)/(2+l), \text{ i.e. } l = \sqrt{3} \text{ (the positive root).}$$

1.35. The O- and o- Notation for Simple Sequences. If a_n, b_n are such that $|a_n/b_n|$ is ultimately bounded, we say that a_n, b_n are of the *same order* when n is large and write $a_n = O(b_n)$.

For example: $(n^2+1)/(2+\cos n\pi) = O(n^2)$.

In particular

(i) if $\lim (a_n/b_n)$ exists, $a_n = O(b_n)$

(ii) if $\lim (a_n/b_n) = 0$, we write $a_n = o(b_n)$ although it is still correct to say $a_n = O(b_n)$

(iii) if a_n is bounded, we can write $a_n = O(1)$

Example. $4/(1+n^2) = o(1/n)$ and is also $O(1)$.

1.4. Limits of a Function of the Real Variable. Suppose for simplicity that a function $f(x)$ is defined for *all* real values of x. Then it is found convenient to use the terms '*neighbourhood*', '*near*', '*large*' and '*small*' in a precise sense like that in which the term 'ultimately' has been used.

1.41. The Terms 'Near', 'Large', 'Small'. A property is said to be true *near* $x = a$ or *in the neighbourhood of* $x = a$, if an interval can be found with a an *internal* point, *throughout* which the property is true. The property may or may not be true at the point $x = a$. If a is an end-point of the interval, the property applies only to the neighbourhood on *the left* (or *right*) of a.

Again, if G can be found such that the property is true for *all* values of $x > G$, it is said to be true for *x large and positive*; and if it is true for all $x < G$ it is true for *x large and negative*.

Finally the phrase 'near O' may be replaced by 'for small x'.

1.42. Limits of a Function. Let $x_1, x_2, \ldots, x_n, \ldots$ be *any* sequence whose limit is a. Then if the sequence $f(x_1), f(x_2), \ldots, f(x_n), \ldots$ tends to a limit l which is *independent* of the particular sequence x_n tending to a, l is called the limit of $f(x)$ when x tends to a and we write $\lim\limits_{x \to a} f(x) = l$ (*Fig. 4* (*a*)).

Now the number a may be regarded as the common limit of two *monotones* a_n, b_n, the former increasing up to a and the latter decreasing down to a (*Fig. 4* (*b*)).

If $\lim f(a_n) = l_1$ for *all* such monotones a_n (where $a_n \to a$), l_1 is called the *limit of* $f(x)$ *on the left of* a and we write $l_1 = f(a - 0)$.

Similarly if $\lim f(b_n)$ exists ($b_n \to a$), and is equal to l_2, then we write $l_2 = f(a + 0)$

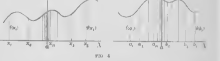


FIG. 4

If $l_1 = l_2$, then $f(a + 0) = f(a - 0) = \lim\limits_{x \to a} f(x)$.

It follows from the above that if $f(x) \to l$ then $|l - f(x)|$ is *small near* $x = a$, and a similar interpretation may be given to l_1, l_2 in terms of the appropriate neighbourhood. This last result may, in fact, be used as a new definition of $\lim f(x)$.

1.43. The Fundamental Theorems on Limits of Functions. It is an immediate consequence of the limit theorems for sequences that if $\lim f(x) = l$, $\lim \phi(x) = l'$ when $x \to a$ then $\lim (f \pm \phi) = l \pm l'$, $\lim (f\phi) = ll'$; $\lim (f/\phi) = l/l'(l' \neq 0)$, but these results can be proved directly from the above alternative definition of $\lim f(x)$. Again, if R denotes a finite number of applications of the fundamental operations, then

$$R(f_1(x), f_2(x), \ldots, f_m(x)) \to R(l_1, l_2, \ldots, l_m)$$

when $f_1(x) \to l_1$, $f_2(x) \to l_2$, $\ldots$, $f_m(x) \to l_m$, and there is no division by zero.

1.44. The O and o Notation for Functions. If $|f(x)/\phi(x)| < K$, where K is independent of x and x is near a, i.e. if $|f/\phi|$ is bounded near a, we write $f(x) = O\{\phi(x)\}$. A similar notation may be used for x large or small.

In particular (i) if $\lim (f/\phi)$ exists, $f = O(\phi)$.

(ii) if $\lim (f/\phi) = 0$, we may write $f = o(\phi)$, but it is consistent to write $f = O(\phi)$.

Example. $(x^4 + x^2)/(1 - x) = O(x^2)$, x small; $= O(x^3)$, x large; and $O\{1/(x - 1)\}$ near $x = 1$.

1.45. Continuity. A function $f(x)$ is said to be *continuous* at $x = a$ if $f(x) - f(a)$ is small near $x = a$ (Cauchy), i.e. if given $\varepsilon (> 0)$, however small, a number $\delta (> 0)$ can be found such that $|f(x) - f(a)| < \varepsilon$ for *all* points in the interval $|x - a| < \delta$. This implies that $\lim\limits_{x \to a} f(x) = f(a)$.

Note: (i) *Heine's* definition of continuity consists in saying that $f(x)$ is continuous at $x = a$, if $\lim f(x) = f(a)$ when $x \to a$ where a is the limit of any one of the infinite sequences that tend to a. To deduce Cauchy's definition from this requires the assumption of a certain selective principle known as the "*Multiplicative Axiom*" (Hobson, '*Real Variable*', *I*, 266).

(ii) If $f(a - 0) = f(a)$, the function is said to be continuous *on the left of* a; and if $f(a + 0) = f(a)$, it is continuous *on the right* of a.

1.46 Geometrical Illustration of Continuity. Let $f(x)$ be continuous at $x = a$. Draw the lines (p), $y = f(a) + \varepsilon$; (q), $y = f(a) - \varepsilon$.

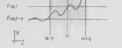


FIG. 5

Continuity implies that we can find a neighbourhood of $x = a$, within which the curve $y = f(x)$ lies entirely between the lines p, q (*Fig. 5*).

1.47. The Polynomial and the Rational Function. A function $P_n(x)$ of the form given by

$$P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_n, \ (n, \text{ positive integer})$$

is called a *Polynomial* (of degree n).

A function $R(x)$ reducible to the form $P_n(x)/Q_m(x)$, where P_n, Q_m are polynomials is called a *Rational Function*. It follows from the fundamental limit-theorems that the polynomial is continuous for *all* values of x and that the rational function is continuous for all values of x, except those that make the denominator vanish.

1.5. Limits at Infinity and Infinite Limits. If x_n is a sequence of *positive* numbers tending to zero, then $\lim (1/x_n)$ is $+\infty$, and therefore if $f(1/\xi)$ tends to l when ξ tends to zero *from the right*, we say that $f(x) \to l$ when $x \to +\infty$ and write

$$\lim_{x \to +\infty} f(x) = l \text{ or } f(+\infty) = l.$$

Similarly, if $f(1/\xi) \to l'$ when $\xi \to 0$ *from the left*, we write

$$\lim_{x \to -\infty} f(x) = l' \text{ or } f(-\infty) = l'.$$

In some cases $l = l'$ and then we may write $\lim_{|x| \to \infty} f(x) = l$.

1.51. Infinite Limits. Let a_n be any monotone increasing to the limit a. If $f(a_n) \to +\infty$ for every such monotone we say that

$$f(x) \to +\infty$$

on the left of $x = a$ and write $f(a - 0) = +\infty$. Similarly meanings

may be given to the relations $f(a - 0) = -\infty$; $f(a + 0) = +\infty$; $f(a + 0) = -\infty$.

In these cases, $f(x)$ is discontinuous at $x = a$.

Examples. (i) Find $\lim\limits_{x \to 1} \dfrac{x^4 - 6x^2 + 8x - 3}{3x^4 - 8x^3 + 6x^2 - 1}$.

Let $x = 1 + h$ and the function is $\dfrac{4 + h}{4 + 3h}$, $(h \neq 0)$; thus the limit is 1, when $h \to 0$, i.e. when $x \to 1$.

(ii) Find the discontinuities of $E(x)$, the greatest integer $\leq x$. Here $E(n - 0) = n - 1$; $E(n) = n = E(n + 0)$.

(iii) Find $\lim (x - a)^{-r}$ when $x \to a$, $(r > 0)$.

Here $f(a + 0) = +\infty$, $f(a - 0) = -\infty$ (r odd); $f(a + 0) = f(a - 0) = +\infty$ (r even).

(iv) Discuss $\lim\limits_{x \to \pm\infty} \dfrac{a_0x^m + a_1x^{m-1} + \ldots + a_m}{b_0x^n + b_1x^{n-1} + \ldots + b_n}$ (m, n positive integers).

If $m = n$, limit is a_0/b_0; if $m < n$, limit is zero.

If $m > n$, limit is $\pm\infty$, the result depending on the sign of a_0/b_0 and on whether $(m - n)$ is even or odd.

Thus $\lim\limits_{x \to -\infty} \dfrac{1 - x^4}{1 + x^3} = +\infty$, $\lim\limits_{x \to \infty} \dfrac{x^3 - x - 1}{x^2 + 1} = +\infty$; $\lim\limits_{x \to -\infty} \dfrac{x^3 - x - 1}{x^2 + 1} = -\infty$.

1.52. Asymptotic Approximations. If $f(x) = \phi(x) + o\{\phi(x)\}$, when x is large, $\phi(x)$ is called an *asymptotic approximation* (not the only one) to $f(x)$. It may sometimes be necessary to specify the *sign* of x.

In some cases, as for example in the case of the rational function, it may be possible to express $f(x)$ in the form

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n + o(1),$$

where n is a positive integer. Then $a_0x^n + \ldots + a_n (= P_n(x))$ may be called the polynomial asymptotic to $f(x)$ and gives a close approximation to $f(x)$, where x is large, since $\lim\limits_{x \to \infty} (f(x) - P_n(x)) = 0$.

When $n = 1$, the polynomial is linear and the corresponding line $y = a_0x + a_1$ is called a rectilinear asymptote or simply an *asymptote*. It is usually possible in this case to obtain a closer approximation that takes the form

$$f(x) = a_0x + a_1 + k/x^m + o(x^{-m}), \quad (m \geq 1,\ k \text{ independent of } x),$$

so that if m is *even*, the curve $y = f(x)$ is *above* (*below*) the asymptote at *both* ends when $k > 0$ $(k < 0)$; but if m is *odd*, the curve is above at one end and below at the other.

If $a_0 = 0$, the asymptote is parallel to the x-axis and if, in addition, $a_1 = 0$, the asymptote is $y = 0$.

Again, near $x = a$, a function $f(x)$ may be expressible in the form $f(x) = k(x - a)^{-m} + o\{(x - a)^{-m}\}$, $(m > 0)$, and since in this case $f(x) \to \pm\infty$ as $x \to a$, we call $x = a$ an asymptote of the curve.

Example. Let $f(x) = \dfrac{(x^3 + 1)}{x^2(1 - x)}$.

Using the result $(1-x)^{-1} = 1 + x + x^2 + \ldots + x^n + O(x^{n+1})$, when x is small, we find

$$f(x) = x^{-3}(1 + x^2)(1 + x + O(x^2)) = x^{-3} + O(x^{-2}),\ x \text{ small},$$
$$f(x) = 2(1-x)^{-1} + O(1) \text{ near } x = 1,$$

and
$$f(x) = -x^{-2}(1 + x^{-2})(1 + x^{-1} + O(x^{-2})) = -x^{-2} + O(x^{-3})$$

when $|x|$ is large.

1.6. Derivatives. If $f(x)$ is defined for all x in the interval $a \leq x \leq b$ and if x_1 is a value in the interval, the function $F(x)$ given by

$$F(x) = \{f(x) - f(x_1)\}/(x - x_1)$$

is defined for all points of the interval except $x = x_1$.

If $F(x)$ tends to a limit when $x \to x_1$, that limit is called the *derivative* (or *differential coefficient*) of $f(x)$ at $x = x_1$. When the derivative exists for a certain domain, it is a function of x, and is usually denoted by $f'(x)$ or dy/dx, where $y = f(x)$.

The limit $F(x_1 + 0)$, if it exists, is called the *derivative on the right* of x_1 (or *progressive derivative*), and $F(x_1 - 0)$ is the *derivative on the left* (or *regressive derivative*).

It is more convenient in practice to take the definition of $f'(x)$ in the form:

$$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$

to which the above is obviously equivalent.

Notes. (i) Continuity of $f(x)$ is *necessary* for the existence of $f'(x)$, for obviously the limit cannot exist unless $f(x+h)$ tends to $f(x)$ as h tends to zero. Continuity is not, however, *sufficient* as may be shown by the following examples.

(a) Let $f(x) = |x|$. This is continuous for all x, but $f'(+0) = 1$, whilst $f'(-0) = -1$.

(b) Let $f(x) = x \sin(1/x)$ (where in this and in other cases we assume for illustrative purposes the properties of elementary functions). Since $|f(x)| \leq |x|$, the function is continuous at $x = 0$, if $f(0)$ is given to be 0.

But $\lim_{h \to 0} \{h \sin(1/h) - 0\}/h$ does not exist because $\sin(1/h)$ oscillates between -1 and $+1$ as $h \to 0$.

(ii) A more general definition of the derivative is given by

$$\lim \{f(x + h_1) - f(x + h_2)\}/(h_1 - h_2)$$

where $h_1, h_2 \to 0$ independently.

This may exist when the ordinary derivative does not.

For example ($h_1 = -h_2 = h$) $\lim_{h \to 0} \{f(x+h) - f(x-h)\}/2h$, (called the *central derivative*) exists when $f(x) = |x|$, its value at $x = 0$ being $\lim (|h| - |-h|)/2h$, i.e. zero.

1.61. Higher Derivatives. If $f'(x)$ has a derivative, this is denoted by $f''(x)$ or d^2y/dx^2 (if $y = f(x)$) and is called the *second derivative*. Similarly $f(x)$ may have third and higher order derivatives, the derivative of the nth order being denoted by

$$f^{(n)}(x) \text{ or } d^ny/dx^n.$$

1.62. Rules for calculating Derivatives. If $f'(x)$, $\phi'(x)$ are known, it is easy to establish the formulae:

$$\frac{d}{dx}(f+\phi) = f' + \phi', \quad \frac{d}{dx}(f\phi) = f'\phi + f\phi', \quad \frac{d}{dx}(f/\phi) = \frac{f'\phi - f\phi'}{\phi^2}.$$

It will be assumed that these results are familiar to the reader.

1.63. The Derivative of $u_1, u_2 \dots u_n$ (u_r being a function of x). If

$$P_r = u_1, u_2 \dots u_r, \text{ then } P_n' = u_n P_{n-1}' + u_n' P_{n-1}.$$

i.e. $$P_n'/P_n = u_n'/u_n + P_{n-1}'/P_{n-1}$$

and by repeated applications of this result we find

$$P_n' = (u_1, u_2 \dots u_n)(u_1'/u_1 + u_2'/u_2 \dots + u_n'/u_n)$$

Note. This formula is, of course, obtained immediately by differentiating $\log P_n = \Sigma \log u_r$.

1.64. The Derivative of x^n (n being a positive or negative integer). In § 1.63, take $u_1 = u_2 = \dots = u_n = x$, then

$$\frac{d}{dx}(x^n) = x^n(n/x) = nx^{n-1} \text{ (n positive)}$$

If $n = -m$ (m positive), $\frac{d}{dx}(x^n) = -mx^{m-1}/x^{2m}$ by the quotient formula, so that $\frac{d}{dx}(x^n) = nx^{n-1}$ also when n is negative.

1.65. Leibniz's Theorem for the nth Derivative of a Product. If u, v are functions of x, then

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n}v + {}^nC_1\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \dots + {}^nC_r\frac{d^{n-r}u}{dx^{n-r}}\frac{d^r v}{dx^r} + \dots + u\frac{d^n v}{dx^n}$$

This may be proved easily by induction, with the use of the identity ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (*e.g. Hardy, Pure Mathematics, p. 101*).

1.66. Derivative of a Function of a Function. If $f(x)$ is continuous, and if for the values of $z(=f(x))$ $F(z)$ is a continuous function of z, then $F(z)$ is a continuous function of x; for $f(x) \to f(x_1)$ when $x \to x_1$ and $F\{f(x)\} \to F\{f(x_1)\}$ when $f(x) \to f(x_1)$.

Also suppose that x_1 is a point where $f(x)$ is *not* equal to $f(x_1)$ in a sufficiently small neighbourhood of x_1; i.e. suppose that $\delta(>0)$ can be found such that $f(x)-f(x_1)$ vanishes only at $x = x_1$ in the interval $|x-x_1| < \delta$, then

$$\frac{F(z)-F(z_1)}{x-x_1} = \frac{F(z)-F(z_1)}{z-z_1}\cdot\frac{z-z_1}{x-x_1} \text{ (except at } x = x_1\text{), since } z - z_1 \neq 0,$$

i.e. $\frac{d}{dx}F(z)$ exists at x_1 and is equal to $F'(z_1).f'(x_1)$, if the derivatives F', f' exist.

1.67. Derivatives of the Rational Function. The above results are obviously adequate for the determination of the derivatives of any order of

$$R(x) = (a_0x^n + a_1x^{n-1} + \ldots + a_n)/(b_0x^m + b_1x^{m-1} + \ldots + b_m)$$

Examples. (i) Find the derivative of $(u_1, u_2 \ldots u_n)/(v_1, v_2 \ldots v_m)$, where u_r, v_s are functions of x. Let $P = \prod_1^n u_r$, $Q = \prod_1^m v_r$, then

$$\frac{d}{dx}\left(\frac{P}{Q}\right) = \frac{P'Q - PQ'}{Q^2} = \frac{P}{Q}\left(\frac{P'}{P} - \frac{Q'}{Q}\right) = \frac{u_1 . u_2 \ldots u_n}{v_1 . v_2 \ldots v_m}\left(\sum_1^n \frac{u_r'}{u_r} - \sum_1^m \frac{v_r'}{v_r}\right)$$

(the result being also obtained by the use of the logarithmic function).

(ii) The derivative of $\{x^3(x-2)\}/(5x+2)^2$ is by (i)

$$\frac{x^3(x-2)}{(5x+2)^2}\left(\frac{3}{x} + \frac{1}{x-2} - \frac{10}{5x+2}\right) = \frac{2x^2(x+1)(5x-6)}{(5x+2)^3}.$$

(iii) Prove that $\frac{d^n}{dx^n}\{(x^2-1)^n\} = 2^n n!$ when $x = 1$.

By Leibniz's Theorem $\frac{d^n}{dx^n}\{(x+1)^n(x-1)^n\}$

$$= (x+1)^n . n! + n(x+1)^{n-1} n!(x-1) + \ldots n!(x-1)^n$$
$$= 2^n n! \text{ when } x = 1$$

1.7. Graphs of Functions of the Real Variable.

It is assumed at this stage that we are dealing with explicit (one-valued) functions of x, although later we shall consider graphs that correspond to implicit relationship between two variables. In order that the graph should bring into evidence the salient features of the functional relation, it should at least indicate

(i) where the function y is *increasing* (or *decreasing*);

(ii) the *stationary* values, if any, including the *relative maxima* and *minima*;

(iii) the neighbourhoods, if any, where y is *large* (or *small*) or has a value associated with a particular problem;

(iv) the behaviour of y when x is large;

(v) the behaviour of y at any exceptional points peculiar to the function.

1.71. The Stationary Values. If $f(x)$ possesses a derivative $f'(a)$ at $x = a$, we may write

$$f(a+h) = f(a) + hf'(a) + o(h)$$

and therefore if $f'(a)$ is *positive* (*negative*), $f(x)$ is *increasing* (*decreasing*) near a.

The line $y - f(a) = (x - a)f'(a)$ is the *tangent* at $P(a, f(a))$ to the curve $y = f(x)$, since this line is easily shown to be the limiting position of PQ when $Q \to P$ along the curve.

If $f'(a) = 0$, the tangent is *parallel* to the x-axis, and the function is therefore said to have there a *stationary value* $f(a)$.

Now when $f'(a) = 0$, it is usually possible to write $f'(x)$ in the form $(x-a)^m\phi(x)$ where $\phi(a) \neq 0$, m is an integer $\geqslant 1$ and $\phi(x)$ is bounded and of constant sign near $x = a$.

If m is *even*, the curve *crosses* the tangent, since $f'(x)$ does not change sign as x increases through the value a. Such a tangent is called *inflexional*, and it is usual to refer to the point $(a, f(a))$ as an *inflexum*.

If m is odd, $f'(x)$ changes sign as x increases through the value a, so that $f(a)$ is a *minimum* when $\phi(a) > 0$ and a *maximum* when $\phi(a) < 0$.

Example. Discuss the stationary values of $x^3(x-2)/(5x+2)^3$

Here
$$\frac{dy}{dx} = \frac{2x^2(x+1)(5x-6)}{(5x+2)^3}.$$

Near $x = 0$, $f'(x) = x^2(+)(-)/(+)$: *Inflexion*
Near $x = -1$, $f'(x) = (x+1)(+)$: *minimum.*
Near $x = 6/5$, $f'(x) = (5x-6)(+)$: *minimum.*

1.72. Graph of the Polynomial. Let
$$y = P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n.$$

If $b = P(a)$, then $y - b$ is of the form $(x-a)^sQ(x)$ where $Q(a) \neq 0$ and $1 \leq s \leq n$ (so that $Q(x)$ is a polynomial of degree $n - s$).

If $s = 1$, the tangent is $(y - b) = (x-a)Q(a)$.

If $s > 1$, the tangent is $y = b$, and the shape of the curve at (a, b) is approximately that of $(y - b) = (x - a)^sQ(a)$.

This is the same as that of $y = Ax^s$ at $(0, 0)$, $(A = Q(a))$. The shape of $y = x^s$ is readily seen by plotting a few points on it in the *first* quadrant and completing it by symmetry. The shape of $y = Ax^s$ can be deduced from that of $y = x^s$.

Thus (i) $y = x^s$ is symmetrical about O if s is odd and it is symmetrical about the y axis if s is even.

(ii) When $A > 0$, the curve $y = Ax^s$ is obtained from $y = x^s$ by increasing the ordinates in the ratio $A : 1$.

(iii) When $A < 0$, $y = Ax^s$ is obtained from $y = -Ax^s$ by taking the reflexion of the latter in the x-axis.

Notes. (i) For a curve to be of practical value, it may be necessary to have different scales on the axes.

(ii) It should be noted that if $m > n$, $(A, B \neq 0)$
$$|Ax^m| > |Bx^n|,\ x \text{ large};\quad |Ax^m| < |Bx^n|,\ x \text{ small}.$$

Examples (i) $y = x^4/10$; (ii) $y = 10x^3$; (iii) $y = -x^5/900$ (*Fig.* 6).

The graph of the polynomial is sufficiently indicated by a knowledge of the stationary values, of the points where $y = 0$ (or any other suitable

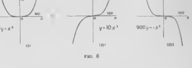


FIG. 6

value c) and of the shapes there; and finally of the shape when x (and in this case y) is large.

The stationary values of $y = P(x)$ are given by the equation $P'(x) = 0$ and if the *real* roots of this equation are known it is a simple matter to complete the graph and to deduce, if required, the real roots of the equation $P(x) = c$. If, however, the roots of $P'(x) = 0$ are not obvious, it may be possible to draw the graph of $y = P'(x)$ by the methods suggested for $y = P(x)$; and we should naturally avail ourselves of the results that occur in the theory of equations if these are more readily applicable than purely geometrical methods.

Note. When $f(x)$ is given by $f(x) = \phi(x) + o(\phi(x))$ near (a, b), the shape of the curve is approximately that of $y = \phi(x)$. If $y \to \infty$ when $x \to a$ it is convenient to refer to this neighbourhood by the symbol (a, ∞). Similarly the symbol (∞, ∞) refers to the neighbourhood in which x, y are both large.

Examples. (i) $y = \frac{1}{8}(35x^4 - 30x^2 + 3)$. (Legendre's Polynomial P_4.)

Symmetry about y-axis. $y = 0$ when $x = \pm 0.86, \pm 0.34$. At (∞, ∞), $8y \sim 35x^4$. At $(1, 1)$, $y' = 10$.

$y' = 0$ at $(0, \frac{3}{8})$ (maximum), and at $(\pm 0.65, -\frac{3}{7})$, (minima) (*Fig.* 7 (*a*)).

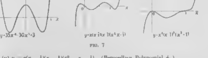


FIG. 7

(ii) $y = x(x - \frac{1}{2})(x - 1)(x^2 - x - \frac{1}{2})$. (Bernoullian Polynomial ϕ_5.)

Symmetry about $(\frac{1}{2}, 0)$, $y = 0$ when $x = 0, 0.5, 1, 1.264, -0.264$. At $(0, 0)$, $y \sim -x/4$, at $(\frac{1}{2}, 0)$, $y \sim (0.15)(x - \frac{1}{2})$: at $(1.264, 0)$, $y \sim (0.39)(x - 1.264)$, $y' = 5x^2(x - 1)^2 - \frac{1}{6}$, giving $(1.16, -0.02)$, (minimum), $(0.23, -0.02)$, (minimum), $(0.77, 0.02)$, (maximum); $(-0.16, 0.02)$, (maximum). At (∞, ∞), $y \sim x^5$ (*Fig.* 7 (*b*)).

(iii) $y = x^3(x - 1)^3(x^2 + 1)$. At $(0, 0)$, $y = -x^3$; $(1, 0)$, $y \sim 2(x - 1)^3$; (∞, ∞), $y \sim x^8$.

Stationary values (other than above), given by $7x^3 - 4x^2 + 5x - 2 = 0$.

Since $21x^2 - 8x + 5 = 0$ has no real roots, the above cubic has only one real root $(x = 0.44)$, giving $y = -0.04$ (minimum) (*Fig.* 7 (*c*)).

1.73. The Graph of the Rational Function. Let the function be $y = P(x)/Q(x)$ where $P(x)$, $Q(x)$ are polynomials. The approximations in the following neighbourhoods should be determined:

(i) At $(a, 0)$ where a is a *real* root of $P(x) = 0$. The shape there is given by an equation of the form $y = A(x - a)^r$ (r integral and positive).

(ii) At (b, ∞) where b is a *real* root of $Q(x) = 0$. Here the approximation is of the form $y = B/(x - b)^s$, (where s is a positive integer > 1).

(iii) When x is large where the approximation may take the form $y = Ax^r$, $A_1x + B + C/x^m$ (where A_1, B may be zero).

(iv) At the stationary values determined by $P'Q = PQ'$. A knowledge of (i), (ii), (iii) often enables us to state the *number* and *approximate position* of the stationary values.

The approximations are therefore of two kinds (a) $y = Ax^s$ which has already occurred in the polynomial and (b) $y = A/x^s$ which is new $(s \geq 1)$.

The graph of $y = 1/x^s$ is readily drawn in the first quadrant: y decreases as x increases and the axes are asymptotes. The curve may be completed by symmetry.

The curve $y = -1/x^s$ is the reflexion of $y = 1/x^s$ in the x-axis.
The curve $y = A/x^s$ is like $y = 1/x^s$ when $A > 0$ and like $y = -1/x^s$ when $A < 0$.

Also it is important to note that $(A, B \neq 0)$.

$$|A/x^m| < |B/x^n|,\ x \text{ small}; \quad |A/x^m| > |B/x^n|,\ x \text{ large}. \quad (m < n > 0)$$

Examples. (i) $y = -2/x^3$, (ii) $y = 1/x^4$ (*Fig.* 8).

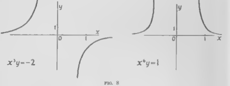


FIG. 8

1.74. The Binomial Expansion for a Negative Integer. In finding approximations to the rational function it will be found necessary to use the expansion for $(1 - x)^{-n}$ when n is a positive integer and x is small. This may be obtained as follows:

The identity $\dfrac{1 - x^{n+r}}{1 - x} = 1 + x + x^2 + \ldots + x^{n+r-1}$ is true if $x \neq 1$, n, r being positive integers.

The $(n - 1)$th derivative of $x^{n+r}/(1 - x)$ can be discontinuous only when $x = 1$ and is obviously of the form $O(x^{r+1})$ when x is small. Thus by differentiating the identity $(n - 1)$ times we find

$$\frac{(n-1)!}{(1-x)^n} = (n-1)! + \frac{n!}{1!}x + \frac{(n+1)!}{2!}x^2 + \ldots + \frac{(n+r-1)!}{r!}x^r + O(x^{r+1})$$

i.e.

$$\frac{1}{(1-x)^n} = 1 + nx + \frac{n(n+1)}{1.2}x^2 + \ldots + \frac{n(n+1)\ldots(n+r-1)}{r!}x^r + O(x^{r+1})$$

Examples. (i) $y = \{x^2(x-1)\}/\{(x+1)^3(x-2)^2\}$.

At $(0, 0)$, $y \sim -x^2/4$; $(1, 0)$, $y = (x-1)/4$; $(-1, \infty)$, $y \sim -\frac{2}{9}(x+1)^{-3}$; $(2, \infty)$, $y \sim \frac{4}{27}(x-2)^{-2}$, (∞, ∞), $y = 1 + 1/x + o(1/x)$. Stationary values when $x^2 - 7x - 6 = 0$, giving $(0{\cdot}77, -0{\cdot}02)$, (minimum); $(-7{\cdot}77, 0{\cdot}04)$, (minimum). (*Fig.* 9 (*a*).)

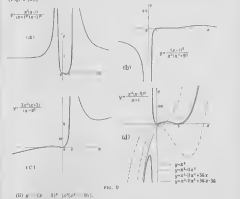


FIG. 9

(ii) $y = (x-1)^3/\{x^2(x^2+9)\}$.

At $(0, \infty)$, $y \sim -\frac{1}{9}x^{-2}$, $(1, 0)$, $y \sim \frac{1}{10}(x-1)^3$, $(\infty, 0)$, $y \sim 1/x$. Only other stationary value $(6, 0{\cdot}038)$, (maximum). (*Fig.* 9 (*b*).)

(iii) $y = 3x^2(x-2)/(x-1)^2$.

At $(0, 0)$, $y = 6x^2$, $(-2, 0)$, $y \sim \frac{1}{2}(x+2)$, $(1, \infty)$, $y \sim -3(x-1)^{-2}$; (∞, ∞), $y \sim 3x + 12 + 21/x$. Stationary values $(4, 32)$, (minimum); $(-1, 0{\cdot}75)$, (maximum). (*Fig.* 9 (*c*).)

(iv) $y = x^3(x-5)^2/(x+1)$.

At $(0, 0)$, $y \sim 25x^3$, $(5, 0)$, $y \sim 25(x-5)^2/6$; $(-1, \infty)$, $y \sim -36/(x+1)$.

At (∞, ∞), $y = (x-2)(x-3)(x-6) + O(1/x)$. Stationary values $(2, 12)$, (maximum); $(-1{\cdot}67, -185)$, (maximum). The various approximations at (∞, ∞), x^3, $x^3 - 11x^2$, $x^3 - 11x^2 + 36x$, $(x-2)(x-3)(x-6)$ are shown in the figure. (*Fig.* 9 (*d*).)

Examples I

Determine the character of the sequences whose nth terms are given in *Examples 1–33.*

1. $\dfrac{4n^3+n+4}{(1-n)(2+3n)}$ 2. $\dfrac{n^4-n^2+1}{(n-1)^4}$ 3. $\dfrac{(n^2+1)^2}{(n-1)(n-2)(n-3)}$ 4. $n^2 - n$

5. $1 - n^2$ 6. $\dfrac{\sin\frac{1}{2}n\pi + 3n}{n}$ 7. $\dfrac{\sin\frac{1}{2}n\pi + 2n^2}{n}$ 8. $\sin^2(\frac{1}{2}n\pi) + 2\cos(\frac{1}{2}n\pi)$

9. $n\sin\frac{1}{2}n\pi$ 10. $\frac{n-1}{n+1}\cos^2(\frac{1}{3}n\pi)$ 11. $\frac{2n^2+1}{1+2n}\cos^2(\frac{1}{3}n\pi)$

12. $\frac{n^2-1}{n^2+1}\cos^2(\frac{1}{2}n\pi)$ 13. $\frac{n^2+1}{2n^2-1}\cos(\frac{1}{2}n\pi)$ 14. $n^2\sin(\frac{1}{2}n\pi)$

15. $\sin\frac{1}{2}n\pi\cos\frac{1}{2}n\pi$ 16. $\frac{2+n^2\cos(\frac{1}{2}n\pi)}{3-4n^2}$ 17. $n^4+n^4\cos(\frac{1}{2}n\pi)$

18. $n^2+n^4\cos(\frac{1}{2}n\pi)$ 19. $\frac{100}{n!}$ 20. n^4x^n 21. $\frac{2^n}{3^n}$ 22. $\frac{3^n}{2^n}$ 23. $(-1)^n\frac{10^n}{n!}$

24. $\frac{1}{2}n-f(\frac{1}{2}n)$, where $f(x)$ is the greatest integer $\leq x$. 25. $\frac{n^3+n^2-1}{n!}$

26. $\frac{x^n}{(n+1)(n+2)}$ 27. $\frac{2^n}{(n+1)!}$ 28. $\frac{2^{4n}}{n!}$ 29. $\frac{n^{10}}{n!}$ 30. $\frac{x^n}{x^n-1}-\frac{n}{n-1}$

31. $\frac{nx^n+x^{n-1}+1}{nx^{n-1}+1}$ 32. $\frac{2^{n^2}}{n!}$ 33. $\frac{(0.5)^{n^2}}{n!}$

34. Prove that in the sequence 1, 6, . . ., a_n, . . . where $a_{n+1}(1+a_n)=12$, a_{2n+1} is an increasing monotone, a_{2n} is a decreasing monotone and that $a_n \to 3$.

35. Show that the sequence 1, 1.4, . . ., a_n, . . ., in which $(2a_n+3)a_{n+1}=4+3a_n$, is monotonic and tends to $\sqrt{2}$.

36. If $a_{n+1}=\sqrt{(6+a_n)}$ and $a_1=2$, show that a_n increases steadily and has the limit 3.

37. Prove that if $a_{n+1}(a_n+2)=4$ and $a_1=1$, then $a_n \to \sqrt{5}-1$.

38. If $a_{n+1}(a_n^2+4)=5$, show that $a_n \to 1$.

39. If $-2<a_1<1$ and $3a_{n+1}=2+a_n^3$, prove that $a_n \to 1$.

40. If $a_1=1$ and $2a_{n+1}(a_n^2+4)=a_n(a_n^2+16)$, prove that a_n is monotonic and tends to 2.

41. If $a_1=2$ and $a_{n+1}(4+3a_n^5)=a_n(6+2a_n^5)$ show that a_n tends to the fifth root of 2.

42. If $a_1=\frac{1}{2}$, $a_2=\frac{1}{3}$ and $a_{n+1}=\frac{2}{3}a_n+\frac{1}{3}a_{n-1}$, $(n>1)$, prove that $a_n \to \frac{3}{8}$.

43. If $a>0$, $b>0$, show that the sequence a, b, $\sqrt{ab}$, . . ., a_n, . . . where $a_{n+1}=\sqrt{(a_n a_{n-1})}$ tends to $\sqrt[3]{(ab^2)}$.

44. In the sequence 2, 8, . . ., a_n, . . ., the law of formation is given by $2a_{2n+1}=a_{2n}+a_{2n-1}$, $a_{2n+1}a_{2n+2}=a_{2n}a_{2n-1}$. Prove that $a_n \to 4$.

45. If a_n is a monotone, show that $(a_1+a_2+\ldots+a_n)/n$ is a monotone of the same kind.

Find the first derivatives of the functions given in *Examples 46–52*

46. $\frac{x^2+1}{(x-1)^2(x-2)^3}$ 47. $\frac{(x^2-1)}{(x-2)^4(x+3)^3}$ 48. $\frac{x^3+4x^2+3x-1}{(x-2)(x-1)}$

49. $\frac{x^2(x-1)^2}{(x+1)^4(x-2)^3}$ 50. $\frac{4x^3-2x-5}{2x^4+3}$

51. $\frac{x^2+1}{x^2-1}$ 52. $\frac{(x-1)^2(x-3)^4}{(x-7)^3}$

Find the *third* derivatives of the functions given in *Examples 53–5*

53. $(x-1)^3(x-2)^2(x-3)^2$ 54. $(x^2-1)^3$ 55. $\frac{x^2+1}{(x^2-1)(x-2)}$

56. Find the nth derivative of $\frac{(x+1)}{(x-1)^2(x-2)}$

57. Show that if $a \neq b$, the nth derivative of $\frac{Lx+M}{(x-a)(x-b)}$ is

$$\frac{(-1)^n n!}{(a-b)}\left\{\frac{La+M}{(x-a)^{n+1}}-\frac{Lb+M}{(x-b)^{n+1}}\right\}$$

and find its value when $a=b$.

58. If $u = (x^2 + px + q)^n$, show that $(x^2 + px + q)u' = n(2x + p)u$ and deduce that the equation $\frac{d}{dx}\left\{(x^2 + px + q)\frac{dy}{dx}\right\} - n(n + 1)y = 0$ is satisfied by $y = \frac{d^n}{dx^n}(x^2 + px + q)^n$.

Sketch the graphs of the polynomials given in *Examples 59–74.*

59. $x^3 + 1$ 60. $x^3 - x + 1$ 61. $(x - 1)(x + 2)(x - 3)$
62. $(x - 1)^2(x + 1)$ 63. $x^4 + 4x - 1$ 64. $(x^2 + 1)(x^2 + 2)$
65. $x^3 - 2x^2 + x - 2$ 66. $(2x + 1)(x^2 + 4)$ 67. $x^3 + 5x - 2$
68. $(x^2 + 1)(x - 1)$ 69. $(x^2 - 1)(x^2 - 4) - 4$
70. $(2x - 1)(x^2 + x + 1)$ 71. $(x^2 - 1)^2(x^2 - 4)^2$ 72. $x^4(x - 1)^3(x + 1)^4$
73. $(1 - x)^2(1 + x)^4$ 74. $x^2(x - 1)^2(x^2 + 1)$

Sketch the graphs of the following sets of polynomials, drawing each set in the same figure (*Examples 75–8*).

75. $1;\ 1 + x;\ 1 + x + \frac{1}{2}x^2;\ 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$
76. $1;\ 1 - \frac{1}{2}x^2;\ 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$
77. $1;\ 1 + 3x;\ 1 + 3x + 3x^2;\ 1 + 3x + 3x^2 + x^3$
78. $x;\ x - \frac{1}{6}x^3;\ x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

Determine the polynomials satisfying the conditions in *Examples 79–88.*

79. $y'' = 2$; $y = 0$ when $x = 0, 1$.
80. $y'' = 4$; $y = 0$ when $x = 1$; $y' = 0$ when $x = 0$.
81. $y'' = 2$; $y = 2$ when $x = 0, 2$.
82. $y'' = 6x$; $y = 0$ when $x = 0$; $y' = 0$ when $x = 1$.
83. $y''' = 6$; $y = 0$ when $x = 0$; $y' = 0$ when $x = 1$; $y'' = 0$ when $x = 2$.
84. $y''' = 12$; $y = 0$ when $x = 1, 2, 3$.
85. $y^{iv} = 1$; $y = 0$, $y' = 0$ when $x = 0, 1$.
86. $y^{iv} = 1$; $y = 0$, $y'' = 0$ when $x = 0, 1$.
87. $y^{iv} = 1$; $y, y' = 0$ when $x = 0$; $y, y'' = 0$ when $x = 1$.
88. $y^{iv} = 1$; $y, y' = 0$ when $x = 0$; $y'', y''' = 0$ when $x = 1$.
89. If the polynomial $P_n(x)$ (*Legendre's*) is defined by the relation

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

prove that

(i) $P_1 = x$; $P_2 = \frac{3}{2}x^2 - \frac{1}{2}$; $P_3 = \frac{5}{2}x^3 - \frac{3}{2}x$; $P_4 = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$.

(ii) $P_n(x) = \frac{(2n)!}{2^n.n!}\left\{x^n - \frac{n(n-1)}{2.(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \dots\right\}$.

(iii) $(1 - x^2)P_n'' - 2xP_n' + n(n + 1)P_n = 0$.

Sketch in the same figure the sets of polynomials given in *Examples 90–4.*

90. $x;\ x^2 - \frac{1}{2};\ x^3 - \frac{3}{4}x;\ x^4 - x^2 + \frac{1}{8}$ (*Tschebyscheff's Polynomials*). (*If $x = \cos\theta$, these are the values of $2^{1-n}\cos n\theta$, $n = 1 - 4$.*)

91. $1;\ x;\ \frac{4}{3}x^2 - \frac{1}{3};\ 2x^3 - x;\ \frac{16}{5}x^4 - \frac{12}{5}x^2 + \frac{1}{5}$. (*If $x = \cos\theta$, these are the values of $\sin n\theta/(n \sin\theta)$, $n = 1 - 5$.*)

92. $1;\ 2x;\ 4x^2 - 2;\ 8x^3 - 12x;\ 16x^4 - 48x^2 + 12$ (*Hermite's Polynomials*). (*The Hermite Polynomial $H_n(x)$ may be defined by the relations:*

$H_n(x) = (-1)^n e^{x^2} d^n(e^{-x^2})/dx^n$ or $H_{n+1} = 2xH_n - H_n'$, $H_0 = 1$.)

93. $1;\ 1 - x;\ 2 - 4x + x^2;\ 6 - 18x + 9x^2 - x^3$ (*Laguerre's Polynomials*). (*The Laguerre Polynomial $L_n(x)$ may be defined by the relations:*

$L_n(x) = e^x d^n(x^n e^{-x})/dx^n$ or $L_{n+1} = xL_n' + (n + 1 - x)L_n$, $L_0 = 1$.)

94. $x;\ x^2 - x;\ x^3 - \frac{3}{2}x^2 + \frac{1}{2}x;\ x^4 - 2x^3 + x^2;\ x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$ (*Bernoulli's Polynomials*). (*The Bernoulli Polynomials $\phi_n(x)$ are the coefficients of $t^n/n!$ in the expansion of $(te^{xt} - t)/(e^t - 1)$. Note that $\phi_4 = \phi_2^2$, $\phi_5 = \phi_3(\phi_2 - \frac{1}{3})$.*)

95. Draw the graph of y where $y = 3x^2$, $(x < 1)$; $y = x^3 + 3x - 1$, $(x > 1)$; and show that y, y', y'' are continuous.

96. If $y = x^4$, $(x < 1)$, $y = 4x^3 - 6x^2 + 4x - 1$ $(x \geq 1)$, show that y, y', y'', y''' are continuous functions.

Sketch the graphs of the functions given in *Examples 97–106*, and point out any discontinuities of the functions or their derivatives:

97. $|1 + x| + |1 - x|$ **98.** $|(x - 1)(x - 2)|$ **99.** $|x + 1| + |x| + |x - 1|$

100. $|x^2 - 1| + |x^2 - 4|$ **101.** $x|x - 1| + (x - 1)|x|$ **102.** $x^2|x - 1|$

103. $|(x - 1)(x - 2)| - |(x - 2)(x - 3)|$

104. $f(x - 1) - f(x + 1)$, where $f(x) = x^2$ when $x > 0$ and $f(x) = 2x$ when $x < 0$.

105. $f(x + 1) - 2f(x) + f(x - 1)$, where $f(x) = 0$, $(x < 0)$; $f(x) = x^2$, $(0 < x < 1)$; $f(x) = 2x - 1$, $(x > 1)$.

106. $(x - 1)f(x - 1) - 2xf(x) + (x + 1)f(x + 1)$, where $f(x) = x$, $(x > 0)$; $f(x) = 0$, $(x < 0)$.

107. Show that $x^4 - x + 1$ does not vanish for any real value of x.

108. If $y = 2x^5 - 15x^4 + 40x^3 - 30x^2 + 10x - 10$, prove that y'' vanishes once only and that y increases steadily with x.

109. Show that the equation $x^3 + 3x + 1 = 0$ has only one real root and that this root lies between $-0·3$ and $-0·4$. Find its value correct to two decimal places.

110. Prove that $x^3 + 5x - 1$ increases steadily with x and find the real root of the equation $x^3 + 5x + 1 = 0$ correct to two significant figures.

111. Sketch the graph of $y = 2x^4 - 20x^2 - 10x + 1$ and show that

(i) y' is a minimum when $x = -1$, $y' = -20$.

(ii) y has a maximum at $(-1·49, 15·8)$ and a minimum at $(-0·25, -0·25)$.

? (iii) y vanishes when $x = -1·96$, $0·14$, $0·76$.

112. If the equation $x^n - npx^{n-1} + q = 0$ has a repeated root prove that either $q = 0$ or $q = (n - 1)^{n-1}p^n$.

113. Show that the function $A(x - a_1)^{m_1}(x - a_2)^{m_2} \ldots (x - a_r)^{m_r}$ where $a_1, a_2, \ldots, a_r$ are real and different and $m_1, m_2, \ldots, m_r$ are positive integers, has $(r - 1)$ non-zero stationary values.

114. Prove that if the quintic $x^5 + ax^2 + bx + c$ has a triple linear factor it never decreases.

115. Prove that the derivative of $x^3(x - 1)^2(x + 1)^4$ vanishes at $(-1, 0)$, $(-0·5\ldots, -0·02\ldots)$, $(0, 0)$, $(0·7\ldots, 0·26\ldots)$, $(1, 0)$.

Find the real solutions, correct to two significant figures, of the equations given in *Examples 116–21*.

116. $x^3 + x^2 = 2$ **117.** $(x - 1)^2(x + 1) = 4$ **118.** $(x^2 + 1)(x^2 - 4) - x = 0$

119. $x^4 - 2x^3 = 4$ **120.** $7x^3 - 8x^2 = 100$ **121.** $x^4 - 3x^3 = 60$

Find, for *Examples 122–31*, the leading term of the approximations to the functions given, in the neighbourhoods mentioned. Also give the asymptotes, where these exist.

122. $x^2(2 - x)^2(1 - 3x)^3$ at ∞ **123.** $\dfrac{x^2(x - 1)}{2x + 2}$ at ∞, -1

124. $\dfrac{x}{(x + 1)(x^2 + 2)}$ at ∞, -1 **125.** $\dfrac{x^2(x - 1)^4}{(2x - 1)^2(x + 2)}$ at ∞, $\frac{1}{2}$, -2

126. $\dfrac{2(x + 1)(x + 2)}{(3x - 1)(x - 2)}$ at ∞, $\frac{1}{3}$, 2 **127.** $\dfrac{(x - 1)(x + 1)^3}{(2x - 1)^2(x^2 - 4)}$ at ∞, $\frac{1}{2}$, 2, -2

128. $\dfrac{x(x^2 + x - 1)}{x^2 + 2x - 3}$ at ∞, -3, 1 **129.** $\dfrac{(2x + 3)(x - 4)^2}{(x - 1)^2}$ at ∞, 1

130. $\dfrac{(x + 1)^2(x - 2)^3}{x^2(x - 1)}$ at ∞, 0, 1 **131.** $\dfrac{3x^2(x + 1)}{(x - 2)^2(x - 1)}$ at ∞, 2, 1

Establish the approximations for x large in *Examples 132–5*.

132. $$\frac{(x^3 + 1)(2x + 3)}{(x - 1)^4(x + 2)} = \frac{2}{x^2} + o\left(\frac{1}{x^2}\right)$$

133. $\dfrac{x^4(2x+1)^2}{(x-1)^2(x+2)^3} = 4x - 12 + \dfrac{45}{x} + o\left(\dfrac{1}{x}\right)$

134. $\dfrac{(x-1)(2x+3)(3x-1)}{(2x+1)(x-4)} = 3x + 11 + \dfrac{79}{2x} + o\left(\dfrac{1}{x}\right)$

135. $\dfrac{16(x-2)^3(x^2+4)}{(2x+1)(x-3)} = 8x^3 - 28x^2 + 70x - 123 + O\left(\dfrac{1}{x}\right)$

Sketch the graphs of the functions given in *Examples 136–56.*

136. $\dfrac{x^2}{x-3}$ **137.** $\dfrac{x^3}{x-3}$ **138.** $\dfrac{3x+2}{(x-1)(x+2)}$ **139.** $\dfrac{x^2}{(x-1)^2(x+2)}$

140. $\dfrac{x^4}{(x-1)^3(x+1)}$ **141.** $\dfrac{x^2-2}{x(x^2+1)}$ **142.** $\dfrac{1}{(x+1)(x+4)}$

143. $\dfrac{1}{(x-1)(x-4)^2}$ **144.** $\dfrac{x}{(x-1)^2(x+4)}$ **145.** $\dfrac{x}{(x+1)(x^2+4)}$

146. $\dfrac{1}{(x^2+1)(x^2-4)}$ **147.** $\dfrac{x^2-1}{(x^2+1)(x^2-4)}$ **148.** $\dfrac{(x-1)(x-5)}{(x-2)(x-3)}$

149. $\dfrac{3x-4}{(x-2)(x-3)}$ **150.** $\dfrac{2x^2-7x-2}{(x-2)(x-3)}$ **151.** $\dfrac{x^2(x-2)}{(x-1)^3}$ **152.** $\dfrac{1}{x^2}-\dfrac{1}{x^3}$

153. $1-\dfrac{3}{4x}-\dfrac{1}{4x^2}$ **154.** $\dfrac{x^4+4}{x^2(x^2+1)}$ **155.** $\dfrac{x^3}{x^2+4}$ **156.** $\dfrac{(x^2-1)^2}{x(x+1)^2}$

157. Prove that $\dfrac{(x-7)(x-1)}{x^2+1}$ always lies between 9 and -1.

158. Prove that there are three real values of x that satisfy the equation $a(x^2+1) = (x+1)(x+3)^2$ if either $7+3\sqrt{6} < a < 16$ or $7-3\sqrt{6} < a < 0$.

159. Sketch in the same diagram the graphs of the functions x^3; x^3+x; x^3+x-2; $x^3+x-2-2/x$.

160. Find the range of values of a for which the equation

$$(a-7)x^4 + 5ax^2 + (4a-13) = 0$$

has (i) 4 real roots, (ii) 2 real roots only.

161. Sketch the graph of the function $y = (x^2-x+1)^3/\{x^2(x-1)^2\}$, and show that for a given value of $y > 27/4$, there are 6 real values of x, such that if x_1 is any one of these values, the other five are $1/x_1$, $1-x_1$, $1/(1-x_1)$, $1-1/x_1$, $x_1/(x_1-1)$.

162 Find the stationary values of $\dfrac{2x^3+3x^2-36x-36}{(x+6)^2(3x^2-8x-12)}$.

163. Prove that $(x-29)(x-13)/(x-4)$ takes all values except those in a certain interval of length 48.

164. The following formula occurs in Laplace's exposition of the theory of Saturn's rings.

$$\frac{\kappa}{\rho} = \frac{\lambda(\lambda-1)}{(\lambda+1)(3\lambda^2+1)}$$

where ρ is the density of the ring, λ the ellipticity of the cross section of the ring, and κ is a constant (>0). Show that the density has a maximum value when λ is approximately equal to 2.594.

Sketch the systems of curves given in *Examples 165–74*, where a is a variable parameter.

165. $y = x^3 - ax$ **166.** $y = x^2(x+a)$ **167.** $y = x(x^2+a)$
168. $y(x+a) = x^2-1$ **169.** $y(x-1) = x^2(x+a)$
170. $y(x+a) = x^2(x-1)$ **171.** $y(x-1) = x^2+a$
172. $y(a-1) = x(x-a)$ **173.** $yx^2 = x^2-a$ **174.** $yx = x^2+a$

Sketch in the same figure the five functions obtained by taking $n = 1, 2, 3, 4, 10$, in *Examples 175–82.*

175. $\dfrac{1}{x+n}$ **176.** $\dfrac{x+n}{2x+3n}$ **177.** $\dfrac{x^2-n^2}{x^2+n^2}$ **178.** $\dfrac{nx}{1+n^2x^2}$ **179.** $\dfrac{(nx+1)^2}{(n^2x^2+1)}$

180. $\dfrac{nx-1}{nx+1}$ **181.** $\dfrac{x^n}{1+x^{2n}}$ **182.** $\dfrac{nx^n}{1+nx^n}$

Solutions

Notation: c, ∞, $-\infty$, OF, OI *mean respectively for Examples 1–33*, '*tends to c*' '*tends to* $+\infty$', '*tends to* $-\infty$', '*oscillates finitely*', '*oscillates infinitely*'

1. $-4/3$ **2.** 0 **3.** ∞ **4.** ∞ **5.** $-\infty$ **6.** 2
7. ∞ **8.** OF **9.** OI **10.** OF **11.** ∞ **12.** 0
13. OI **14.** OI **15.** 0 **16.** OF **17.** ∞ **18.** OI
19. 0 **20.** 0, $(|x| < 1)$; ∞, $(x > 1)$; OI, $(x < -1)$ **21.** 0
22. x **23.** 0 **24.** OF **25.** 0
26. ∞, $(x > 1)$, 0, $(|x| < 1)$, OI, $(x < -1)$ **27.** 0 **28.** 0
29. 0 **30.** $1/x$, $(|x| > 1)$, 1, $(|x| < 1)$
31. x, $(|x| > 1)$, 1, $(|x| < 1)$. **32.** ∞ **33.** 0

46. $\dfrac{7-x+x^2-3x^3}{(x-1)^3(x-2)^4}$ **47.** $\dfrac{2-3x-7x^2}{(x-2)^5(x+3)^3}$

48. $1-\dfrac{1}{2(x-1)^2}+\dfrac{1}{2(x+3)^2}$ **49.** $\dfrac{x^4(x-1)^2(2x^3+15x^2-11x-10)}{(x+1)^5(x-2)^3}$

50. $\dfrac{2(3+18x^2-20x^3-6x^4-4x^6)}{(2x^4+3)^3}$ **51.** $\dfrac{-6x^4}{(x^3-1)^3}$

52. $\dfrac{3(35-17x)(x-1)^2(x-3)^3}{(x-7)^6}$ **53.** $12(10x^3-60x^2+116x-72)$

54. $24x(5x^2-3)$ **55.** $\dfrac{6}{(x-1)^4}-\dfrac{2}{(x-1)^4}-\dfrac{10}{(x-2)^6}$

56. $(-1)^n.n!\left\{\dfrac{3}{(x-2)^{n+1}}-\dfrac{3}{(x-1)^{n+1}}-\dfrac{2(n+1)}{(x-1)^{n+2}}\right\}$

57. $(-1)^n.n!\{L(x-na)+(n+1)M\}\dfrac{1}{(x-a)^{n+2}}$

79. $x(x-1)$ **80.** $(x-1)(2x+3)$ **81.** x^2-2x+2
82. $x(x^2-3)$ **83.** $x(x-3)^2$ **84.** $2(x-1)(x-2)(x-3)$
85. $\frac{1}{24}x^4(x-1)^2$ **86.** $\frac{1}{24}x(x-1)(x^2-x-1)$
87. $\frac{1}{48}x^3(x-1)(2x-3)$ **88.** $\frac{1}{24}x^2(x^2-4x+6)$
97–106. The functions are continuous. Also:
97. y' is discontinuous at ± 1. **98.** y' at 1, 2.
99. y' at 0, -1. **100.** y' at ± 1, ± 2. **101.** y' at 0, 1.
102. y' at 1. **103.** y' at 1, 2, 3. **104.** y' at ± 1.
105. y' continuous, y'' discontinuous at 0, ± 1, 2.
106. y' continuous, y'' discontinuous at 0, ± 1. **109.** 0·32
110. $-0{\cdot}20$ **116.** 1 **117.** 2·1 **118.** 1·9, 2·1
119. 2·3, $-1{\cdot}1$ **120.** 1·5, 1·3 **121.** 2·0 **122.** $27x^7$

123. $\frac{1}{2}x^2$, $\dfrac{1}{x+1}$, asymptote $x = -1$.

124. $\dfrac{1}{x^2}$, $-\dfrac{1}{3(x+1)}$, asymptotes $y = 0$, $x = -1$

125. $\frac{1}{2}x^2$, $\dfrac{27}{32(2x+1)^2}$, $\dfrac{36}{(x-2)}$, asymptotes $x = -\frac{1}{2}$, $x = 2$.

126. $\frac{1}{2}$, $\dfrac{-56}{13(2x-1)}$, $\dfrac{24}{3(x-2)}$, asymptotes $y = \frac{1}{2}$, $x = \frac{1}{2}$, $x = 2$.

127. $\frac{1}{4}$, $\frac{9}{20(2x-1)^2}$, $\frac{3}{4(x-2)}$, $\frac{-3}{100(x+2)}$, asymptotes $y = \frac{1}{4}$, $x = \frac{1}{2}$, $x = 2$, $x = -2$.

128. x, $\frac{21}{4(x+3)}$, $\frac{3}{4(x-1)}$, asymptotes $y = x - 1$, $x = -3$, $x = 1$.

129. $2x$, $\frac{45}{(x-1)^2}$, asymptotes $y = 2x - 9$, $x = 1$. **130.** x^4, $-\frac{32}{x^3}$, $\frac{972}{(x-1)}$, asymptotes $x = 0$, $x = 1$.

131. 3, $\frac{4}{(x-2)^2}$, $\frac{4}{9(x-1)}$, asymptotes $y = 3$, $x = 2$, $x = 1$.

160. 4 real roots if $2 < a < 3\frac{1}{2}$; 2 only if $3\frac{1}{2} < a < 7$.

162. Maxima at $x = -3, 2, 6$; minima at $x = -2, 0$.

CHAPTER II

MEAN VALUE THEOREM. FUNCTIONS OF SEVERAL VARIABLES. TAYLOR'S THEOREM WITH REMAINDER.

2. Sets of Points. Suppose that we have a set of points, infinite in number on a straight line (the x-axis for example). If they are all on the right of a fixed point L, the set is said to be *bounded on the left* (or *below*). If they are all on the left of a fixed point G, the set is *bounded on the right* (or *above*). If G, L both exist, the set is said to be *bounded*.

Example. The set: $0, \ldots, 0{\cdot}0001, 0{\cdot}001, 0{\cdot}01, 0{\cdot}1, 1, 1{\cdot}9, 1{\cdot}99, 1{\cdot}999, \ldots$ is bounded. Take $G = 2$, and L any negative number.

No loss of generality in the description of these sets usually arises if we confine our attention to bounded sets, since, for example, by such a relation as

$$y = \frac{x}{2\sqrt{(x^2+1)}} + \frac{1}{2} \text{ (the positive radical being chosen),}$$

we establish a one-one correspondence between the x axis and the interval $(0, 1)$ of the y-axis. Also the *order* of the points is unaltered by such a transformation.

2.01. The Process of Bisection. This is a process that is frequently used to establish properties of sets. Let $f(x)$ be a function of the points

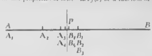


FIG. 1

x of a closed interval AB, and suppose that $f(x)$ possesses a property in the interval AB, which is of such a kind that it must be true for at least one of the closed sub-intervals AC, CB where C is any point of AB. Call an interval EF for which the property is true a *suitable* interval. Bisect AB at K, then one at least of the intervals AK, KB is suitable. Denote such a one by A_1B_1 (*Fig. 1*), bisect it and obtain similarly a suitable interval A_2B_2. If this process is continued, we obtain two monotones $A, A_1, A_2, \ldots, B, B_1, B_2, \ldots$, the former increasing, the latter decreasing (in the broad sense), and both bounded. They must therefore both tend to limits, and since $\lim A_nB_n = \lim (AB/2^n) = 0$,

they have a common limit P. By this process, therefore, we have obtained a point P *near* which $f(x)$ has the given property; i.e. given $\epsilon\ (>0)$, however small, at least one point P exists such that $f(x)$ possesses the property throughout the interval $|x - x_p| < \epsilon$. It is naturally assumed that the property specified for AB is one that is not necessarily satisfied for *every* sub-interval of AB. For example, it may be given that $f(x)$ possesses both positive and negative values in AB. Whilst this property is obviously not necessarily satisfied in every sub-interval, the process shows that there must exist at least one point, near which the property is satisfied.

2.02. Limiting Points. If a point P exists (not necessarily belonging to the set) near which there is an infinite number of points of the set, P is called a *limiting point* (or *point of accumulation*). This means that given $\epsilon\ (>0)$, however small, an infinity of points of the set lie in the interval $|x - x_p| < \epsilon$. There need not be an infinity of points on both sides of P in this interval and they may all lie on one side. Thus the end points of an open interval are limiting points. In the above example (§ 2), 0, 2 are limiting points, the latter not belonging to the set.

A bounded set of points (infinite in number) must contain at least one limiting point. For by the process of bisection, there must exist at least one point P near which an infinity of points of the set exists.

2.021. Upper Limits and Bounds. In general, a set contains more than one limiting point; the greatest of these is called the *upper limit*, and the least the *lower limit*.

Note. When the set is unbounded above, we say for completeness that the upper limit is $+\infty$, similarly the lower limit is $-\infty$ when the set is unbounded below.

Example. $-2, -1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \ldots, \cdot 1, \cdot 11, \cdot 111, \ldots, 2, 2\frac{1}{2}, 2\frac{3}{4}, 2\frac{7}{8}, \ldots, 3, 4$. The limiting points are $0, \frac{1}{9}, 3$. 0 is the lower limit and 3 is the upper limit.

The *least* number which is *not less* than every number of the set is called the *upper bound*; and the *greatest* number which is not greater than every number of the set is called the *lower bound*. In the above example, -2 is the lower bound and 4 the upper bound.

If the upper (lower) bound *belongs* to the set, it may be called the *maximum* (*minimum*). The maximum (minimum) is greater (less) than or equal to the upper (lower) limit. If the upper (lower) bound does not belong to the set, it is the same as the upper (lower) limit.

The simplest way in which sets of points arise is through functional relationship. Thus if $a \leq x \leq b$ and $y = f(x)$, the numbers $f(x)$ form a set of points; and if, for an infinite interval, x has only the values $1, 2, 3, \ldots, n, \ldots, f(x)$ is a simple sequence, which thus constitutes the most elementary set of points.

2.022. The Simple Sequence. (i) If the sequence a_n is convergent, the set a_n has only one limiting point, viz. the limit of the sequence.

(ii) If the sequence a_n is bounded but not convergent, the set a_n must have at least two limiting points. The upper limit is in this

case denoted by $\overline{\lim}\, a_n$ and the lower limit by $\underline{\lim}\, a_n$. The difference $(\overline{\lim}\, a_n - \underline{\lim}\, a_n)$ is called the *Oscillation*.

(iii) If $\overline{a_n} \to +\infty$, we may call $+\infty$ the (only) limiting point of the set, and if $a_n \to -\infty$, then $-\infty$ is the only limiting point.

(iv) If one of the extreme limits is infinite ($\pm$), and there is at least one other limiting point (finite or infinite), the sequence oscillates infinitely.

Note. In such a sequence as $1, -1, 1, -1, \ldots$, the numbers $1, -1$ are limiting points, since they occur an infinite number of times.

Examples. (i) $2, 3, 4, 5, \ldots$; lower bound, 2 (minimum), upper bound (and limit), $+\infty$.

(ii) $0, 10^{1-n}, 3 - 10^{1-n}$ $(n = 1, 2, 3, \ldots)$; lower bound, 0 (minimum), lower limit, 0, upper bound, 3; no maximum; upper limit, 3.

(iii) $\cos \frac{1}{3}n\pi + \frac{1}{n}\cos n\pi$; lower bound, $-4/3$ (minimum); upper bound, $7/6$ (maximum); four limiting points, $\pm 1, \pm \frac{1}{2}$; upper limit, 1; lower limit, -1, only limit belonging to set, $-\frac{1}{2}$; oscillation, 2.

(iv) $n - 20\cos(\frac{1}{2}n\pi)$; lower bound, -14 (minimum); no finite limit, upper bound (and limit), $+\infty$.

2.03. Derived Sets. The set E', which consists of the limiting points of E, is called the *derived set* or *first derivative* of E. If E' possesses an infinite number of points, it also possesses a derivative E''. Similarly there may be any number of higher derivatives. If any derivative $E^{(n)}$ contains a finite number of points, the next derivative $E^{(n+1)}$ is void. In this case E is said to be of the *first species* (of the nth order). If there is an infinite number of derivatives, the set is of the *second species*.

Examples. (i) The set $\left(\frac{1}{m} - \frac{1}{n} + \frac{1}{p}\right)$ where m, n, p take all positive integers for their values. Denoting the set by E, we have $E' = \left(0, \frac{1}{m}, \frac{1}{m} - \frac{1}{n}\right)$, $E'' = \left(0, \frac{1}{m}\right)$; $E''' = 0$; $E^{(iv)}$ void. E is of the first species and third order.

(ii) If E is the set of *rational* numbers in $(0, 1)$, then E' is the set of *real* numbers in $(0, 1)$, so that $E' = E'' = E''' = \ldots$ and E is of the second species.

2.04. Sum. The set consisting of every point that belongs to at least one of a given sets $E_1, E_2, \ldots E_n$ is called the *sum* (or greatest common measure) of the sets and is written $E_1 + E_2 + \ldots + E_n$.

2.041. Product. The set consisting of every point that belongs to *all* the sets $E_1, E_2, \ldots E_n$ is called the *product* (or greatest common divisor) of the sets and is written $E_1.E_2.E_3.\ldots.E_n$.

2.042. Complement. If E is a set of points in a given interval, the set of points of that interval not belonging to E is called the *complement* of E for that interval and written $C(E)$.

Example. The set of points belonging to none of the sets $E_1, E_2, \ldots, E_n$, in an interval is $C(E_1 + E_2 + \ldots + E_n)$ and is the same as $C(E_1).C(E_2).\ldots.C(E_n)$.

2.043. Closed Sets. A set containing all its limiting points is said to be *closed*.

Example. The set $0, \frac{1}{2^n}, \frac{1}{4} + \frac{1}{2^n}, \frac{1}{2} + \frac{1}{2^n}, \frac{3}{4} + \frac{1}{2^n}$ is closed, its limiting points being $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

2.044. Isolated Set. If a set contains none of its limiting points it is said to be *isolated*.

Examples. (i) The set $\frac{1}{4n}, \frac{1}{2} - \frac{1}{4n}, \frac{3}{4} + \frac{1}{4n}$ is isolated.

(ii) EE' is void if E is isolated.

(iii) $E - EE'$ is isolated.

2.045. *Set Dense in Itself.* If every point of a set is a limiting point, the set is said to be *dense in itself.*

Example. The rational numbers in $(0, 1)$ form a set dense in itself.

2.046. *Set Everywhere Dense.* A set E is said to be *everywhere dense* in an interval if every sub-interval (however small) contains points of E. There are therefore limiting-points (not necessarily belonging to the set) in every sub-interval; and the derivative of E consists of the given interval.

A set that is everywhere dense must be dense in itself, but the converse is not necessarily true. Thus the set of rational points is everywhere dense and is also dense in itself, but the set of real points given by $0 < x < \frac{1}{3}$, $\frac{2}{3} < x < 1$, whilst dense in itself is not everywhere dense in $(0, 1)$.

2.047. *Set Non-dense.* A set is said to be *non-dense* in an interval if no sub-interval is everywhere dense.

2.048. *Perfect Set.* A set that is dense in itself and closed is said to be *perfect.*

Thus the set of real points specified by $0 \leqslant x \leqslant \frac{1}{3}$, $\frac{2}{3} \leqslant x \leqslant 1$, is perfect but the set of rational points in $(0, 1)$ is not perfect.

All the derivatives of a perfect set E are identical with E.

2.049. *Enumerable Sets.* If the points of a set can be placed in 1–1 correspondence with the integers $1, 2, 3, \dots, n, \dots$, it is said to be *enumerable.*

The sum of a finite number of enumerable sets is enumerable; for if $x_1, x_2, \dots$; $y_1, y_2, \dots$; are two enumerable sets (as indicated by the notation), the sum may be arranged as

$$x_1, y_1, x_2, y_2, \dots$$

and is therefore enumerable. Similarly the sum of a finite number of enumerable sets is enumerable. Moreover, the sum of an enumerable infinity of enumerable sets is enumerable. For the nth member of the mth set may be denoted by x_{mn} and the sum may be arranged as

$$x_{11}, x_{12}, x_{21}, x_{13}, x_{22}, x_{31}, \dots$$

grouping together those terms for which $m + n$ is the same number k and taking $k = 2, 3, 4, \dots$.

Examples. (i) The set of all rational numbers in $(0, 1)$ is enumerable, since they can be arranged in groups of the same denominator, thus

$$0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{5}{6}, \dots$$

It follows that the set of *all* rational numbers is enumerable since the number of intervals $m \leqslant x \leqslant m + 1$ (m being a positive or negative integer or zero) is enumerable and the set of rational points in each interval is enumerable.

(ii) The set of real numbers in $(0, 1)$ is not enumerable. If it were enumerable, the numbers could be arranged as a sequence $x_1, x_2, \dots, x_n, \dots$. Suppose that each number x_n is expressed as an infinite decimal (a terminating decimal being completed with an infinity of zeros and a recurring nine being excluded). Let c_n be the figure in the nth place of the decimal for x_n. If $c_n = 0$, let $c_n' = 1$, and if $c_n \neq 0$, let $c_n' = 0$ $(n = 1, 2, 3, \dots)$.

Then the decimal $\cdot c_1' c_2' c_3' \dots c_n' \dots$, which lies between 0 and 1 is not identical with any x_n. The set is therefore not enumerable.

2.05. *Open Sets.* The complement of a closed set is called an *open set.*

Example. The set of real points given by

$$0 < x < \tfrac{1}{3}, \quad \tfrac{1}{2} < x < \tfrac{2}{3}, \quad \tfrac{3}{4} < x < 1$$

is open, since the set 0, $\frac{1}{3} \leqslant x \leqslant \frac{1}{2}$, $\frac{2}{3} \leqslant x \leqslant \frac{3}{4}$ is closed.

2.051. Characteristic Property of an Open Set

An open set consists of an enumerable (or finite) set of non-overlapping intervals.

Let P be a point of an open set E in the interval (a, b). There must be points of E in the neighbourhood of P, for otherwise P would be a limiting point of $C(E)$ which would therefore not be closed. Thus there must exist positive numbers ϵ_1, ϵ_2 such that the interval $x - \epsilon_1 < x < x + \epsilon_2$ consists entirely of points of E. Let δ_1, δ_2 be the upper bounds of ϵ_1, ϵ_2 respectively. Then the open interval $x - \delta_1 < x < x + \delta_2$ consists entirely of points of E. The end points $x - \delta_1, x + \delta_2$ must belong to $C(E)$ since δ_1, δ_2 are upper bounds. Similarly every point of E falls into an open interval and the intervals do not overlap since the end points do not belong to E. There can only be a *finite* number of intervals whose lengths lie between $\frac{1}{m}(b-a)$ and $\frac{1}{m+1}(b-a)$ where m is a positive integer. The intervals can therefore be arranged in the finite groups specified by

$$\frac{1}{m}(b-a) < \delta < \frac{1}{m+1}(b-a), \quad m = 1, 2, 3, \ldots$$

where δ is the length of an interval. The number of intervals is therefore enumerable.

Since the sum of two open intervals is an open interval (or two open intervals), we deduce that the sum of any finite number (or of an enumerable infinity) of open sets is an open set; and since the common points of two open intervals (if they overlap) lie in an open interval, we conclude that the product $E_1 . E_2 . \ldots E_n$ of a *finite* number of open sets is open.

2.06. The Measure of an Open Set. The measure of an open interval $x_1 < x < x_2$ is defined to be $x_2 - x_1$ the length of the interval; and the measure of an *open set* (within an interval $a < x < b$) is defined to be the sum of the lengths of its intervals. Since the number of intervals is infinite (enumerable) in general, it is the sum S of an infinite series (convergent since $S < (b-a)$).

2.061. Exterior and Interior Measure of a Set. The *exterior measure* of a set E is defined to be the lower bound of the measures of all open sets that contain E.

Denoting it by $m_e(E)$, we have obviously

$$0 \leq m_e(E) \leq b - a,$$

if the set lies in the interval (a, b).

The *interior measure* $m_i(E)$ of the set E is defined by the relation $m_i(E) + m_e(CE) = b - a$. It follows from this definition that

$$m_e(CE) + m_i(E) = b - a.$$

If $m_i(E) = m_e(E)$, the set E is said to be *measurable* and the common value of $m_i(E)$ and $m_e(E)$ is called its *measure*. We then denote the measure of the set E by the symbol $m(E)$. If E is measurable,

$$\begin{aligned} m_e(CE) &= (b - a) - m(E) \\ &= m_i(CE). \end{aligned}$$

Thus CE is measurable and its measure is $(b-a) - m(E)$.

The interior measure cannot be greater than the exterior measure. For let F, G be open sets that contain E and CE respectively. Every point of (a, b) is interior to an interval of F or of G (or of both), and we may show by the process of bisection that a finite set of intervals can be selected from those of F and G that together contain all the points of (a, b). For if such a finite set did not exist, we could show that there existed at least one point P that was not interior to an interval of F or of G, and this would contradict the statement that every point of (a, b) is interior to an interval of F or of G.

Let the sum of the lengths of the intervals of F, G be denoted by μ_1, μ_2 respectively and let μ denote the sum of the lengths of the intervals of the selected finite set. Then $\mu_1 + \mu_2 > \mu$ and $\mu > b - a$, i.e. $\mu_1 + \mu_2 > b - a$. Therefore the lower bound of $\mu_1 + \mu_2$ is $> b - a$

i.e. $$m_e(E) + m_e(CE) > b - a$$

or $$m_e(E) > m_i(E) > 0.$$

It follows from this result that if $m_e(E) = 0$, then $m_i(E) = 0$, so that such a set is measurable and its measure is zero.

Notes. The next development in the theory of measure consists in the establishment of two fundamental theorems.

(i) If E_r is measurable ($r = 1, 2, 3, \ldots$), then $\sum_1^\infty E_r = E$ is measurable and $m(E) \leq \sum_1^\infty m(E_r)$.

(ii) If E_r is measurable, ($r = 1, 2, 3, \ldots$), then $\prod_1^\infty(E_r)$ is measurable.

It should be remarked that in the above, the measure of an open set has been given a special definition on which the definition of the measure of any set has been based. It may be shown that these definitions of an open set are consistent by means of the first fundamental theorem, which may be established on the basis of the special definition of the open set.

(*The summarized description given above of the meaning of measure is based on the account given in Titchmarsh, 'Theory of Functions', X, where proofs of the fundamental theorems may be found.*)

Examples. (i) An enumerable set is measurable and its measure is zero.

Let the set be arranged as the sequence of points $x_1, x_2, \ldots, x_n, \ldots$

Take intervals $x_n - \epsilon_n < x < x_n + \epsilon_n$, where $\epsilon_n = \epsilon_1/2^{n-1}$. If ϵ_1 is given, these intervals in general overlap and the sum of the lengths of the first n is $4\epsilon_1(1 - 2^{-n})$. The n points $x_1, x_2, \ldots, x_n$ may therefore be enclosed in a set of $m(< n)$ non-overlapping intervals of total length $< 4\epsilon_1(1 - 2^{-n})$. If $n \to \infty$ and $\epsilon_1 \to 0$, we see that $m_e(E) = 0$; i.e. the measure of the set is zero.

(ii) An isolated set of points is enumerable. Let x be a point of an isolated set in (a, b). Then an interval $x - \epsilon < x < x + \epsilon$ exists (where $\epsilon > 0$) containing no points of the set except x. Otherwise x would be a limiting point of the set. Thus each point of the given set can be associated with one of a set of open non-overlapping intervals. Since this open set of intervals is enumerable, the original set of points is also enumerable.

(iii) If E', the derivative of a set E is enumerable, so also is E. For $E - EE'$ is enumerable, since it is isolated. But EE' is a component of E' and is therefore enumerable. Thus E also is enumerable. The converse is not necessarily true. For example, the set of rational points is enumerable but its derivative is not.

(iv) A set of the first species is enumerable. For if E is of the first species and the nth order, $E^{(n)}$ has a finite number of points. Thus $E^{(n-1)}$ is enumerable and so also by (iii) are $E^{(n-2)}, E^{(n-3)}, \ldots, E', E$. It is not true, however, that every set of the second species is not enumerable.

(v) The segment $(0, 1)$ is divided into m equal parts, ($m > 2$), and the open interval $1/m < x < 2/m$ is removed. Each of the remaining $(m - 1)$ intervals is similarly divided into m equal parts and the second (open) interval removed. The process is continued indefinitely, and the points that remain form a closed set E. The sum of the lengths of the open intervals removed is obviously

$$1 - \lim_{n \to \infty}\left(1 - \frac{1}{m}\right)^n \text{ i.e. } 1.$$

The measure of the set is therefore zero. The set E is non-dense since its complement is everywhere dense. It is, however, dense in itself and is therefore perfect. Also it is not enumerable. For if, for definiteness, we take $m = 10$, the set E consists of all decimals $(0 \leq x < 1)$ that do not contain the digit unity together with those that contain a digit unity followed by an infinity of zeros. (It is assumed that terminating decimals are completed by an infinity of zeros and that a recurring 9 is excluded.) If these decimals could be arranged in a sequence $x_1, x_2, \ldots, x_n, \ldots$, denote by c_n the nth digit in the expression for x_n. Let $c_n' = 0$ when $c_n \neq 0$ and let $c_n' = 2$ when $c_n' = 0$. Then the decimal $0{\cdot}c_1'\,c_2' \ldots c_n' \ldots$ does not belong to the sequence but belongs to E. Thus the set cannot be arranged in a sequence

and is not enumerable. A similar proof may be obtained for other values of m, by expressing the numbers of E in the scale of m. The particular set obtained by taking $m = 3$ is sometimes called *Cantor's Ternary Set.*

$|x_1 - x_0| < \delta$
$x_1, x_2 \in (x_0 - \delta, x_0 + \delta)$

2.1. Continuous Functions. We have already seen that a function $f(x)$ is continuous at a point x_0 of the interval $a \leq x \leq b$ if, given ε (> 0), we can find δ (> 0) such that $|f(x) - f(x_0)| < \varepsilon$ for all x of the interval that lie in $|x - x_0| < \delta$.

It is obviously necessary and sufficient for continuity at $x = x_0$, that $|f(x_1) - f(x_2)| < \varepsilon$ for all x_1, x_2 of the interval that lie in $|x - x_0| < \delta$.

The functional relation forms a set of points, which, we shall see, is perfect like the points of the interval.

Notes. (i) If the above inequality is altered to $f(x) - f(x_0) < \varepsilon$, the point x_0 is called a point of *upper semi-continuity* and if it is altered to $f(x_0) - f(x) < \varepsilon$, then x_0 is a point of *lower* semi-continuity: and the function is in each case called a *semi-continuous* function. Both inequalities must be satisfied for continuity.

(ii) If M, m are the upper and lower bounds of a function $f(x)$ in an interval, $M - m$ is called the *oscillation* (or *fluctuation*) of $f(x)$ in that interval. Thus if $f(x)$ is continuous at x_0, the oscillation of $f(x)$ near x_0 is small.

If, given ε (> 0), a number δ (> 0) can be found such that for every enumerable (or finite) set of non-overlapping intervals in (a, b) of total length $< \delta$, the sum of the oscillations of $f(x)$ is less than ε, the function is said to be *absolutely continuous* in the interval. Thus absolute continuity refers to the interval as a whole, and whilst absolute continuity obviously implies ordinary continuity, the converse is not necessarily true.

(iii) If the domain of continuity of $f(x)$ consists of an unenumerable set of points, which is everywhere dense but is *not* closed, $f(x)$ is said to be *point-wise discontinuous.*

2.11. Properties of a Continuous Function.

Let $f(x)$ be continuous in the interval $a \leq x \leq b$.

I. Given ε (> 0), it is possible to divide (a, b) into a *finite* number of sub-intervals, such that $|f(x_1) - f(x_2)| < \varepsilon$ where x_1, x_2 are any two points in any sub-interval.

For if this were not true, it would be possible by the process of bisection, to find a point P near which $|f(x_1) - f(x_2)|$ could not be made less than ε. This contradicts the hypothesis of continuity at P.

Since the inequality is satisfied *throughout* all the sub-intervals, we say that the continuity of $f(x)$ is *uniform.* Thus *continuity implies uniform continuity.*

II. $f(x)$ is *bounded* in the interval. For by I, we can divide (a, b) into n intervals within each of which $|f(x_1) - f(x_2)| < \varepsilon$: so that, if x is in the rth interval, $|f(x) - f(a)| < r\varepsilon$, i.e. $f(x)$ is bounded.

III. $f(x)$ has an upper limit M which is a *maximum* and a lower limit m which is a *minimum*.

For if M is the upper bound, $[f(x) - M]^{-1}$ is unbounded since $|f(x) - M|$, if not vanishing, can be made as small as we please. Thus $[f(x) - M]^{-1}$ is not continuous; but since $f(x) - M$ is continuous, $[f(x) - M]^{-1}$ can be discontinuous only when $f(x) = M$. There must be therefore at least one such point. Similarly there is at least one point for which $f(x) = m$. The upper and lower bounds are therefore limiting points belonging to the set of points $f(x)$.

IV. If $f(a)$, $f(b)$ are of opposite signs, $f(x)$ vanishes at least once within the interval.

For, by the process of bisection, there is at least one point ξ near which $f(x)$ has opposite signs. If $f(\xi)$ were not zero, the sign of $f(x)$ would by the hypothesis of continuity be invariable near ξ. Therefore $f(\xi) = 0$.

V. $f(x)$ takes, at least once, every value inclusive between M, m its upper and lower bounds. For $f(x) - k$ where $M > k > m$ has both signs in (a, b). Therefore $f(x) = k$ at least once in the interval.

VI. If $f(x)$ increases (or decreases) *steadily* between $f(a)$ and $f(b)$ and is defined for all points in (a, b), it is continuous in (a, b).

A function is said to increase steadily between $f(a)$ and $f(b)$, if, whilst increasing in the broad sense, it takes *every* value between $f(a)$ and $f(b)$.

There must be a neighbourhood of any point x_0, within which $f(x)$ increases from $f(x_0) - \varepsilon$ to $f(x_0) + \varepsilon$, i.e. the function is continuous at x_0. Similarly a function that decreases steadily is continuous.

2.12. Rolle's Theorem. If $f(x)$ is continuous in $a \leq x \leq b$, possesses a derivative in $a < x < b$ and vanishes at $x = a$ and $x = b$, then $f'(x)$ vanishes at least once in $a < x < b$. For (i) if $f(x) = 0$ throughout $]a, b)$, the theorem is true; (ii) if $f(x) > 0$ at any point of the interval, $f(x)$ attains a maximum $f(\xi)$ at some point ξ in $a < x < b$.

Hence $f(\xi + h) - f(\xi)$ is always negative $(a \leq \xi + h \leq b)$. Therefore the progressive derivative $f'(\xi + 0)$ cannot be positive and the regressive derivative $f'(\xi - 0)$ cannot be negative. But these deriva-

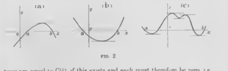


FIG. 2

tives are equal to $f'(\xi)$, if this exists and each must therefore be zero, i.e. $f'(\xi) = 0$. Similarly $f'(x)$ vanishes at least once if $f(x) < 0$ at any point. We have here the geometrical result that if the curve given by $y = f(x)$ meets the x axis at $x = a$, $x = b$, and if there is a unique tangent at each point, the tangent is parallel to the x-axis at some point interior to the interval (*Fig. 2*).

2.13. The Mean Value Theorem If $f(x)$ is continuous in $a \leq x \leq b$ and if $f'(x)$ exists in $a < x < b$, then, for at least one point $x = c$ of the interval $a < x < b$

$$f(b) - f(a) = (b - a)f'(c).$$

Let $F(x) = (a - b)f(x) + (b - x)f(a) + (x - a)f(b)$. Then $F(x)$ satisfies the conditions of Rolle's Theorem; $F(a) = 0 = F(b)$.

$F'(x)$ exists and is equal to $(a-b)f'(x) - f(a) + f(b)$. This vanishes for $x = c$ where $a < c < b$, i.e. $f(b) - f(a) = (b-a)f'(c)$.

Geometrically, this means that the chord joining the two points P, Q of co-ordinates $\{a, f(a)\}$, $\{b, f(b)\}$ respectively, is parallel to the tangent at some point R of the curve between P and Q. (*Fig. 3.*)

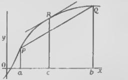


FIG. 3

Corollary. If $b = a + h$, a number between a and b may be denoted by $a + \theta h$, $(0 < \theta < 1)$; and the theorem becomes $f(a+h) = f(a) + hf'(a + \theta h)$ for some number θ in the interval $0 < \theta < 1$.

2.2. Functions of Two Variables. If x, y are two independent variables, real and continuous, belonging respectively to the intervals

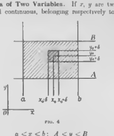


FIG. 4

$$a < x < b; \quad A < y < B$$

and if a third variable z is known when x and y are given, then z is a function of the two variables specified in a rectangle of the x-y plane. (*Fig. 4.*)

2.21. Continuity. A function $z = f(x, y)$ is said to be continuous at (x_0, y_0) if, given ε, we can find δ such that for all points (x, y) of the square specified by $|x - x_0| < \delta$, $|y - y_0| < \delta$, the inequality

$$|f(x, y) - f(x_0, y_0)| < \varepsilon$$

is true. This might be described briefly by saying that *near* (x_0, y_0), $|f(x, y) - f(x_0, y_0)|$ is *small*.

Notes. (i) The neighbourhood need not necessarily be taken as a square, but there is no loss in generality if we do so.

(ii) Not only is $|f(x, y) - f(x_0, y_0)|$ small near (x_0, y_0) but also $|f(x_1, y_1) - f(x_2, y_2)|$ where (x_1, y_1), (x_2, y_2) are near (x_0, y_0).

2.22. *Double Sequences.* In the same way that simple sequences are associated with functions of one variable, so we may expect *double* sequences to be associated with functions of two variables. An aggregate of numbers (a_{mn}) in which m, n may take all positive integers for their values is called a double sequence. The terms may be arranged in the following array:

$$\begin{array}{llllll}
a_{11}, & a_{12}, & a_{13}, & \ldots & a_{1n}, & \ldots \\
a_{21}, & a_{22}, & a_{23}, & \ldots & a_{2n}, & \ldots \\
a_{31}, & \cdot & \cdot & & \cdot & \\
\cdot & \cdot & \cdot & & & \\
a_{m1}, & a_{m2}, & \ldots & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

The suffix m denotes the *row* and n the *column*; and a finite number of terms may be omitted without altering the essential character of the sequence (i.e. its behaviour when m, n tend to infinity).

2.23. *Limit of a Double Sequence a_{mn}.* The sequence a_{mn} is said to tend to a limit l as m, n tend *independently* to infinity if, given ε, we can find N such that $|a_{mn} - l| < \varepsilon$ for *all* $m, n > N$, and we write

$$\lim_{m, n} a_{mn} = l.$$

It is, however, *sufficient* (and necessary) that $|a_{mn} - a_{NN}| < \varepsilon$ for all $m, n > N$.

This is necessary, for, if l exists, $|a_{mn} - l|$ and $|a_{NN} - l|$ are small. It is sufficient, for the condition shows that the *simple* sequence a_{mm} converges to the limit l. Thus $|a_{NN} - l|$ is small ultimately and therefore also $|a_{mn} - l|$.

2.24. *Repeated Limits of a Double Sequence a_{mn}.* The convergence of a_{mn} implies the convergence of a_{mn} when m, n tend to infinity in any particular way, although the converse is not true.

Example. If $a_{mn} = 2^{m-n}$, a_{mn} is obviously not convergent, whilst the sequence obtained by putting $n = 2m$, converges to zero as m (and therefore n) tends to infinity.

If $\phi(n) \to \infty$ when $n \to \infty$ and if $a_{mn} \to l$, then $a_{mn} \to l$ when $m = \phi(n)$ and $n \to \infty$.

There is a particular way in which m, n tend to ∞ that is important in the theory of double sequences. This consists in letting m (or n) tend to infinity before n (or m) tends to infinity. The sequences $a_{1n}, a_{2n}, \ldots, a_{mn}, \ldots$ where m is fixed are simple sequences, that may or may not possess limits (even when the double sequence converges). Suppose, however, that $\lim a_{mn}$ exists. It is a function of m, say $f(m)$. Then

n
$\downarrow$

$\lim\limits_{m} f(m) = l$ if $a_{mn} \to l$. For $|a_{mn} - l|$ is small and also $|\lim\limits_{n} a_{mn} - a_{mn}|$ is small, when m, n are large, i.e. $|f(m) - l|$ is small, or, $f(m) \to l$.

The limit $\lim\limits_{m} f(m)$ may be denoted by $\lim\limits_{m} \lim\limits_{n} a_{mn}$ and is called a *repeated* limit. Similarly if $\lim\limits_{m} a_{mn}$ exists we may have the repeated limit $\lim\limits_{n} \lim\limits_{m} a_{mn}$, which is equal to l if $a_{mn} \to l$.

Note. When $\lim\limits_{n} a_{mn}$ does not exist, a_{mn} must have an upper limit $\overline{\lim\limits_{n}}\, a_{mn}$ and a lower limit $\underline{\lim}\, a_{mn}$: so that if the double sequence converges to a limit l, we must have

$$\lim_{m} \overline{\lim_{n}}\, a_{mn} = \lim_{m} \underline{\lim_{n}}\, a_{mn} = l.$$

For completeness therefore we may include this case in the above by regarding the symbol $\lim\limits_{m} \lim\limits_{n} a_{mn}$ as inclusive of $\lim\limits_{m} \overline{\lim\limits_{n}}\, a_{mn}$.

Examples. (i) $a_{mn} = (-1)^{m+n}\left(\dfrac{1}{m} + \dfrac{1}{n}\right)$.

Here $\overline{\lim\limits_{n}}\, a_{mn} = \dfrac{1}{m}$, $\underline{\lim\limits_{n}}\, a_{mn} = -\dfrac{1}{m}$ and $\lim\limits_{mn} a_{mn} = 0$:

$$\overline{\lim_{m}}\, a_{mn} = \frac{1}{n},\quad \underline{\lim_{m}}\, a_{mn} = -\frac{1}{n} \text{ and } \lim_{m} \overline{\lim_{n}}\, a_{mn} = \lim_{n} \overline{\lim_{m}}\, a_{mn} = 0.$$

(ii) $a_{mn} = (m - n)^2/(m + n)^2$.

Here $\lim\limits_{m} \lim\limits_{n} a_{mn} = 1 = \lim\limits_{n} \lim\limits_{m} a_{mn}$. But a_{mn} is not convergent since a_{mm} is always zero. Thus the repeated limits may exist and be equal whilst the double limit does not exist.

2.25. *Double Monotones.* If $a_{mn} < a_{m,n+1}$ and $a_{mn} < a_{m+1,n}$ for all values of m, n, the sequence is called an increasing double monotone. As for simple sequences, a *bounded* monotone is convergent. For the simple sequence a_{nn} is monotonic and therefore converges to a limit l. But a_{mn} lies between a_{mm} and a_{nn} and must therefore tend to l. It is obvious that in a bounded double monotone, the repeated limits exist and are equal to the double limit.

2.26. *Limits of a Function $f(x, y)$.* If ξ_m is a sequence tending to x_0 and η_n a sequence tending to y_0, and if the double sequence $f(\xi_m, \eta_n)$ tends to a limit l which is the same for all sequences ξ_m, η_n that tend to x_0, y_0 respectively, then l is called the limit of $f(x, y)$ as $x \to x_0$, $y \to y_0$ and we write

$$\lim_{x_0, y_0} f(x, y) = l.$$

If $l = f(x_0, y_0)$, the function is said to be *continuous* at (x_0, y_0), for then, near (x_0, y_0), $|f(x, y) - f(x_0, y_0)|$ is small.

Although x, y may tend to x_0, y_0 in any manner, there is no loss in generality in taking two monotones for x, one increasing and one decreasing and having a common limit x_0. Similarly two monotones may be taken for y with a common limit y_0. One of the x-sequences may then

be associated with one of the y-sequences thus distinguishing the four quadrants bounded by the lines $x = x_0$, $y = y_0$.

Let (ξ_m), (η_n) be the two monotones that *decrease* respectively to x_0, y_0 (*Fig. 5*). Then if $f(\xi_m, \eta_n)$ tends to a limit *independent* of the particular monotones selected, this limit is written $f(x_0 + 0, y_0 + 0)$ and if this is equal to $f(x_0, y_0)$, the function is said to be *continuous for the quadrant* under consideration. There are obviously similar definitions for $f(x_0 + 0, y_0 - 0)$, $f(x_0 - 0, y_0 + 0)$, $f(x_0 - 0, y_0 - 0)$ for the other quadrants, and for continuity all the four limits must be equal to $f(x_0, y_0)$.

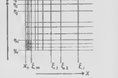


FIG. 5

The continuity of $f(x, y)$ at (x_0, y_0) implies that $f(x, y)$ tends to $f(x_0, y_0)$ when x, y tend to x_0, y_0 in any particular way. For example, if $\phi(t)$ is a continuous function of t tending to x_0 when t tends to t_0 and $\psi(t)$ a continuous function of t tending to y_0 when t tends to t_0, then $f\{\phi(t), \psi(t)\}$ is a continuous function of t at $t = t_0$.

Again, as for double sequences, the repeated limits $\lim\limits_{x \to x_0} \lim\limits_{y \to y_0} f(x, y)$, $\lim\limits_{y \to y_0} \lim\limits_{x \to x_0} f(x, y)$ exist, when $f(x, y)$ is continuous and are equal to $f(x_0, y_0)$ although a modification of this statement is necessary when either $\lim\limits_{y \to y_0} f(x, y)$ or $\lim\limits_{x \to x_0} f(x, y)$ does not exist. Thus if $f(x, y)$ is a continuous function of both variables at (x_0, y_0), $f(x, y_0)$ is a continuous function of x at x_0 and $f(x_0, y)$ a continuous function of y at y_0. Conversely, however, the continuity of $f(x, y_0)$ at x_0 and the continuity of $f(x_0, y)$ at y_0 does *not* imply the continuity of $f(x, y)$ at (x_0, y_0).

Examples. (i) Let $f(x, y) = (x + y) \sin\left(\frac{1}{x} + \frac{1}{y}\right)$, $(x \neq 0, y \neq 0)$,

$$f(x, 0) = f(0, y) = 0.$$

When $x \neq 0$, $y \neq 0$, $|f(x, y)| \leq |x + y|$ and therefore $f(x, y)$ is a continuous function of both variables at $(0, 0)$, actually in this case the limits $\lim\limits_{x \to 0} f(x, y)$, $(y \neq 0)$ and $\lim\limits_{y \to 0} f(x, y)$, $(x \neq 0)$ do not exist.

(ii) Let $f(x, y) = \{(x - 2y)(2x - y)\}/(x^2 + y^2)$, $f(0, 0) = 2$.

Here $f(x, 0) = f(0, y) = f(0, 0) = 2$ so that $f(x, 0)$, $f(0, y)$ are both continuous. However $f(x, mx) = (1 - 2m)(2 - m)/(1 + m^2)$ when $x \neq 0$ and tends to this value when $x \to 0$. This function of m can have any value between $-\frac{1}{2}$ and $9/2$ by a suitable choice of m. The function is therefore not continuous at $(0, 0)$.

(iii) Let $f(x, y) = x^2y/(x^4 + y^2)$, $f(0, 0) = 0$.

Then $f(x, 0) = 0 = f(0, y)$ are continuous functions of x, y respectively. Also $f(x, mx) = mx/(m^2 + x^2)$, when $x \neq 0$, and when $x \to 0$ with m fixed, $f(x, mx) \to 0$. The function is therefore *continuous in every direction* from $(0, 0)$, nevertheless, it is not a continuous function of (x, y) for if x, y vary along the curve $x = kt$, $y = t^2$,

$f(x, y) = k^2/(1 + k^4)$ (when $t \neq 0$) and tends to this value when $t \to 0$. This function of k can have any value between 0 and $\frac{1}{2}$ by a proper choice of k.

2.27. *Properties of a Continuous Function of Two Variables.*—These are analogous to those of functions of one variable and may be established by similar methods. Corresponding to linear sets of points we have *plane* sets, which possess at least one limiting point. A continuous function may be shown to be *uniformly* continuous and *bounded*. It may be shown that it has a maximum M and a minimum m and that if k is a number between m, M inclusive, the equation $f(x, y) = k$ is satisfied for at least one pair of values (x, y).

2.28. *The Polynomial and the Rational Function.* A function that consists of a finite number of terms of the type $c_{mn}x^my^n$ may be called a *Polynomial* in x, y, where c_{mn} are independent of x, y, and m, n are positive integers. The degree of the polynomial is the greatest value of $m + n$ that occurs. The polynomial is obviously continuous for all values of x, y since $x^m \to x_0^m$ when $x \to x_0$ and $y^n \to y_0^n$ when $y \to y_0$.

A function reducible to the form $P(x, y)/Q(x, y)$ where P, Q are polynomials is called a *Rational Function* of x, y, and is obviously continuous for all values of x, y except those that satisfy the equation $Q(x, y) = 0$.

Example. $\dfrac{(x^2 - y^2 + 2x - y - 3)}{(x - 2)(x^2 + y^2 - 4y + 4)(y^2 - 2x)}$ is continuous except along the line $x = 2$, the parabola $y^2 = 2x$ and at the point $(0, 2)$.

2.3. Differentials. *Functions of One Variable.* Let $y = f(x)$ be a continuous function of x and let $y + \delta y$ correspond to $x + \delta x$, so that $\delta y = f(x + \delta x) - f(x)$.

If δy can be expressed in the form $A\delta x + o(\delta x)$, where A is independent of δx, $f(x)$ is said to be *differentiable*.

Example. Since $(x + \delta x)^n - x^n = nx^{n-1}\delta x + o(\delta x)$, the function x^n is differentiable.

$A\delta x$ is called the *differential* of y and written dy.

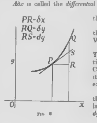


FIG. 6

In *Fig.* 6, P is the point (x, y), Q the point $(x + \delta x, y + \delta y)$.

Now $\delta y/\delta x = A + \{o(\delta x)\}/\delta x$ and therefore dy/dx exists and is equal to A. We may therefore write $dy = f'(x)\delta x$.

In the figure, dy is SR, δy is QR. Thus when a function $f(x)$ is differentiable, it possesses a derivative $f'(x)$. Conversely, if it possesses a derivative, it is differentiable, for if $\lim \delta y/\delta x$ exists, δy is of the form $A\delta x + o(\delta x)$.

If $y = x$, $d(x) = \delta x$ and we can therefore write $dy = f'(x)dx$, this relation being actually more general than $dy = f'(x)\delta x$, for δx is an arbitrary in-

crement, whilst dx is a differential, which, for example, is equal to $\frac{dx}{dt}dt$, when x is expressed as a function of a new variable t.

2.31. Functions of Two Variables. Let $z = f(x, y)$ be continuous throughout a certain domain and let $z + \delta z$ correspond to $(x + \delta x, y + \delta y)$.

Then $\delta z = f(x + \delta x, y + \delta y) - f(x, y)$, which $\to 0$ when $\delta x, \delta y \to 0$. If δz can be expressed in the form $A\delta x + B\delta y + o(\delta\rho)$ when $\delta x = \delta\rho\cos\theta$, $\delta y = \delta\rho\sin\theta$ and A, B are functions of x, y independent of δx, δy, the the function $f(x, y)$ is said to be *differentiable.*

Example. Since $(x + \delta x)^2(y + \delta y)^3 - x^2y^3 = 2xy^3\delta x + 3x^2y^2\delta y + \kappa$ where κ is equal to the sum of a finite number of terms of the form $C\delta x^r\delta y^s$, (C independent of δx, δy, and $r + s > 1$), and since $(\delta x)^r(\delta y)^s = (\delta\rho)^{r+s}\cos^r\theta\sin^s\theta$ (so that $\kappa = o(\delta\rho)$), it follows that the function x^2y^3 is differentiable.

The expression $A\delta x + B\delta y$ is called the *differential* of z and written dz. Also since $d(x) = \delta x$ and $d(y) = \delta y$, we may use the more general relation $dz = Adx + Bdy$.

2.32. Partial Derivatives. Let $z\ (= f(x, y))$ be differentiable and let $\delta y = 0$, then $\delta z = A\delta x + o(\delta x)$; i.e. $\lim \delta z/\delta x$, when $\delta x \to 0$, exists and is equal to A. Thus A is the derivative of z with regard to x when y is constant, and has a meaning even when z is not a differentiable function of the *two* variables. This derivative is called the *first partial derivative* with regard to x and is written $\frac{\partial z}{\partial x}$ or z_x. Similarly B is the first partial derivative with regard to y, x being constant, and is denoted by $\frac{\partial z}{\partial y}$ or z_y.

We therefore obtain the fundamental relation

$$dz = z_x dx + z_y dy \ (\text{or } f_x dx + f_y dy).$$

Notes. (i) Continuity is necessary but *not sufficient* for differentiability.

(ii) If $f(x, y)$ is differentiable, it possesses the derivatives f_x, f_y but the converse is not necessarily true.

Example. Let $f(x, y) = (x + y)\sin\left(\frac{1}{x} + \frac{1}{y}\right)$, $(x \neq 0,\ y \neq 0)$, with $f(x, 0) = f(0, y) = f(0, 0) = 0$.

Then $f(\delta x, \delta y) - f(0, 0) = (\delta x + \delta y)\sin\left(\frac{1}{\delta x} + \frac{1}{\delta y}\right)$; but since $\sin\left(\frac{1}{\delta x} + \frac{1}{\delta y}\right)$ does not tend to a limit when $\delta x, \delta y \to 0$, the function (which is continuous) is not differentiable.

Also $f_x(0, 0) = \lim \{f(\delta x, 0) - f(0, 0)\}/\delta x = 0$ and similarly $f_y(0, 0)$ exists.

(iii) It can be shown, however, that if f_x exists at (x, y) and at *all* points *near* (x, y) and is a continuous function, and if f_y exists at (x, y), then $f(x, y)$ is differentiable. (*Ref.*: Young, '*Cambridge Tract*', *No. 11.*)

2.33. The Derivative along a Curve. Let $x = x(t)$, $y = y(t)$ where $x(t)$, $y(t)$ are differentiable.

Then $\delta x = \frac{dx}{dt}\delta t + o(\delta t)$; $\delta y = \frac{dy}{dt}\delta t + o(\delta t)$.

But $\delta z = z_x \delta x + z_y \delta y + o(\delta\rho)$ if z is differentiable

i.e. $$\delta z = \left(z_x \frac{dx}{dt} + z_y \frac{dy}{dt}\right)\delta t + o(\delta t)$$

since $(\delta\rho)^2 = \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}(\delta t)^2 + o\{(\delta t)^2\}$ i.e. $\delta\rho = O(\delta t)$.

Thus z is a differentiable function of t possessing the derivative

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}.$$

Note. This is the rate of change of z with respect to t along the curve. In particular, if $t = s$ = arc of the curve measured from some fixed point on it, $\frac{dz}{ds} = z_x \frac{dx}{ds} + z_y \frac{dy}{ds} = z_x \cos\theta + z_y \sin\theta$, (where θ is the angle that the tangent to the curve makes with OX); and this may be called the *derivative of z in the direction θ.*

2.34. *Change of Variable.* Let x, y be expressed as differentiable functions of two new variables u, v.

Then $$dx = x_u du + x_v dv,\ dy = y_u du + y_v dv,$$

i.e. $$dz = (z_x x_u + z_y y_u)du + (z_x x_v + z_y y_v)dv$$

which shows that z is a differentiable function of u, v with derivatives z_u, z_v given by

$$z_u = z_x x_u + z_y y_u;\quad z_v = z_x x_v + z_y y_v.$$

2.4. Functions of Several Variables. If $z, x_1, x_2, \ldots, x_n$ are $(n + 1)$ variables such that when $x_1, x_2, \ldots, x_n$ are given, z is determined, z is a function of the n variables $x_1, x_2, \ldots, x_n$ which may be written $z(x_1, x_2, \ldots, x_n)$.

Definitions of derivatives and differentials are obvious extensions of those associated with functions of two variables. The function is continuous at $(c_1, c_2, \ldots, c_n)$ in the domain specified by $a_r \leq x_r \leq A_r$ $(r = 1$ to $n)$ if, given ε, we can find δ (> 0), such that

$$|z(x_1, x_2, \ldots, x_n) - z(c_1, c_2, \ldots, c_n)| < \varepsilon$$

for all x_r that satisfy $|x_r - c_r| < \delta$.

The differential dz is given by

$$dz = \frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2 + \ldots + \frac{\partial z}{\partial x_n}dx_n.$$

2.41. *Functions of Functions.* When x_r $(r = 1$ to $n)$ is itself a differentiable function of m other variables u_s $(s = 1$ to $m)$, then, as in the simpler case, z is a differentiable function of the u-variables, possessing derivatives $\frac{\partial z}{\partial u_s}$ where

$$\frac{\partial z}{\partial u_s} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial u_s} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial u_s} + \ldots + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial u_s}\ (s = 1, 2, \ldots, m)$$

2.5. Higher Partial Derivatives. If f_x is differentiable, it possesses derivatives $\frac{\partial}{\partial x}(f_x)$, $\frac{\partial}{\partial y}(f_x)$ which may be denoted by f_{xx}, f_{xy} or $\frac{\partial^2 f}{\partial x^2}$,

$\dfrac{\partial^2 f}{\partial x \partial y}$ respectively. Similarly, if f_y is differentiable, it possesses derivatives $\dfrac{\partial}{\partial x}(f_y)$, $\dfrac{\partial}{\partial y}(f_y)$, which may be denoted by f_{yx}, f_{yy} or $\dfrac{\partial^2 f}{\partial y \partial x}$, $\dfrac{\partial^2 f}{\partial y^2}$ respectively. For the functions that usually occur, it will be found that $f_{xy} = f_{yx}$ (except possibly for particular values of x, y).

Examples. (i) Let $f(x, y) = ax^4y + bx^3y^3 + cxy^5$.

Then $\quad f_x = 4ax^3y + 3bx^2y^3 + cy^5; \qquad f_y = ax^4 + 3bx^3y^2 + 5cxy^4;$

$f_{xx} = 12ax^2y + 6bxy^3; \; f_{xy} = 4ax^3 + 9bx^2y^2 + 5cy^4 = f_{yx}; \; f_{yy} = 6bx^3y + 20cxy^3.$

(ii) Let $f(x, y) = x^2 \arctan\left(\dfrac{y}{x}\right) - y^2 \arctan\left(\dfrac{x}{y}\right)$, with

$$f(x, 0) = f(0, y) = f(0, 0) = 0.$$

The function arc tan k is the angle between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$ and is therefore finite. Since arc tan y/x, arc tan x/y $(x, y \neq 0)$ are finite, $f(x, y)$ is easily seen to be continuous at $(0, 0)$.

$f_x = 2x \arctan(y/x) - y$; $f_x(0, y) = -y$; $f_x(x, 0) = f_x(0, 0) = 0$, so that $f_x(x, y)$ is continuous and possesses at $(0, y)$ the derivative with regard to y, $f_{xy}(0, y) = \lim\limits_{\delta y \to 0} \{-(y + \delta y) + y\}/\delta y = -1$.

Similarly, $f_y = x - 2y \arctan(x/y)$; $f_y(0, y) = f_y(0, 0) = 0$; $f_y(x, 0) = x$, and therefore $f_y(x, y)$ is a continuous function possessing the derivative

$$f_{yx}(x, 0) = \lim_{\delta x \to 0} \{(x + \delta x) - x\}/\delta x = 1.$$

In this example, $f_{xy}(0, 0)$ is not equal to $f_{yx}(0, 0)$; in general

$$f_{xy}(x, y) = (x^2 - y^2)/(x^2 + y^2)$$

which is not continuous at $(0, 0)$ and may tend to any value between -1 and $+1$.

2.51. *The Equivalence of f_{xy} and f_{yx}.* *Sufficient* conditions for the truth of the relation $f_{xy} = f_{yx}$ can be given in various ways, but the simplest set of conditions is used in the following theorem:

If $f(x, y)$, $f_x(x, y)$, $f_y(x, y)$, $f_{xy}(x, y)$, $f_{yx}(x, y)$ are continuous at (x, y), then $f_{xy} = f_{yx}$.

The continuity of f_{xy}, f_{yx} implies that of f_x, f_y.

Let $E = f(x + \delta x, y + \delta y) - f(x + \delta x, y) - f(x, y + \delta y) + f(x, y)$,

$$F(x, y, \delta y) = f(x, y + \delta y) - f(x, y),$$
$$G(x, y, \delta x) = f(x + \delta x, y) - f(x, y);$$

then $E = F(x + \delta x, y, \delta y) - F(x, y, \delta y)$

$= \dfrac{\partial F}{\partial x}(x + \theta_1 \delta x, y, \delta y)\delta x, \; (0 < \theta_1 < 1)$ (*by the Mean Value Theorem*)

$= \{f_x(x + \theta_1 \delta x, y + \delta y) - f_x(x + \theta_1 \delta x, y)\}\delta x$

$= f_{xy}(x + \theta_1 \delta x, y + \theta_2 \delta y)\delta x \delta y, \; (0 < \theta_2 < 1)$ (*Mean Value Theorem*).

Similarly $E = G(x, y + \delta y, \delta x) - G(x, y, \delta x)$

$= G_y(x, y + \theta_3 \delta y, \delta x)\delta y, \; (0 < \theta_3 < 1)$

$= f_{yx}(x + \theta_4 \delta x, y + \theta_3 \delta y)\delta x \delta y, \; (0 < \theta_4 < 1)$

i.e. $\quad f_{xy}(x + \theta_1 \delta x, y + \theta_2 \delta y) = f_{yx}(x + \theta_4 \delta x, y + \theta_3 \delta y)$.

Hence $f_{xy}(x, y) = f_{yx}(x, y)$ since both are continuous.

Note It is, however, sufficient to assume—
(i) f_x, f_y differentiable (*Young*), or (ii) f_x, f_y, f_{xy} to exist and be continuous (*Schwarz*) (*Ref.*, *De la Vallée-Poussin*, '*Cours d'Analyse*', *I*, *153*)

Similar results hold for higher derivatives and also for the higher derivatives of functions of several variables. In particular, when the derivatives that occur are continuous, the differentiation may be effected in any order of the variables. The third order derivatives of $f(x, y)$ are written $f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}$ or $\dfrac{\partial^3 f}{\partial x^3}, \dfrac{\partial^3 f}{\partial x^2 \partial y}, \dfrac{\partial^3 f}{\partial x \partial y^2}, \dfrac{\partial^3 f}{\partial y^3}$; and there is a corresponding notation for the fourth and higher derivatives of $f(x, y)$ and also for the higher derivatives of functions of several variables.

2.52. Change of Variables in Higher Derivatives. The rules already established may be applied to determine expressions for the higher derivatives when a change of the variables is made. Whilst the method is simple, the formulae are usually long, and it will therefore be sufficient to consider a particular case as an illustration.

Let $V(x, y, z)$ be a function of x, y, z and let x, y, z be expressed as functions of any number of variables u, v, etc. Let us find V_{uu}, V_{uv} in terms of the derivatives of V with regard to x, y, z.

$$V_u = V_x x_u + V_y y_u + V_z z_u; \quad V_v = V_x x_v + V_y y_v + V_z z_v.$$

$$\frac{\partial}{\partial u}(V_x x_u) = x_u \frac{\partial}{\partial u}(V_x) + V_x \frac{\partial}{\partial u}(x_u) = x_u(V_{xx} x_u + V_{xy} y_u + V_{xz} z_u) + V_x x_{uu}$$

with similar results for $\dfrac{\partial}{\partial u}(V_y y_u)$ and $\dfrac{\partial}{\partial u}(V_z z_u)$.

Thus
$$V_{uu} = V_{xx} x_u^2 + V_{yy} y_u^2 + V_{zz} z_u^2 + 2V_{yz} y_u z_u + 2V_{zx} z_u x_u + 2V_{xy} x_u y_u + V_x x_{uu} + V_y y_{uu} + V_z z_{uu}.$$

Similarly
$$V_{uv} = V_{xx} x_u x_v + V_{yy} y_u y_v + V_{zz} z_u z_v + V_{yz}(y_u z_v + y_v z_u) + V_{zx}(z_u x_v + z_v x_u) + V_{xy}(x_u y_v + x_v y_u) + V_x x_{uv} + V_y y_{uv} + V_z z_{uv}$$

Examples. (i) If $x = a + u + v$, $y = b + cu - cv$ and V is a function of x, y, find $V_x, V_y, V_{xx}, V_{xy}, V_{yy}$ in terms of the derivatives of V with regard to u, v.
Here $2u = x - a + (y - b)/c$, $2v = x - a - (y - b)/c$.

$$V_x = V_u u_x + V_v v_x = \tfrac{1}{2}(V_u + V_v); \quad V_y = V_u u_y + V_v v_y = \frac{1}{2c}(V_u - V_v);$$

$$V_{xx} = \tfrac{1}{2}(V_{uu} u_x + V_{uv} v_x + V_{uv} u_x + V_{vv} v_x) = \tfrac{1}{4}V_{uu} + \tfrac{1}{2}V_{uv} + \tfrac{1}{4}V_{vv}.$$

Similarly $V_{xy} = \dfrac{1}{4c}(V_{uu} - V_{vv})$; $V_{yy} = \dfrac{1}{4c^2}(V_{uu} - 2V_{uv} + V_{vv})$.

(ii) If $x = uv$, $y = uvw$, and z is a function of x, y, find z_{uvw}. Here $z_w = uvz_y$;
$z_{vw} = u^2vz_{xy} + u^2vwz_{yy} + uz_y$; and

$$z_{uvw} = (z_{xxy}u^2v^2 + z_{xyy}u^2v^2w + 2z_{xy}uv) + (z_{xyy}u^2v^2w + z_{yyy}u^2v^2w^2 + 2z_{yy}uvw) + (z_{xy}uv + z_{yy}uvw + z_y)$$
$$= x^2z_{xxy} + 2xyz_{xyy} + y^2z_{yyy} + 3xz_{xy} + 3yz_{yy} + z_y.$$

(iii) If $V = 3x^2 + 2y^2 + \phi(x^2 - y^2)$, prove that $yV_x + xV_y = 10xy$.
$V_x = 6x + 2x\phi'(u)$, $V_y = 4y - 2y\phi'(u)$, (where $u = x^2 - y^2$), from which the result follows.

(iv) If $V = x^2 + y^2 + \phi(xy) + \psi(y/x)$, prove that

$$x^2V_{xx} - y^2V_{yy} + xV_x - yV_y = 4(x^2 - y^2).$$

$V_x = 2x + y\phi'(u) - (y/x^2)\psi'(v)$, $V_y = 2y + x\phi'(u) + (1/x)\psi'(v)$, where $u = xy$, $v = y/x$.

Thus $V_{xx} = 2 + y^2\phi''(u) + (y^2/x^4)\psi''(v) + (2y/x^3)\psi'(v)$,

$V_{yy} = 2 + x^2\phi''(u) + (1/x^2)\psi''(v)$, giving the required result.

(v) *Homogeneous Functions.* $F(x_1, x_2, \ldots, x_n)$ is called a *homogeneous function* of degree m if $F(t\xi_1, t\xi_2, \ldots, t\xi_n) = t^mF(\xi_1, \xi_2, \ldots, \xi_n)$. Thus $x^3 + y^3 + 3x^2y$; $(x^3 + y^3 + z^3)/(x + y + z)^3$, $x^2y^7 + 2z^4u^5$ are homogeneous functions of degrees 3, 0, 9 respectively.

Euler's Theorem states that for such a function F

$$x_1\frac{\partial F}{\partial x_1} + x_2\frac{\partial F}{\partial x_2} + \ldots + x_n\frac{\partial F}{\partial x_n} = mF.$$

Let $x_r = t\xi_r (r = 1 \text{ to } n)$ so that $\dfrac{dx_r}{dt} = \xi_r$ (t being regarded as the only variable).

Thus
$$\frac{d}{dt}F = \sum_1^n\frac{\partial F}{\partial x_r}\cdot\frac{dx_r}{dt} = \frac{1}{t}\sum_1^n x_r\frac{\partial F}{\partial x_r}$$

But
$$F = t^mF(\xi_1, \xi_2, \ldots, \xi_n) \text{ and therefore } \frac{dF}{dt} = \frac{m}{t}F.$$

i.e.
$$\sum_1^n x_r\frac{\partial F}{\partial x_r} = mF.$$

2.6. Taylor's Expansion with Remainder. The object in this expansion is to express a function near $x = a$ approximately as a polynomial. It is more convenient here to obtain the result by the mean value theorem, although there is some advantage in using another method involving integration in which the remainder is expressed as an integral. (*See: Darboux's 'Formula', § 11.31, Ex.*)

2.61. Taylor's Expansion. Functions of One Variable. Let $f(x)$ be a function which with its first n derivatives is continuous in

$$a \le x \le a + h \text{ (or } a + h \le x \le a + h)$$

and which possesses an $(n + 1)$th derivative in $a < x < a + h$

Let
$$F(x) = f(a + h) - f(x) - \frac{a + h - x}{1!}f'(x)$$
$$- \ldots - \frac{(a + h - x)^n}{n!}f^{(n)}(x) - (a + h - x)^mK$$

where $m > 0$ and K is independent of x. Then $F(a + h) = 0$. Also $F(a) = 0$ if K satisfies the equation

$$h^mK = f(a + h) - f(a) - hf'(a) - \ldots - \frac{h^n}{n!}f^{(n)}(a).$$

$F(x)$ is continuous in $a \le x \le a + h$ and its derivative exists in

$$a < x < a + h.$$

Therefore, by Rolle's Theorem, $F'(x) = 0$ when $x = a + \theta h$ where θ is some number satisfying $0 < \theta < 1$.

But $F'(x) = -\dfrac{(a+h-x)^n}{n!}f^{(n+1)}(x) - m(a+h-x)^{m-1}K$

and since $(a+h-x) \neq 0$ when $x = a + \theta h$, we have

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + R_n$$

where R_n (called the *Remainder* after $(n+1)$ terms) is equal to

$$\frac{h^{n+1}(1-\theta)^{n-m+1}}{m.\,n!}f^{(n+1)}(a+\theta h) \quad (\textit{Schlömilch}).$$

In particular

(i) (taking $m = n + 1$), $R_n = \dfrac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta h)$ (*Lagrange*).

(ii) (taking $m = 1$), $R_n = \dfrac{h^{n+1}}{n!}(1-\theta)^n f^{(n+1)}(a+\theta h)$ (*Cauchy*)

If we write x for h and 0 for a we obtain *Maclaurin's Expansion*

$f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \ldots + \dfrac{x^n}{n!}f^{(n)}(0) + R^n$, where

(i) $R_n = \dfrac{x^{n+1}}{(n+1)!}f^{(n+1)}(\theta x)$, (*Lagrange*) or

(ii) $R_n = \dfrac{x^{n+1}(1-\theta)^n}{n!}f^{(n+1)}(\theta x)$, (*Cauchy*).

An important case that arises in practice is one in which the $(n+1)$th derivative is *bounded* at $x = a$ in Taylor's expansion and at $x = 0$ in Maclaurin's expansion, for then, in the latter, for example, if n is fixed and x is *small*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + O(x^{n+1})$$

2.62. Taylor's Expansion for n = 1. (The Second Mean Value Theorem.) The expansion for $n = 1$ is

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta h)$$

In this case, let us assume for simplicity that $f''(x)$ is continuous at $x = a$ and in the neighbourhood.

In *Fig. 7*, P is the point $\{a, f(a)\}$, Q the point $\{a+h, f(a+h)\}$;

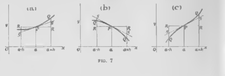


FIG. 7

the tangent at P meets the ordinate at Q in S and the parallel through P to OX meets that ordinate in R.

The height of Q above the tangent is $f(a+h) - f(a) - hf'(a)$ i.e. $\frac{1}{2}h^2 f''(a + \theta h)$.

If $f''(a) > 0$, so also is $f''(a + \theta h) > 0$ if h is small. In this case the curve is *above* the tangent (*Fig.* 7 (*a*)). If $f''(a) < 0$, the curve is *below* the tangent (*Fig.* 7 (*b*)). If $f''(a) = 0$ and $f''(x)$ changes sign as x passes through a, the curve *crosses* the tangent (*Fig.* 7 (*c*)). A point where the curve crosses the tangent is called an *inflexion*; and the problem of determining the inflexions is therefore the same as that of determining the maxima and minima of $f'(x)$; for at such a point a, $f''(a)$ is zero and $f''(x)$ changes sign as x passes through a.

Example. If $y(1+x^2) = (1-x)$, then $y'(1+x)^2 = x^2 - 2x - 1$ and $y''(1+x^2)^3 = -2(x+1)(x^2 - 4x + 1)$. There are, therefore, inflexions at $x = -1, 2 \pm \sqrt{3}$ (these being simple roots of $y'' = 0$).

Note. The mean value theorem gives also a method (*Newton's*) of approximating to a root of the equation $f(x) = 0$. Taking a typical case, let us suppose that $f(a) > 0$, $f(b) < 0$ and that $f'(x)$ does not vanish in the interval $b < x < a$. The derivative $f'(x)$ is therefore of constant sign in (b, a) (in this case positive). There is one root x and one only of $f(x) = 0$ in the interval and

$$0 = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(c), \quad (x < c < a).$$

The error in taking $x = a - f(a)/f'(a)$ is of the order $\{f(a)\}^2 f''(c)/\{f'(a)\}^3$, and is therefore small in comparison to $f(a)/f'(a)$, if $f(a)$ is small. Suppose for example that $f''(x) > 0$ in (b, a), then since $a > a - f(a)/f'(a) > x$ the sequence $a_1, a_2, \ldots$ where $a_{n+1} = a_n - f(a_n)/f'(a_n)$ is monotonic and decreases steadily to x. Similarly if $f''(x) < 0$, the sequence increases steadily to x. In the former $a_1 > x$ and in the latter $a_1 < x$. If $f''(x) = 0$ (when $f(x) = 0$), the terms are alternately greater than and less than x, and tend to x provided $x - a$ is sufficiently small.

2.63. Taylor's Expansion. Functions of Two Variables. Let $f(x, y)$ and all its partial derivatives up to and including those of order n be differentiable near (a, b): and let $F(t) = f(a + ht, b + kt)$.

Expand $F(t)$ in powers of t by Maclaurin's formula. To obtain this expansion, we have $F(0) = f(a, b)$; $F'(t) = f_x h + f_y k$ and therefore $F'(0) = hf_a + kf_b$ where f_a, f_b denote the values of f_x, f_y respectively at (a, b). Similarly $F''(0) = h^2 f_{aa} + 2hkf_{ab} + k^2 f_{bb}$, which is conveniently written $\left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b}\right)^2 f$.

Thus, continuing the differentiation, we find

$$f(a + ht, b + kt) = f(a, b) + t\left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b}\right)f + \frac{t^2}{2!}\left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b}\right)^2 f + \ldots + \frac{t^n}{n!}\left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b}\right)^n f + R_n$$

where

$$R_n = \left[\frac{t^{n+1}}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x, y)\right]\binom{x = a + \theta th}{y = b + \theta th}, \quad (0 < \theta < 1).$$

Putting $t = 1$, we have

$$f(a+h,\ b+k) = f(a,\ b) + hf_a + kf_b + \frac{1}{2!}\left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b}\right)^2 f + \ .\ .$$
$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b}\right)^n f + R_n,$$

where $\quad R_n = \left[\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x,\ y)\right]\begin{pmatrix} x = a + \theta h \\ y = b + \theta k \end{pmatrix}.$

Writing $a = 0 = b$, $h = x$, $k = y$, we obtain the Maclaurin Expansion

$$f(x,\ y) = f(0,\ 0) + \{xf_x(0,\ 0) + yf_y(0,\ 0)\} + \ .\ .\ .$$
$$+ \frac{1}{n!}\left\{\left(x\frac{\partial}{\partial \xi} + y\frac{\partial}{\partial \eta}\right)^n f(\xi,\ \eta)\right\}\begin{pmatrix} \xi = 0 \\ \eta = 0 \end{pmatrix}$$
$$+ \frac{1}{(n+1)!}\left\{\left(x\frac{\partial}{\partial \xi} + y\frac{\partial}{\partial \eta}\right)^{n+1} f(\xi,\ \eta)\right\}\begin{pmatrix} \xi = \theta x \\ \eta = \theta y \end{pmatrix}.$$

2.64. Taylor's Expansion. Functions of Several Variables. The formula for two variables may be extended in an obvious way to functions of any number of variables. In particular,

$$f(a+h,\ b+k,\ c+l) = f(a,\ b,\ c) + \left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b} + l\frac{\partial}{\partial c}\right)f + \ .\ .\ .$$
$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b} + l\frac{\partial}{\partial c}\right)^n f + \frac{1}{(n+1)!}\left\{\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^{n+1} f(x,\ y,\ z)\right\}$$

where in the last term $a + \theta h$, $b + \theta k$, $c + \theta l$ are substituted respectively for x, y, z after differentiation.

Examples II

In the sets of points given in *Examples 1–14*, where, unless otherwise stated, m, n take all positive integers for their values obtain (i) the upper and lower bounds, (ii) the upper and lower limits, if any (iii) the finite limiting points, if any (iv) the maximum and minimum if they exist

1. $3^m + 2^n$ **2.** $(1{\cdot}5)^m + (0{\cdot}5)^n$ **3.** $(1{\cdot}01)^m - (0{\cdot}9)^n$ **4.** $3^m - (-2)^n$

5. $2^m + (-3)^n$ **6.** $(0{\cdot}9)^m - (-0{\cdot}8)^n$ **7.** $(-3)^m + (-2)^n$

8. $\frac{1}{n},\ \frac{n+2}{n+1},\ -n,\ 2-2^{-n}$ **9.** $2^m,\ 2^{-n}$

10. $2\sin\frac{1}{2}n\pi + 3\sin\frac{1}{3}n\pi + \sin\frac{1}{4}n\pi$ **11.** $x^n,\ x^{-n}\ (x \neq 0)$

12. $\frac{(-1)^n}{n} - \frac{(-1)^m}{m}$ **13.** $\frac{n-2}{n-3} + \frac{4m-1}{m+4}$

14. $2^m + 2^n$, all integer values of m, n, positive and negative, including zero.

State the limiting-points, infinite and finite, of the sets of points given in *Examples 15–21*, where m, n, p take all positive integers for their values.

15. $\frac{m}{n}$ **16.** $\frac{m-n}{m+n}$ **17.** $\frac{mn}{(m+n)^2}$ **18.** $x^n,\ y^m$

19. $\frac{n+2}{n+3} + \frac{2m-3}{m+4} + \frac{3p+5}{p+1}$

20. The limiting points of *Example 19*.

21. The limiting points of *Example 20*.

22. If G is set given by $3^{-m} + 5^{-n} + 7^{-p}$ (m, n, p all positive integers), find the derived sets of G, showing that G is of the first species and third order

23. If G is a set of points, and $G_1 = GG'$, $G_2 = G - G_1$, show that G_2G_2' is void.

24. If $f(x) = \sin\left(\frac{1}{x_1}\right)$ where $x_1 = \sin\left(\frac{1}{x_2}\right)$ and $x_2 = \sin\left(\frac{1}{x}\right)$, show that the zeros of $f(x)$ form a set of points of the third order.

25. Prove that the set of all numbers that are the roots of all equations of the type $p_0x^n + p_1x^{n-1} + \dots + p_n = 0$ where $p_0, p_1, p_2, \dots, p_n$ are integers positive or negative, or zero is enumerable.

26. From the interior of the interval $(0, 1)$, an open interval of length $f_1\ (<1)$ is removed. From the remaining intervals of length x_1, x_2, open intervals of length f_2x_1, f_2x_2 respectively are removed from the interior. The process is continued indefinitely with the intervals that remain. Show that the measure of the set of points that remain is $\lim_{n\to\infty}(1-f_1)(1-f_2)\dots(1-f_n)$. Find the measures in the following cases: (i) $f_n = c$, all n, (ii) $f_n = \frac{1}{n+1}$, (iii) $f_n = \frac{1}{4n^2}$, (iv) $f_n = 2^{-n}$.

27. Is it possible for a function to be continuous in the rational domain and to be unbounded?

Show that when the mean value theorem $f(a+h) - f(a) = hf'(a+\theta h)$ is applied to the functions given in *Examples 28–31*, the values of θ are as stated $(h \neq 0)$.

28. $f(x) = x^2$, $\theta = \frac{1}{2}$.

29. $f(x) = x^3$; $\theta = -\frac{a}{h} \pm \left(\frac{1}{3} + \frac{a}{h} + \frac{a^2}{h^2}\right)^{\frac{1}{2}}$, but the positive sign must be taken when $3a + h > 0$ and the negative sign when $3a + 2h < 0$.

30. $f(x) = x^4$; $\theta = -\frac{a}{h} + \left(\frac{1}{4} + \frac{a}{h} + \frac{3a^2}{2h^2} + \frac{a^3}{h^3}\right)^{\frac{1}{3}}$.

31. $f(x) = \frac{1}{x}$; $\theta = -\frac{a}{h} - \frac{(a^2+ah)^{\frac{1}{2}}}{h}$, $(a + h < 0)$; $\theta = -\frac{a}{h} + \frac{(a^2+ah)^{\frac{1}{2}}}{h}$, $(a > 0)$. Explain why the interval $a + h > 0 > a$ is omitted.

32. Show that the radius of the circle whose centre is on the normal to the curve $y = f(x)$ at $x = a$ and which passes through the point whose abscissa is $a + h$ is $(1+\lambda^2)^{3/2}/\mu - h\lambda(1+\lambda^2)^{\frac{1}{2}} - \frac{1}{2}\mu h^2(1+\lambda^2)^{\frac{1}{2}}$ where $\lambda = f'(a)$, $\mu = f''(a + \theta h)$, $(0 < \theta < 1)$.

33. The breaking weight W of a cantilever beam is given by the formula $Wl = kbd^2$, where b is the breadth, l the length, d the depth and k a constant depending on the material of the beam. If the breadth is increased by 2 per cent, and the depth by 5 per cent, by how much per cent should the length be altered so as to keep the breaking weight unchanged?

34. A physical constant c is given by the formula $c = \frac{x^2+y^2}{x^2-y^2}$. Find the relative errors in c in the two cases (i) $x = 10$, $y = 1$, (ii) $x = 10$, $y = 5$ when the relative errors of x, y are $\frac{1}{100}$ in both cases.

35. Prove that if in a triangle ABC, the area is calculated from the elements a, B, C, the measurements of which have errors δa, δB, δC respectively, the approximate error δS in the area is given by

$$\delta S/S = 2\delta a/a + c\,\delta B/(a \sin B) + b\,\delta C/(a \sin C).$$

36. Find the derivative of $(x^2/a^2) + (y^2/b^2)$ at (x_1, y_1) in the direction making an angle θ with the x-axis. Prove that its greatest value is along the normal to the ellipse $(x^2/a^2) + (y^2/b^2) = (x_1^2/a^2) + (y_1^2/b^2)$, and is equal to

$$2\{(x_1^2/a^4) + (y_1^2/b^4)\}^{\frac{1}{2}}.$$

Find the points of inflexion of the curves given in *Examples 37–43*.

37. $y(x^2 + 1) = 1$ **38.** $y = (x^2 - 1)^3$ **39.** $y = 3x^6 - 48x^5 + 196x^4$

40. $y(x^2 + 1) = x^3$ **41.** $y(x - 1)(x - 2) = 3x - 2$

42. $xy(x + 3) = 1 - 6x - 3x^2$ **43.** $y(x^2 - 2x + 2) = x^3 + 11$

44. Prove that the curve $Ay(x-\alpha)(x-\beta) = ax^2 + bx + c$ has one point of inflexion at $x = \lambda$, where $(\lambda-\alpha)(a\beta^2+b\beta+c)^{\frac{1}{3}} = (\lambda-\beta)(a\alpha^2+b\alpha+c)^{\frac{1}{3}}$.

45. If $pv = R\theta$ determines θ as a function of p, v, prove that $\theta = R\theta_p\theta_v$.

46. Show that if $x^2 + 2xy = c(z^2 - 2yz)$, where c is a constant,

$$(2y^2z - x^2y - xz^2)dx + (x^2 - z^2)dy + (yz^2 + x^2z - 2xy^2)dz = 0.$$

47. If $x = 3u - 4v - 2$, $y = 2u + 3v - 1$, $V = x^2 - y^2$, show that

$$V_u = 6x - 4y,$$

when V is expressed as a function of u, v.

48. If $V = x^3 + 8y^3 - 18xy$, find the real values of x, y that satisfy the equations $V_x = V_y = 0$.

49. If $P = x^2y - y^3 - y^2z$, $Q = xy^2 - x^3 - x^2z$, $R = xy^2 + x^2y$, show that

$$P(Q_z - R_y) + Q(R_x - P_z) + R(P_y - Q_x) = 0.$$

50. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = x^3 + y^3 + z^3 - 3xyz$, prove that

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0.$$

51. If V, P, Q, R, μ are functions of x, y, z satisfying the relations $V_x = \mu P$, $V_y = \mu Q$, $V_z = \mu R$, $(\mu \neq 0)$, show that

$$P(Q_z - R_y) + Q(R_x - P_z) + R(P_y - Q_x) = 0.$$

Discuss the continuity of the functions given in *Examples 52–4*, where in each case the value of the function at $(0, 0)$ is 0.

52. $\dfrac{x^4 + y^4}{x^2 + y^2}$ **53.** $\dfrac{x^3 - y^3}{x - y}$ **54.** $\dfrac{xy(y + x^2)}{y^2 + x^2}$

55. Transform the relation $V_{xx} - 8xy^2V_x - 8y(1 - y^2)V_y - 16x^2y^2V = 0$ by means of the change of variables $u = 2xy$, $vy = 1$.

Find V_x, V_y, V_z, V_u, V_{xy}, V_{xyz}, V_{xyzu} for the functions V given in *Examples 56–9*.

56. $\dfrac{x^2y^2u}{a^2 - z^2}$ **57.** $\dfrac{x^2y^2}{u^2 + z^2}$ **58.** $\dfrac{xyzu}{(x - y)(z - u)}$

59. $\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{u} + \dfrac{u}{x} + \dfrac{xy}{zu}$

60. If $F = \dfrac{x}{x^2 + y^2}$, $G = \dfrac{y}{x^2 + y^2}$, show that $F_y = G_x$, $F_x = -G_y$,

$\nabla^2F = \nabla^2G = 0$ where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

61. If V, a function of r, θ is expressed as a function of ρ, ϕ by means of the equations: $\rho r = c^2$, $\phi = 2\pi - \theta$, prove that

$$\rho^2V_{\rho\rho} + \rho V_\rho + V_{\phi\phi} = r^2V_{rr} + rV_r + V_{\theta\theta}.$$

62. If $V = x\phi(y/x) + \psi(y/x)$, show that $x^2V_{xx} + 2xyV_{xy} + y^2V_{yy} = 0$.

63. If V is a function of x, y and x, y are functions of u, v such that $x_u = y_v$, $x_v = -y_u$, prove that $(V_{uu} + V_{vv})/(V_{xx} + V_{yy}) = x_u^2 + x_v^2 = y_u^2 + y_v^2$.

64. If V is a homogeneous function of three variables x, y, z of the mth degree, prove that $x^2V_{xx} + y^2V_{yy} + z^2V_{zz} + 2xyV_{xy} + 2yzV_{yz} + 2zxV_{zx} = m(m - 1)V$.

65. Prove that if $z_y = F(z_x)$, then $z_{xx}z_{yy} = z_{xy}^2$.

66. If $z = xF(x + y) + G(x + y)$, show that $z_{xx} - 2z_{xy} + z_{yy} = 0$.

67. If $z = F(y + m_1x) + G(y + m_2x)$ and m_1, m_2 are the roots of the quadratic equation $am^2 + 2hm + b = 0$, then $az_{xx} + 2hz_{xy} + bz_{yy} = 0$.

68. Prove that the function $z = x^n\phi(y/x) + (1/x^n)\psi(y/x)$ satisfies the equation $x^2z_{xx} + 2xyz_{xy} + y^2z_{yy} + xz_x + yz_y = n^2z$.

69. If $\dfrac{\partial X}{\partial t} = c(\gamma_y - \beta_z)$; $\dfrac{\partial Y}{\partial t} = c(\alpha_z - \gamma_x)$; $\dfrac{\partial Z}{\partial t} = c(\beta_x - \alpha_y)$ and

$\dfrac{\partial \alpha}{\partial t} = c(Y_z - Z_y)$, $\dfrac{\partial \beta}{\partial t} = c(Z_x - X_z)$, $\dfrac{\partial \gamma}{\partial t} = c(X_y - Y_x)$, c being constant,

and if $\alpha_x + \beta_y + \gamma_z$, $X_x + Y_y + Z_z$ are both functions of t only, show that $X, Y, Z, \alpha, \beta, \gamma$, all satisfy the equation $\Box^2 V = 0$ where

$$\Box^2 = \frac{\partial^2}{\partial t^2} - c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

70. If $\{F(x) + G(y)\}^2 e^z = 2F'(x)G'(y)$, prove that $z_{xy} = e^z$.

71. If $z = \phi(x - y) + \psi(x + y) - x\{\phi'(x - y) + \psi'(x + y)\}$
$+ \frac{1}{4}x^2\{\phi''(x - y) + \psi''(x + y)\}$,
show that $x(z_{xx} - z_{yy}) = 4z_x$.

72. Prove that a homogeneous polynomial $V(x, y, z)$ of the second degree which satisfies the equation $V_{xx} + V_{yy} + V_{zz} = 0$ is a linear combination of the five polynomials $xy, yz, zx, x^2 - y^2, x^2 - z^2$. Find corresponding polynomials of the third degree.

Solutions

1. ∞, 5, $-\infty$, none, none, none, 5.
2. ∞, 2; $-\infty$, none, none, none, 2
3. ∞, $(1{\cdot}01)^{10} + (0{\cdot}9)^{10}$; $-\infty$, none, none, none, $(1{\cdot}01)^{10} + (0{\cdot}9)^{10}$.
4. ∞, 1; $-\infty$, none; none; none, 1
5. ∞, $-\infty$, ∞, $-\infty$; none, none, none.
6. 1·45, 0; 0, 0; 0, 1·45, none.
7. ∞, $-\infty$; ∞, $-\infty$, none, none, none.
8. 2, $-\infty$; 2, $-\infty$, 0, 1, 2, none, none
9. ∞, 0; ∞, 0; 0; none, none.

10. $\frac{5}{2}\sqrt{3}$, $-\frac{5}{2}\sqrt{3}$; $\frac{5}{2}\sqrt{3}$, $-\frac{5}{2}\sqrt{3}$; $\pm\sqrt{3}$, $-\frac{5}{2}$, $\pm\frac{5}{2}\sqrt{3}$, -2, $\pm\frac{\sqrt{3}}{2}$, 0, $-\frac{5}{2}\sqrt{3}$, $\frac{5}{2}\sqrt{3}$.

11. $(x = 0, x = 1)$, ∞, 0; ∞, 0; 0, none, none, $(x = 0, x = -1)$, ∞, $-\infty$; ∞, $-\infty$; 0; none, none; $(x = 1)$, 1, 1; 1, 1; 1; 1, 1; $(x = -1)$, 1, -1; 1, -1, 1, -1, 1, -1.

12. 1, -2; $\frac{1}{2}$, -1; $(-1)^n/n$, 0; 1, -2.

13. 5, $1\frac{2}{3}$, 5, 2; $4 + \frac{n-2}{n-3}$, $1 + \frac{4m+1}{m+4}$, 5; none, $1\frac{2}{3}$.

14. ∞, 0; ∞, 0; 2^n, 0, none, none.

15. All real numbers with ∞, 0 as upper and lower limits respectively.

16. All real numbers in the interval $-1 < x < 1$.

17. All real numbers in $0 < x \leq \frac{1}{2}$.

18. Various possibilities: (i) ∞, (ii) 1, (iii) 0, (iv) ∞, 1, (v) ∞, 0, (vi) ∞, $-\infty$, (vii) 1, 0, (viii) 1, -1, (ix) ∞, 1, -1, (x) ∞, 1, $-\infty$, (xi) ∞, 0, $-\infty$, (xii) ∞, 1, -1, $-\infty$, (xiii), 0, 1, -1.

19. $1 + \frac{2m-3}{m+4} + \frac{3p+5}{p-1}$, $\frac{n+2}{n+3} + 2 + \frac{3p+5}{p+1}$, $\frac{n+2}{n+3} + \frac{2m+3}{m+4} + 3$, $3 + \frac{3p-5}{p+1}$, $\frac{n+2}{n+3} + 5$, $4 + \frac{2m+3}{m-4}$, 8

20. $3 + \frac{3p+5}{p-1}$, $\frac{n+2}{n+2} + 5$, $4 + \frac{2m-2}{m+4}$, 6 21. 6

22. G': $\frac{1}{2^m} + \frac{1}{5^n}$, $\frac{1}{5^n} + \frac{1}{7^p}$, $\frac{1}{2^m} + \frac{1}{7^p}$, $\frac{1}{3^m}$, $\frac{1}{5^n}$, $\frac{1}{7^p}$, 0, G'': $\frac{1}{2^m}$, $\frac{1}{5^n}$, $\frac{1}{7^p}$, 0; G''', 0

25. *See* Hobson, '*Real Variable*', *I*, § 59.

26. (i) 0, (ii) 0, (iii) $2/\pi$, (iv) 0·29 (approx.)

27. Yes, $\frac{1}{x^2 - 2}$ for example. 33. 12 per cent. 34. 0·0006, 0·02

36. $2\frac{x_1}{a^2}\cos\theta + 2\frac{y_1}{b^2}\sin\theta$ **37.** $(\pm\frac{1}{2}\sqrt{3}, \frac{1}{2})$ **38.** $(\pm\frac{1}{4}\sqrt{3}, \frac{1}{4})$

39. $x = 5$ or 7 **40.** $(-\sqrt{3}, -\frac{3}{2}\sqrt{3}), (0,0)$ **41.** $x = 0$

42. $x = 1$ **43.** $x = \frac{1}{4}, 7 \pm 3\sqrt{3}$ **48.** $x = 0, y = 0$ or $x = 3, y = \frac{3}{2}$

52. Continuous. **53, 54.** Discontinuous at $(0, 0)$.

55. $V_{uu} + 2uv^2V_u + 2v(1 - v^2)V_v + u^2v^2V = 0$

56. $V_x = \frac{2xy^2u}{a^2 - z^2}, V_y = \frac{2x^2yu}{a^2 - z^2}, V_z = \frac{2x^2y^2uz}{(a^2 - z^2)^2}, V_u = \frac{x^2y^2}{(a^2 - z^2)},$

$V_{xy} = \frac{4xyu}{(a^2 - z^2)}, V_{xyz} = \frac{8xyzu}{(a^2 - z^2)^2}, V_{xyzu} = \frac{8xyz}{(a^2 - z^2)^2}.$

57. $V_x = \frac{2xy^3}{(z^3 + u^3)}, V_y = \frac{3x^2y^2}{z^3 + u^3}, V_z = -\frac{3x^2y^3z^2}{(z^3 + u^3)^2}, V_u = -\frac{3x^2y^3u^2}{(z^3 + u^3)^2},$

$V_{xy} = \frac{6xy^2}{(z^3 + u^3)}, V_{xyz} = -\frac{18xy^2z^2}{(z^3 + u^3)^2}, V_{xyzu} = \frac{72xy^2z^2u^2}{(z^3 + u^3)^3}$

58. $V_x = \frac{y^2zu}{(x + y)^2(z + u)}, V_y = \frac{x^2zu}{(x + y)^2(z + u)}, V_z = \frac{xyu^2}{(x + y)(z + u)^2},$

$V_u = \frac{xyz^2}{(x + y)(z + u)^2}, V_{xy} = \frac{2xyzu}{(x + y)^3(z + u)}, V_{xyz} = \frac{2xyu^2}{(x + y)^3(z + u)^2},$

$V_{xyzu} = \frac{4xyzu}{(x + y)^3(z + u)^3}.$

59. $V_x = \frac{1}{y} - \frac{u}{x^2} + \frac{y}{zu}, V_y = -\frac{x}{y^2} + \frac{1}{z} + \frac{x}{zu}, V_z = -\frac{y}{z^2} + \frac{1}{u} - \frac{xy}{z^2u},$

$V_u = -\frac{z}{u^2} + \frac{1}{x} - \frac{xy}{u^2z}, V_{xy} = -\frac{1}{y^2} + \frac{1}{zu}, V_{xyz} = -\frac{1}{uz^2}, V_{xyzu} = \frac{1}{u^2z^2}.$

70. Take the relation as $z = \log 2 + \log F'(x) + \log G'(y) - 2\log(F(x) + G(y))$.

72. xyz, $3x^2y - y^3$, $3xy^2 - x^3$, $3y^2z - z^3$, $3yz^2 - y^3$, $3xz^2 - x^3$, $3zx^2 - z^3$.

CHAPTER III

IMPLICIT FUNCTIONS OF ONE VARIABLE. ALGEBRAIC CURVES. CONTOUR LINES.

3. Implicit Functions of One Variable. If $f(x, y)$ is a function of two variables, the relation $f(x, y) = 0$, may, under certain conditions, determine y as a function of x. Functions defined in this way are called *implicit*, although *explicit* forms may sometimes be obtained for them. A simple illustration is provided by the inverse function.

3.01. Inverse Functions. Let the relation connecting x, y be $x = F(y)$ where $F(y)$ is a function whose properties are known. If this determines y as a function of x, the symbol $F^{-1}(x)$ is sometimes used for y; and since more than one function y may satisfy the equation, this symbol is, in general, ambiguous. The function $F^{-1}(x)$ is called the *function inverse* to $F(x)$. If $F(y)$ increase steadily from $F(B)$ to $F(A)$ as y increases

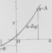


FIG 1

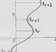


FIG 2

from B to A, then y is a continuous function of x in the interval $F(B) < x < F(A)$ for y increases steadily from B to A as x increases from $F(B)$ to $F(A)$ (*Fig 1.*) Similarly a continuous function of x is determined when $F(y)$ decreases steadily from $F(B)$ to $F(A)$. More generally, when $F(y)$ does not increase (or decrease) steadily for all values of y, it is usually possible to divide the range of variation of y into intervals given by $A_r \leq y \leq A_{r+1}$, within each of which $F(y)$ is monotonic. To *each* such interval, there will correspond a single-valued continuous function of x. (*Fig. 2.*)

A further specification is necessary in order to make y definite. Thus a function defined by the relation $x = \sin y$ is made definite by defining $\sin^{-1} x$ (or arc sin x) as that value that satisfies $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$. When more than one value of y is determined by a value of x, the various values are called *Branches*

3.02. The Derivative of an Inverse Function. If $x = F(y)$ and $F'(y)$ exists and is of constant sign near (a, b), then $F(y)$ is monotonic near (a, b) and therefore a continuous function y of x exists. Also since $\lim\left(\frac{y-b}{x-a}\right).\lim\left(\frac{x-a}{y-b}\right) = 1$ when one of the limits exists, the other limit exists, if the former is not zero, i.e. since $\frac{dx}{dy}$ exists and is not zero, so also does $\frac{dy}{dx}$ and its value is $1/\left(\frac{dx}{dy}\right)$. When

$$\frac{dx}{dy} \to 0, \quad \left|\frac{dy}{dx}\right| \to +\infty$$

3.03. Rational Indices. Consider the relation $x = y^q$, where q is an integer. As y increases from 0, x increases steadily from 0. (*Fig. 3.*) Thus the relation determines for $x > 0$ a unique positive function y.

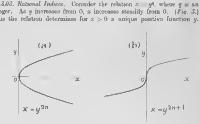


FIG. 3

By the laws of indices, this function may be denoted by $x^{1/q}$, a symbol we shall regard as non-ambiguous. If p, q are integers with no common factor, the function $x^{p/q}$ may be defined as $(x^{1/q})^p$ and satisfies the equation $y^q = x^p$. Finally, the function x^a, $(x > 0)$, where a is *rational* and *negative*, is (by the laws of indices) equal to $1/x^{-a}$.

It should be noted, however, that the equation $y^q = x^p$, when q is even, determines for $x > 0$, *two* real functions, viz. $\pm\, x^{p/q}$; and, when q is odd, a single function for *all* x. Thus when $x > 0$, the function is $x^{p/q}$ and when $x < 0$, it is $-(-x)^{p/q}$ when p is odd and $(-x)^{p/q}$ when p is even.

3.04 The Derivative of x^a, a rational. Let $a = p/q$, then $y^q = x^p$ and $qy^{q-1}y' = px^{p-1}$

i.e.
$$\frac{d}{dx}(x^a) = \frac{p}{q}x^{p-1}(x^{p/q})^{1-q} = ax^{a-1}.$$

3.05. Expansion of $(1+x)^\alpha$, α rational, x small. Using Maclaurin's expansion, we find

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1\cdot 2}x^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + R_n$$

where $R_n = \dfrac{x^{n+1}}{(n+1)!}\alpha(\alpha-1)\ldots(\alpha-n)(1+\theta x)^{\alpha-n-1} = O(x^{n+1})$ for a *fixed* n since $1+\theta x \neq 0$ when x is small $(0<\theta<1)$.

3.06. The Graph of $y^m = ax^n$. In this equation we suppose that m is a positive integer and that n is an integer that may be positive or negative. In view of future applications, however, we will no longer assume that m, n have no common factors. The graph is easily drawn by finding a quadrant in which the curve lies and completing the curve by symmetry.

Notes (i) The curve touches OX at O if $n - m > 0$; and touches OY at O if $m - n > 0$.

(ii) If $n < 0$, the axes $x = 0$, $y = 0$ are asymptotes.

(iii) If $a < 0$, m, n positive *even* integers, then $(0, 0)$ is the *only* real point on the curve; whilst if $a < 0$, m an even integer, n a *negative even* integer, there are no real points in the finite part of the plane.

Some typical cases are shown in *Fig. 4.*

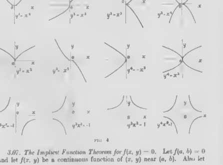


FIG. 4

3.07. The Implicit Function Theorem for $f(x, y) = 0$. Let $f(a, b) = 0$ and let $f(x, y)$ be a continuous function of (x, y) near (a, b). Also let

$f(x, y)$ be a function that increases (or decreases) steadily with y for any fixed x in the neighbourhood. Then $f(a, b - \epsilon), f(a, b + \epsilon)$ have opposite signs, when ϵ is small, so that $f(x, y)$ vanishes for a value of y between $b - \epsilon, b + \epsilon$, for any x near $x = a$ (*Fig.* 5); for, since $f(x, y)$ is continuous, $f(a, b - \epsilon)$ and $f(a, b + \epsilon)$ have the same signs respectively as $f(x, b - \epsilon)$ and $f(x, b + \epsilon)$ when $x - a$ is small. Also $f(x, y)$ can vanish *once* only for a given x when $b - \epsilon < y < b + \epsilon$, since $f(x, y)$ increases steadily with y. Thus the function y so defined is unique and since ϵ is arbitrarily small, the function is *continuous*. A similar proof holds if $f(x, y)$ decreases steadily with y. In particular, it is *sufficient* for the existence of y that f_y should exist and be *not zero* at (a, b).

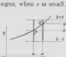


FIG. 5

Again, if $f(x, y)$ is differentiable, since $f(a + \delta x, b + \delta y) = 0$, we have $f_a \delta x + f_b \delta y + o(\delta\rho) = 0$ (where $\delta x = \delta\rho \cos\theta$, $\delta y = \delta\rho \sin\theta$) thus verifying that $\delta y \to 0$ when $\delta x \to 0$ and showing that $\frac{dy}{dx}$ which is equal to $\lim\limits_{\delta x \to 0}\left(-\frac{f_a}{f_b} - \frac{1}{f_b}\frac{o(\delta\rho)}{\delta x}\right)$ has the value $-\frac{f_a}{f_b}$ (since $f_b \neq 0$).

The second and higher derivatives of y may also be calculated by direct differentiation of $f(x, y) = 0$, their existence being dependent on that of the higher partial derivatives of $f(x, y)$.

For example, $f_x + f_y\frac{dy}{dx} = 0$; $f_{xx} + 2f_{xy}\frac{dy}{dx} + f_{yy}\left(\frac{dy}{dx}\right)^2 + f_y\frac{d^2y}{dx^2} = 0$ give $(f_y)^3\frac{d^2y}{dx^2} = -(f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2)$.

3.1. Algebraic Functions. The function y, if it exists, that satisfies the equation

$$f(x, y) = P_0(x)y^m + P_1(x)y^{m-1} + \ldots + P_r(x)y^{m-r} + \ldots + P_m(x) = 0$$

where $P_0, P_1, \ldots, P_m$ are polynomials, is called an *Implicit Algebraic Function*. Since for a given x, y may have more than one value, further specification is necessary to make the branch definite. By the implicit function theorem, however, if $f(a, b) = 0$, and if $f_b \neq 0$, there is one branch that tends to the value b when x tends to a.

Note. It is, of course, possible for f_y to be zero and yet a unique function to exist (at least when *real* variables are considered). Thus there is a unique function at $(0, 0)$ in each of the two cases $y^3 - x^5 = 0$, $y^3 - x^3 - x^4$.

3.11. Explicit Algebraic Functions. A function y that is expressed in terms of a *finite* number of fundamental operations together with the operation of *root-extraction* (i.e. that indicated by the *rational* index), is called an *explicit algebraic function*. It is easy to show that such a function satisfies an algebraic equation of the type given in § 3.1, but since it is tedious to write out a proof of a general character, it is sufficient for us to illustrate the result by particular examples.

Examples. (i) $y = (x-1)^{\frac{1}{3}} + \{x/(x+1)\}^{\frac{1}{2}}$.
Here $[y - \{x/(x+1)\}^{\frac{1}{2}}]^3 = (x-1)$, from which we find
$$(x+1)\{(x+1)y^3 + 3xy - x^2 + 1\}^2 = x\{3y^2(x+1) + x\}^2.$$
(ii) $y = x + \sqrt{(x^2-1)} + \sqrt{(x^2-x)} - \sqrt{(x^2+x)}$.
Therefore $(y-x)^2 - 2(y-x)\sqrt{(x^2-1)} + (x^2-1) = 2x^2 - 2x\sqrt{(x^2-1)}$ from which we find $y^4 - 4xy^3 + 2y^2 + 4xy + 1 = 0$.

The converse of the above result is not true, in general, as it is not possible to express the roots of a general algebraic equation (of degree higher than 4) in terms of the processes indicated. Sometimes, the form of a particular branch of an algebraic function suggests the correct expressions for the other branches.

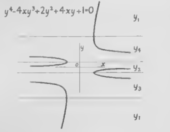


FIG. 6

Thus in Example (ii) above, we have shown that $y^4 - 4xy^3 + 2y^2 + 4xy + 1 = 0$ is satisfied by
$$y_1 = x + \sqrt{(x^2-1)} + \sqrt{(x^2-x)} - \sqrt{(x^2+x)}$$
An inspection of the method by which the algebraic equation was obtained shows that the other three branches are
$$y_2 = x - \sqrt{(x^2-1)} - \sqrt{(x^2-x)} - \sqrt{(x^2+x)},$$
$$y_3 = x - \sqrt{(x^2-1)} + \sqrt{(x^2-x)} - \sqrt{(x^2+x)},$$
$$y_4 = x - \sqrt{(x^2-1)} - \sqrt{(x^2-x)} + \sqrt{(x^2+x)}.$$
These are shown in *Fig. 6*, the curve being easily drawn by our previous methods by writing the relation as $4x = (y^2+1)^2/\{y(y^2-1)\}$, thus expressing x as a rational function of y.

3.2. Algebraic Curves. As in the simpler cases, the functional relationship is made clear if we plot pairs of values (x, y) that satisfy the equation, the corresponding curve being called *algebraic*. In dealing with this geometrical aspect, however, it is better to regard the relationship as a mutual one between x and y, rather than one in which one of the variables is necessarily taken as dependent.

The condition that has arisen, viz $f_y = 0$, if interpreted *algebraically*,

is the condition that $f(x, y) = 0$, regarded as an equation in y, should not have a *multiple* root. Interpreted geometrically, since dy/dx is equal to $-f_x/f_y$, it is the condition that the tangent to the curve should not be parallel to OY. The interpretation of the case $f_x = f_y = f = 0$ arises naturally in the course of our subsequent work.

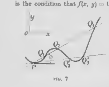


FIG. 7

3.21. Ordinary Points. Let P, of co-ordinates (a, b) be a point on the curve $f(x, y) = 0$. (*Fig. 7.*) The co-ordinates of a point Q on the line through P making an angle θ with OX are given by

$$x = a + r\cos\theta,\ y = b + r\sin\theta, \text{ where } r \text{ is } PQ$$

If this point is on the curve, we have

$$f(a,b) + r(f_a\cos\theta + f_b\sin\theta) + \frac{r^2}{2!}(f_{aa}\cos^2\theta + 2f_{ab}\sin\theta\cos\theta + f_{bb}\sin^2\theta)$$
$$+ \dots + \frac{r^n}{n!}\left(\cos\theta\frac{\partial}{\partial a} + \sin\theta\frac{\partial}{\partial b}\right)^n f = 0$$

by Taylor's Expansion ($f(x, y)$ being of degree n).

This use of θ is not necessary, but it makes the exposition simpler.

Since $f(a, b) = 0$, one root of this equation is of course zero and the remaining points of intersection $Q_1, Q_2, \dots$ are determined by the real roots of the equation

$$(f_a\cos\theta + f_b\sin\theta) + \frac{r}{2!}(f_{aa}\cos^2\theta + 2f_{ab}\sin\theta\cos\theta + f_{bb}\sin^2\theta)$$
$$+ \dots + \frac{r^{n-1}}{n!}\left(\cos\theta\frac{\partial}{\partial a} + \sin\theta\frac{\partial}{\partial b}\right)^n f = 0.$$

One, at least, of these roots tends to zero if $f_a\cos\theta + f_b\sin\theta \to 0$, thus verifying that the gradient of the tangent is $-f_a/f_b$ and also that the equation of the tangent is

$$(x-a)f_a + (y-b)f_b = 0.$$

Thus if $f_a = 0, f_b \neq 0$, the tangent is parallel to OX, and if $f_a \neq 0, f_b = 0$, the tangent is parallel to OY.

When f_a, f_b are *not both* zero, (a, b) is called an *Ordinary Point.*

3.22. Singular Points. If $f_a = 0 = f_b$, *every* line through (a, b) gives at least two zero roots for r, and the point (a, b) is therefore called *Singular.*

3.23. Double Points. If $f_a = 0 = f_b$ and f_{aa}, f_{ab}, f_{bb} are *not all* zero, the point (a, b) is called a *Double Point.* In that case, the equation giving the other values of r is

$$\frac{1}{2!}(f_{aa}\cos^2\theta + 2f_{ab}\cos\theta\sin\theta + f_{bb}\sin^2\theta) + \frac{r}{3!}(\quad) + \dots + \frac{r^{n}}{n!}(\quad) = 0.$$

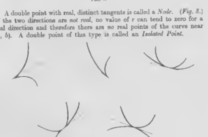
One, at least, of these values tends to zero when θ tends to a value that satisfies the equation

$$f_{aa}\cos^2\theta + 2f_{ab}\cos\theta\sin\theta + f_{bb}\sin^2\theta = 0.$$

If the two directions determined by this quadratic in $\tan\theta$ are *real* and *distinct*, the corresponding lines satisfy the geometrical concept of tangency and are therefore called the *tangents at the double point*.

nodes
FIG. 8

A double point with real, distinct tangents is called a *Node*. (*Fig. 8.*) If the two directions are *not real*, no value of r can tend to zero for a real direction and therefore there are *no* real points of the curve near (a, b). A double point of this type is called an *Isolated Point*.



cusps
FIG. 9

Note. A node is sometimes called a *crunode* and an isolated point an *acnode* or *conjugate point*.

If the two directions are *coincident*, there is a single real tangent and such a point may or may not be isolated. In this case the point is called a *Cusp* (or *Cuspidal Point*) and it will be seen in the examples that the cusp may be *simple* or *double*. (*Fig. 9.*)

A double point is therefore a node, a cusp or an isolated point according as $f_{ab}^2 >, =$ or $< f_{aa}f_{bb}$.

3.24. Multiple Points of Order m. If all the derivatives of f up to and including those of order $(m-1)$ vanish, but not all those of order m, the point is called a *Multiple Point* of Order m.

Thus, for a triple point, for example, we deduce, as in the simpler cases, that the directions of the tangents are given by

$$f_{aaa}\cos^3\theta + 3f_{aab}\cos^2\theta\sin\theta + 3f_{abb}\cos\theta\sin^2\theta + f_{bbb}\sin^3\theta = 0.$$

3.25. Summary. A simple way of summarizing the above results is obtained by taking the *origin* to be the point under consideration. Let the equation therefore be

$$a_{10}x + a_{01}y + \frac{1}{2!}(a_{20}x^2 + 2a_{11}xy + a_{02}y^2) + \frac{1}{3!}(a_{30}x^3 + \ldots) + \ldots = 0.$$

(i) If a_{10}, a_{01} are not both zero, O is an *ordinary* point, the tangent at which is $a_{10}x + a_{01}y = 0$.

(ii) If $a_{10} = 0 = a_{01}$, but *not all* the numbers a_{20}, a_{11}, a_{02} are zero, then O is a *double point*, the tangents at which, if real, are given by $a_{20}x^2 + 2a_{11}xy + a_{02}y^2 = 0$.

The point is a *node*, a *cusp* or an *isolated point* according as

$$a_{11}{}^2 >, = \text{ or } < a_{20}a_{02}.$$

(iii) If $a_{10} = 0 = a_{01} = a_{20} = a_{11} = a_{02}$, but not all the numbers a_{30}, a_{21}, a_{12}, a_{03} are zero, O is a *triple point*, the tangents at which are given by $a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3 = 0$.

Similarly we may write down the tangents at O, when O is a *multiple point* of any order.

3.26. The Shape at any Ordinary Point. Let the origin be the point under consideration and let the equation of the curve be

$$ax + by + Ax^2 + 2Bxy + Cy^2 + \ldots = 0$$

where a, b are not both zero.

Note. For any other point on the curve, we may suppose the origin transferred there by a change of axes.

If $b \neq 0$, $y = -ax/b + o(x)$ and therefore by substitution

$$y = -ax/b - x^2(Ab^2 - 2Bab + Ca^2)/b^3 + o(x^2)$$

i.e.

$$y = -ax/b + Kx^2 + o(x^2), \text{ where } Kb^3 = -Ab^2 + 2Bab - Ca^2, (K \neq 0).$$

The constant K may vanish, however, and so in general, the curve departs from the tangent like

$$y = -ax/b + Lx^s, \ (s > 2,\ b \neq 0).$$

If $a = 0$, the approximation is $y = Lx^s$. If $b = 0$ (so that $a \neq 0$), the approximation is $x = My^s$, $(s > 2)$.

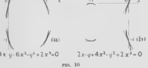
Examples. (i) Find the approximations for $4x - y - 6x^2 - y^2 + 2x^3 = 0$ at $(0, 0)$, $(1, 0)$, $(0, -1)$, $(1, -1)$.

At $(0, 0)$, $y = 4x - 6x^2 - (4x)^2 = 4x - 22x^2$.

At $(1, -1)$, take $x = 1 + X$, $y = -1 + Y$ and find

$$-2X + Y - Y^2 \ldots = 0.$$

i.e. $(y + 1) = 2(x - 1) + 4(x - 1)^2$. Similarly at $(1, 0)$, $y = 2(x - 1) - 4(x - 1)^2$ and at $(0, -1)$, $y - 1 = -4x - 22x^2$. (*Fig. 10 (a)*.)



$4x - y - 6x^2 - y^2 + 2x^3 = 0$ $\qquad$ $2x - y + 4x^2 - y^2 + 2x^3 = 0$

FIG. 10

(ii) Find the approximations to $2x - y + 4x^2 - y^2 + 2x^3 = 0$ at $(0, 0)$, $(0, -1)$, $(-1, 0)$, $(-1, -1)$. The results are: $y = 2x + 2x^2$, $y + 1 = -2x - 2x^2$; $y = 2(x + 1)^2$; $(y + 1) = 2(x + 1)^2$. (*Fig. 10 (b)*.)

3.27. *Shape at a Multiple Point.* (1) At a *node*: the equation has the form:

$$(px + qy)(lx + my) + \phi_3(x, y) + \phi_4(x, y) + \ldots = 0$$

where $\phi_r(x, y)$ is a homogeneous polynomial of degree r. Near the tangent $px + qy = 0$, we have $y = -px/q + o(x)$, if $q \neq 0$, and therefore one branch of the curve departs from its tangent like

$$y = -px/q + Lx^s, \ (s \geq 2).$$

If $q = 0$, the approximation is of the form $x = My^s$, $(s \geq 2)$. In particular, when $q \neq 0$, the approximation is

$$(px + qy)(l - mp/q) + x^2\phi_3(1, -p/q) = 0, \ \{\phi_3(1, -p/q) \neq 0\}.$$

When $px + qy$ is a factor of $\phi_3(x, y)$ but *not* of $\phi_4(x, y)$, then

$$px + qy = O(x^3)$$

for $\phi_3(x, y) = O(x^s)$, $(s > 5)$, i.e. the approximation is

$$(px + qy)(l - mp/q) + x^3\phi_4(1, -p/q) = 0.$$

More generally, the approximation is

$$(px + qy)(l - mp/q) + x^k\phi_k(1, -p/q) = 0$$

where ϕ_k is the first ϕ_r that does not contain the factor $px + qy$. In the same way, the approximation near the other tangent $lx + my = 0$ is obtained.

Examples. (i) Find the approximations at $(0, 0)$ to

$$x^2 - y^2 + x^3 - y^3 + 3x^2y = 0.$$

The tangents at $(0, 0)$ are $x - y = 0$ and $x + y = 0$.

Near $x - y = 0$, $(x - y)2x + 3x^2.x = 0$, i.e. $y = x + \frac{3}{2}x^2$.

Near $x + y = 0$, $(x + y)2x - 2x^3 = 0$, i.e. $y = -x - x^2$. (*Fig. 11 (a)*.)

(ii) Find the approximations at $(0, 0)$ to $3xy + 5x^3 + 4y^4 = 0$. The tangents are $x = 0$, $y = 0$.

Near $x = 0$, $3xy + 4y^4 = 0$, i.e. $3x = -4y^3$.

Near $y = 0$, $3xy + 5x^3 = 0$, i.e. $3y = -5x^2$. (*Fig. 11 (b)*.)

(2) At an *isolated point*; the terms of the second degree do not break up into real factors and there are no real points in the neighbourhood.

Examples (i) $4x^2 - 4xy + 2y^2 + x^3 - y^3 = 0$. (ii) $x^2 - 4xy + 5y^2 + x^3 = 0$. (iii) $4y^2 - x^4 - x^2$.

(3) At a *cusp*. The coefficients of x^2, y^2 cannot *both* be zero; let us therefore assume for definiteness that there is a term in y^2 and that the equation is

$$(y - mx)^2 + \phi_3 + \phi_4 + \ldots + \phi_n = 0.$$

Since $y - mx$ may be a factor of $\phi_3, \phi_4, \ldots, \phi_k$ $(k < n)$, we deduce that the effective approximation takes the form

$$(y - mx - \lambda_1 x^2 - \lambda_2 x^3 - \ldots - \lambda_{r-1} x^r)^2 = Kx^t, \ (t > 2r).$$

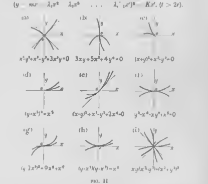


FIG. 11

If $K < 0$, t even, the origin is isolated with a *real* tangent $y = mx$; otherwise there are real points near O given approximately by

$$y = mx + \lambda_1 x^2 + \ldots + \lambda_{r-1} x^r \pm (Kx^t)^{\frac{1}{2}}.$$

Examples. (i) $(x + y)^2 + x^3 - y^3 = 0$.

Here $y = O(x)$, and the approximation is $(x + y)^2 + 2x^3 = 0$
i.e. $y = -x \pm \sqrt{(-2x^3)}$. *Keratoid Cusp.* (*Fig. 11* (c).)

(ii) $(y - x^2)^2 = x^5$. *Rhamphoid Cusp.* (*Fig. 11* (d).)

(iii) $(x - y)^2 + (x^3 - y^3) + 2x^4 = 0$.

Here $y = x + kx^2$ where $k^2 - 3k + 2 = 0$, or $y = x + x^2$, $y = x + 2x^2$. *Double Cusp.* (*Fig. 11* (e).)

(iv) $y^2 - x^4 - xy^2 + x^5 = 0$. Here $y = \pm x^2$. *Double Cusp.* (*Fig. 11* (*f*).)

(v) $(y - 2x^2)^2 = 9x^6 + x^7$, giving $y = 2x^2 \pm 3x^3$. *Double Cusp.* (*Fig. 11* (*g*).)
(vi) $(y - x^2)(y - x^3) = x^5$, giving $y = x^2$, $y = x^3$. *Double Cusp.* (*Fig. 11* (*h*).)
(vii) $(y + x^2)^2 + x^6 = x^7$. The origin is *isolated.*
(viii) $(y + x)^2 - (y + x)^2(2x - y) + x^4 = 0$. Here $y + x = kx^2 + o(x^2)$ where $k^2 + 1 = 0$, i.e. the origin is *isolated.*

(4) At a *multiple point*: similar methods may be applied but the details for the general curve are tedious.

Example. $xy(x - y)(x + y) = (x^2 + y^2)^3$.
O is a quadruple point with 4 real tangents, $x = 0$, $y = 0$, $x = \pm y$.
Near $x = 0$, $xy(-y)(-y) = y^6$, i.e. $x = -y^3$. Near $y = 0$, $y = x^3$. Near $x = y$, $x^2(x - y)2x = 8x^6$, i.e. $y = x - 4x^3$; similarly near $x = -y$, we have $y = -x - 4x^3$. (*Fig. 11* (*i*).)

3.3. Graph of the Algebraic Function. A representation adequate for picturing the functional relationship is obtained if the shape of the corresponding curve is determined—

(*a*) at critical points in the finite part of the plane;

(*b*) at places where one, at least, of the variables is large.

Under (*a*), we should determine (i) where the tangent is parallel to an axis, and (ii) the singular points and the approximations there.

In general, the approximation is usually of the type

$$(y - b)^r = A(x - a)^s$$

where r, s may be positive or negative integers; but it should be remembered that this may be the common approximation of two or more branches of the curve and that further approximation may be necessary to separate these branches. In the two paragraphs that follow, examples are given that admit of an obvious treatment.

3.31. The Relation $y^m = P(x)/Q(x)$, where P, Q are polynomials with no common factor.

Find the shapes at points (*real*) where y is small, where y is large and where x is large.

Obtain the points, other than those that have already occurred, where a tangent is parallel to OX; these are given by

$$P'(x)Q(x) = P(x)Q'(x).$$

Their number and approximate position are often deducible from a knowledge of the roots of $P(x) = 0$ and $Q(x) = 0$; and if only a rough graph is required, it is unnecessary to find them; but their determination not only provides a verification of previous work, it also gives an idea of relative dimensions.

Examples. Find the approximations to the following curves when x is large:

(i) $y^3 = \dfrac{(x + 1)}{(x - 1)^5}$; $y^3 = \dfrac{1}{x^4}\left(1 + \dfrac{1}{x}\right)\left(1 - \dfrac{1}{x}\right)^{-5} = \dfrac{1}{x^4} + o\left(\dfrac{1}{x^4}\right)$, $(\infty, 0)$.

(ii) $y^5 = \dfrac{x^5 + 1}{(8x - 3)(4x + 1)^4}$; $y^5 = \dfrac{1}{32}\left(1 + \dfrac{1}{x^5}\right)\left(1 - \dfrac{3}{8x}\right)^{-1}\left(1 + \dfrac{1}{4x}\right)^{-1}$,

i.e. $y = \dfrac{1}{2} + \dfrac{1}{80x}$, $(\infty, \tfrac{1}{2})$.

(iii) $y^4 = \dfrac{x^3 - 1}{(x + 1)^2(x + 2)}$; $y^4 = 1 - \dfrac{4}{x} + o\left(\dfrac{1}{x}\right)$ or $y = \pm\left(1 - \dfrac{1}{x}\right)$, $(\infty, \pm 1)$.

(iv) $y^3 = \dfrac{x^4}{x-1}$; $y = x + \dfrac{1}{3} + \dfrac{2}{9x} + o\left(\dfrac{1}{x}\right)$, (∞, ∞). Here $y = x + \dfrac{1}{3}$ is an asymptote and the curve is above (below) this asymptote when x is large and positive (negative).

(v) $y^3 = \dfrac{(x+1)^5}{x(x-1)^2}$; $y^3 = x^2 + o(x^2)$, (∞, ∞).

(vi) $y^2 = \dfrac{(x-1)^5}{(x+1)^3}$; $\pm y = x - 4 + \dfrac{15}{2x} + o\left(\dfrac{1}{x}\right)$, (∞, ∞).

(vii) $y^4 = \dfrac{(x-1)^3(2-x)^3}{(x-3)^2} = -x^4 + o(x^4)$. *No real points.*

Draw the complete curve for the following examples:

(i) $y^2 = \dfrac{x^3}{(x-1)^2}$; $(0, 0)$, $y^2 = x^3$; $(-1, \infty)$, $y^2 = \dfrac{1}{(x+1)^2}$; $(\infty, 0)$, $y^2 = \dfrac{1}{x}$. Parallel to OX when $x = -2$, $y = \pm\sqrt{\dfrac{4}{27}} = \pm 0{\cdot}385$; x cannot be -1; symmetry about OX. (*Fig.* 12 (a).)

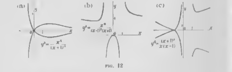


FIG. 12

(ii) $y^2 = \dfrac{x^4}{(x-1)^2(x+1)}$: find the approximations at $(0, 0)$, $(1, \infty)$, $(-1, \infty)$, $(\infty, 1)$. In particular at $(\infty, 1)$, $y = 1 + \dfrac{2}{3x}$. The tangent is parallel to OX at $(-2, 0{\cdot}84)$. (*Fig.* 12 (b).)

(iii) $y^4 = \dfrac{(x+1)^6}{x(x-1)}$; find the approximations at $(-1, 0)$, $(0, \infty)$, $(1, \infty)$, (∞, ∞). In particular at (∞, ∞), $\pm y = x + \dfrac{7}{4} + \dfrac{29}{32x}$; the tangent is parallel to OX at $(1{\cdot}59, \pm 4{\cdot}24)$. (*Fig.* 12 (c).)

3.32. The Relation $A(y)/B(y) = C(x)/E(x)$ where A, B, C, E are polynomials.

Let $F(y) = A(y)/B(y)$, $G(x) = C(x)/E(x)$. Then it will usually be sufficient to consider—

(i) places where $F(y)$, $G(x)$ are simultaneously small or simultaneously large; (ii) places where x or y is large or both x, y are large, if they have not occurred in (i); (iii) singular points that have not occurred in (i), (ii) and also the points where the tangent is parallel to an axis.

Thus the curve is parallel to OY when $F'(y) = 0$, $G'(x) \neq 0$, it is parallel to OX when $F'(y) \neq 0$, $G'(x) = 0$; but if (a, b) is a point where $F'(b) = 0 = G'(a)$, $F(b) = G(a)$, that point is singular and by writing

the given relation in the form $F(y) - F(b) = G(x) - G(a)$, the approximations there are easily determined.

Examples. (i) Show that the curve $4xy(x-1)(1-y) = 17y^2 - 17y + 4$ has a singular point and find the approximation there.

Write the equation $4x(x-1) = -17 - 4/\{y(y-1)\}$, $(f(x, y) = 0)$. Then $f_x = 0 = f_y$ when $x = \frac{1}{2}$, $y = \frac{1}{2}$ and this point is on the curve. The relation is equivalent to $(2x-1)^2 = \dfrac{4(2y-1)^2}{y(1-y)}$, showing that the singular point is a node with tangents $2x - 1 = \pm 4(2y-1)$.

(ii) It may be found similarly that the curve

$$x(2y-1)(y-2) = y(x-2)(2x^3 - 4x^2 + 4x - 1)$$

has a double cusp at $(1, 1)$. The equation may be written $x(y-1)^2 = y(x-1)^4$.

(iii) Find approximations for $4x^2(y-1)(y+1)(x+1) = 3(x-1)^2(y+2)$ at $(-1, \infty)$, $(0, \infty)$, (∞, c).

At $(-1, \infty)$, $4y = 24/(x+1)$; at $(0, \infty)$, $4y = -3/x^2$.

When $x \to \infty$, $4(y-1)(y+1) \to 3(y+2)$, i.e. $y \to 2$ or $-5/4$.

Writing the equation as $\dfrac{(y-2)(4y+5)}{(y-2)} = \dfrac{3(4x^2-3x+1)}{x^2(x+1)}$ we find

$$(y-2)\frac{13}{4} = -\frac{12}{x}, \quad (4y+5)\frac{13}{3} = \frac{12}{x}$$

giving the closer approximations $y = 2 - \dfrac{48}{13x}$, $y = -\dfrac{5}{4} + \dfrac{9}{13x}$.

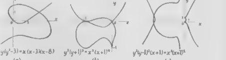
Draw the complete curve for the following relations.

(i) $18y(y^2 - 3) = x(x-3)(x-8)$.

$(0, 0)$, $-54y = 24x$; $(0, \sqrt{3})$, $108(y - \sqrt{3}) = 24x$. $(0, -\sqrt{3})$, $108(y + \sqrt{3}) = 24x$.

Similar results at $(3, 0)$, $(3, \pm\sqrt{3})$, $(8, 0)$, $(8, \pm\sqrt{3})$.

There is a node at $(6, 1)$ with tangents $54(y-1)^2 = 7(x-6)^2$. The tangent is parallel to OX at $(8, -2)$, $(1{\cdot}33, -1{\cdot}5)$, $(1{\cdot}33, 0{\cdot}3)$, $(1{\cdot}33, 1{\cdot}5)$. The tangent is parallel to OY at $(-1, 1)$, $(8{\cdot}7, -1)$. And at (∞, ∞) the first approximation is $y = 0{\cdot}38x$, the next is $y = 0{\cdot}38x - 1{\cdot}4$. (*Fig. 13* (*a*).)



$18y(y^2-3) = x(x-3)(x-8)$ (*a*) $y^2(y+1)^3 = x^2(x+1)^4$ (*b*) $y^2(y-1)^2(x+1) = x^4(x+7)^2$ (*c*)

FIG. 13

(ii) $y^2(y+1)^3 = x^2(x+1)^4$. Obtain the approximations at $(0, 0)$, $(0, -1)$, $(-1, 0)$, $(-1, -1)$, (∞, ∞). (*Fig. 13* (*b*).)

(iii) $y^2(y-1)^2(x+1) = x^4(x+7)^2$. Find the approximations at $(0, 0)$, $(0, 1)$, $(-7, 0)$, $(-7, 1)$, $(-1, \infty)$, (∞, ∞). In particular at (∞, ∞), $y = x + \dfrac{15}{4}$ and $y = -x - \dfrac{11}{4}$. There is symmetry about the line $y = \frac{1}{2}$. (*Fig. 13* (*c*).)

(iv) $50y^2(2y + 1)^4(x + 1)^4 = x^2(8y - 3)$. (*Fig.* 14.)

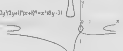


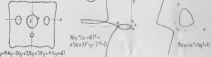
FIG. 14

(v) $(y - 4)(y - 1)(y + 2)(y + 3)(y + 4)(y + 6)$
$$+ (x^2 - 1)(x^2 - 4)(x^2 - 9)(x^2 - 16) = 576$$

No large values of x, y. Symmetry about OY. (*Fig.* 15 (*a*).)

(vi) $16y^4(x + 6)^2 + x^2(x + 3)^2(y - 2)^2 = 0$. (*Fig.* 15 (*b*).)

(vii) $9xy = (x^2 + 1)(y^2 + 1)$. This example illustrates the importance of finding where the tangents are parallel to an axis, since there is an oval within the first quadrant. Actually $f_x = 0$ at (0·79, 4·54 or 0·22) and $f_y = 0$ at (2 or 0·22 or 2·22, 1) and (−0·22, −1). (*Fig.* 15 (*c*).)



(*a*) (*b*) (*c*)

FIG. 15

3.33. *Newton's Polygon* (or *Analytical Polygon*). When the variables cannot be separated as in the two preceding paragraphs, and when it is not obvious what terms give the correct approximations in significant neighbourhoods, these approximations may be obtained by using a subsidiary diagram known as *Newton's Polygon*.

Let the origin, which may be a singular point, be on the curve and let the equation of the curve be

$$\Sigma A_{rs}x^r y^s = 0 \ (A_{00} = 0)$$

where there is at least one non-zero term $A_{m0}x^m$ and at least one non zero term $A_{0n}y^n$. In a subsidiary diagram (*Fig.* 16), take rectangular axes $\Omega\xi$, $\Omega\eta$ and plot all the pairs of values (r, s), *that occur* in the given equation. Draw through these points a polygon that is not re-entrant, such that every vertex is one of the plotted points and such that every plotted

point is either within the polygon or on one of its sides. The sides may be classified into *eight* types as follows (*Fig. 17*):

Type: Fig. 17	*Equation*		*Inequality for points not on side*
(i) AB, BC	$p\xi + q\eta = 1$,	$p > 0,\ q > 0$	$p\xi + q\eta > 1$
(ii) GH, HL	$p\xi + q\eta = 1$,	$p > 0,\ q > 0$	$p\xi + q\eta < 1$
(iii) DE, EF	$p\xi + q\eta = 1$,	$p < 0,\ q > 0$	$p\xi + q\eta < 1$
(iv) MN, NP	$p\xi + q\eta = 1$,	$p > 0,\ q < 0$	$p\xi + q\eta < 1$
(v) CD	$\xi = 0$		$\xi > 0$
(vi) PA	$\eta = 0$		$\eta > 0$
(vii) FG	$q\eta = 1$,	$q > 0$	$q\eta < 1$
(viii) LM	$p\xi = 1$,	$p > 0$	$p\xi < 1$

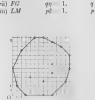


FIG. 16

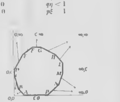


FIG. 17

The characteristic property of the polygon is as follows:

The terms of the equation that correspond to the points on a particular side give the first approximation to the curve in a particular neighbourhood; and the particular neighbourhoods to be associated with the different sides are given by the scheme

(i) $(0, 0)$, (ii) (∞, ∞), (iii) $(0, \infty)$, (iv) $(\infty, 0)$, (v) $(0, c)$, (vi) $(c, 0)$, (vii) (c, ∞), (viii) (∞, c).

Note. The different types may not, of course, all occur. There may be more than two points on a side, and to any side, as we shall see, there may correspond more than one branch. The constant c, for example, may have many values.

Let K_1, K_2 be two points of a side $p\xi + q\eta = 1$ and let these correspond to $Ax^{r_1}y^{s_1}$, $Bx^{r_2}y^{s_2}$ respectively so that

$$pr_1 + qs_1 = pr_2 + qs_2 = 1$$

Let $x = O(\lambda^p)$, $y = O(\lambda^q)$ (where λ is chosen, later, to be small or large). Then $Ax^{r_1}y^{s_1} = O(\lambda)$ and $Bx^{r_2}y^{s_2} = O(\lambda)$ so that these terms are of the *same* order in the neighbourhood $\{O(\lambda^p), O(\lambda^q)\}$. Also, if (r_3, s_3) is a point *not* on the line, the corresponding term $Cx^{r_3}y^{s_3} = O(\lambda^{pr_3+qs_3})$.

Thus for type (i), $pr_2 - qs_2 > 1$ and therefore for λ *small*,

$$O(\lambda^{pr_2+qs_2}) = o(\lambda)$$

and so (i) gives an approximation at (0, 0) since $p, q > 0$. Similarly for type (ii), by taking λ *large* we find that the corresponding terms give the approximation for (∞, ∞). In the same way the other results may be established. In particular, (v), (vi) give the points where the curve crosses the axes of y and x respectively; and (vii), (viii) give respectively the asymptotes parallel to the axes of y and x.

Also, the polygon gives the *next* approximations to the curve in the appropriate neighbourhoods, since it follows from the above proofs that if the line containing a particular side is moved parallel to itself until it meets another plotted point (or a number of points simultaneously), then the term corresponding to that point (points) provides the next approximation. This closer approximation may be required when the first is linear or involves repeated factors. It may also be used to obtain the tangents at the points where the curve crosses the axes.

In the following examples, the complete curve is in each case indicated by a dotted line in the corresponding diagram, but the reader at the present stage should regard these dotted lines merely as tentative.

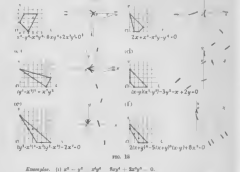


FIG. 18

Examples. (i) $x^2 - y^2 - x^4y^4 - 8xy^4 + 2x^3y^3 = 0$.

$(0, 0)$, $x^2 - y^2 = 0$, with closer approximation $x^2 - y^2 = 8xy^4$, i.e. $(x - y)2x = 8x^5$ and $(x + y)2x = 8x^5$ giving $y = x - 4x^4$ and $y = -x + 4x^4$;

(∞, ∞), $(-x^4y^4 - 8xy^4) + 2x^3y^3 = 0$ gives $3y(x + 2) = 2$;

$(0, \infty)$, $8xy^2 = -1$; $(\infty, 0)$, $x^3y^4 = 1$. (*Fig. 18* (a).)

(ii) $2x - y^4 - x^3y + x^2 = 0$

Find the approximations at $(0, 0)$, $(c, 0)$, $(\infty, 0)$, (∞, ∞).

In particular, $(c, 0)$, $x(2 + x) = yx^3$ gives $4y = x + 2$ $(c = -2)$; (∞, ∞), $(x^3y - y^4) + x^2 = 0$, i.e. $y(x + y)(x^2 - xy + y^2) = x^2$ giving the real asymptote $x + y = 0$, with closer approximation $x + y = -\frac{1}{3x}$. (*Fig. 18* (*b*).)

(iii) $(y^2 - x^3)^2 = x^7y^2$. Find approximations at $(0, 0)$, $(\infty, 0)$, (∞, ∞). In particular at $(0, 0)$ $(y^2 - x^3)^2 = x^7y^2$ gives $y^2 = x^3 \pm x^5$. (*Fig. 18* (*c*).)

(iv) $(x - y)^2(x^2 + xy + y^2) - 3y^2 - x + 2y = 0$.

Find approximations at $(0, 0)$, $(c_1, 0)$, $(0, c_2)$, (∞, ∞).

In this case $c_1 = 1$; c_2 may be 1 or -2. (*Fig. 18* (*d*).)

(v) $(y^2 - x^3)^2 - x^4(y^2 - x^3) - 2x^8 = 0$. If the terms that give the second approximation vanish when the first approximation is substituted, a temporary change of variable may be used to obtain the correct second approximation. Thus at $(0, 0)$, $y^2 = x^3$ in this example and the terms that should give the second approximation vanish. Take therefore $Y = y^2 - x^3$ and the new equation is $Y^2 - x^4Y - 2x^8 = 0$. From this we deduce that $Y = -x^4$ or $2x^4$, i.e. $y^2 = x^3 - x^4$, or $y^2 = x^3 + 2x^4$. (*Fig. 18* (*e*).)

(vi) $2(x + y)^4 - 5(x + y)^2(x - y) + 8x^3 = 0$.

At (∞, ∞), the first approximation gives $x + y = 0$ and this makes the terms giving the next approximation vanish. Let $x + y = Y$ and find

$$2Y^4 - 5Y^2(2x - Y) + 8x^3 = 0$$

By drawing the Newton polygon for the equation in (x, Y) we find that at (∞, ∞), $(Y^2 - x)(Y^2 - 4x) = 0$, i.e. $x + y = \pm\sqrt{x}$ and $x + y = \pm 2\sqrt{x}$. (*Fig. 18* (*f*).)

3.34. *Summary of General Method for $f(x, y) = 0$.* (i) Draw Newton's polygon. If O is *not* on the curve, the approximations for one or more of the neighbourhoods $(c, 0)$, $(0, c)$, $(\infty, 0)$, $(0, \infty)$, (c, ∞), (∞, c), (∞, ∞) are obtained. If O is on the curve, we obtain also the approximation at $(0, 0)$.

(ii) Solve the equations $f = f_x = 0$; and obtain the points (a, b) where the tangent is parallel to OX (if $f_b \neq 0$). Solve the equations $f = f_y = 0$; and obtain the points (a, b) where the tangent is parallel to OY (if $f_a \neq 0$). If $f = f_a = f_b = 0$ the point is *singular*.

(iii) If (a, b) is singular, take $x = a + X$, $y = b + Y$ and draw that part of the polygon for $f(a + X, b + Y) = 0$ that gives the approximation at $X = 0$, $Y = 0$.

Examples. (i) $x^3 = y^4 - xy^2$.

$(0, 0)$, $y^2 = -x^2$. $(0, \infty)$, $xy = 1$, (∞, ∞), $y^3 = x^3$.

$f_a = 0$ when $32x^7 = 81$, i.e. at $(1{\cdot}14, 1{\cdot}31)$; $f_b = 0$ when $3125x^7 = -256$, i.e. at $(-0{\cdot}70, -1{\cdot}14)$. (*Fig. 19* (*a*).)

(ii) $x^4y^4 - x^3 - y^4 + xy^2 = 0$.

Find approximations at $(0, 0)$, (*two*). $(\infty, \pm 1)$; $(\pm 1, \infty)$.

$f_x = 0$ when $9x^4 - 16x^3 + 3 = 0$, giving $(1{\cdot}25, \pm 0{\cdot}97)$, $(0{\cdot}46, \pm 0{\cdot}50)$.

$f_y = 0$ when $4x^4 - 4x^3 + 1 = 0$, giving $(0{\cdot}5, \pm 0{\cdot}52)$, $(0{\cdot}92, \pm 1{\cdot}24)$. Also when $x = 1$, $y = \pm 1$, gradient $\pm\frac{1}{2}$. Symmetry about OX. (*Fig. 19* (*b*).)

(iii) $x^3 - y^3 + x^2y^2 - 4x^2y^3 = 0$. (*Fig. 19* (*c*).)

(iv) $y^2 - 4x^2y + x^6 - x^5 + y^3 = 0$. (*Fig. 19* (*d*).)

(v) $(y - x^2)^2 = x^4y$. Find approximations at $(0, 0)$, $(\infty, 1)$, (∞, ∞).

If $x = t - \frac{1}{t}$, $y = (t^2 - 1)^2$. $\frac{dx}{dt}$ is never zero, and $\frac{dy}{dt} = 0$ only at $(0, 0)$. Curve crosses $y = 1$ (the asymptote) at $x = \pm 1/\sqrt{2}$. Symmetry about OY and y cannot be < 0. (*Fig. 20* (*a*).)

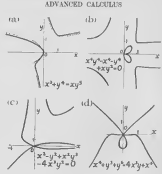

(a)
x³+y⁴=xy⁵
(b)
x⁴y²−x⁴−y⁴+xy²=0
(c)
x²−y²+x²y³−4x³y²=0
(d)
x⁴+y³+y⁶=4x²y+x⁶

FIG. 19

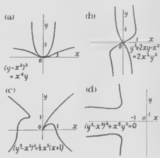

(a)
(y−x²)² = x⁴y
(b)
y²+2xy−x² =2x²y²
(c)
(y³−x²)²=½x⁵(x+1)
(d)
(y²−x³)²+x⁵y²=0

FIG. 20

(vi) $y^3 - x^3 - 2x^2y^2 + 2xy = 0$

Approximation for (∞, c) is not real, and the tangent is never parallel to OX. (*Fig.* 20 (*b*).)

(vii) $(y^2 - x^2)^2 = \frac{1}{4}x^4 + \frac{1}{4}x^5$. (*Fig.* 20 (*c*).)

(viii) $(y^2 - x^2)^2 + x^3y^2 = 0$; in this example the origin is isolated and x must be ≥ -2. (*Fig.* 20 (*d*).)

3.4. Unicursal Curves. If the co-ordinates of a point (x, y) on a curve can be expressed rationally in terms of a parameter t, the curve is said to be *unicursal*. The complete curve is obtained by allowing t to vary from $-\infty$ to $+\infty$. Such a curve is algebraic since the elimination of t between the equations giving x and y obviously gives an algebraic equation in (x, y). Conversely, however, the general algebraic curve is not unicursal, the conditions that it should be so being associated, as we shall find, with the number of singular points possessed by the curve. It should be remembered that the determination of analytical conditions of the type indicated may lead to results that have no real geometrical significance, and that geometrical ideas are used merely to amplify the exposition. The analysis of algebraic functions requires the use of the complex variable for an adequate treatment, and our illustrations here are therefore confined to those for which the results obtained are real.

3.41. The Number of Points required to specify a Curve of Degree n. The number of coefficients in the equation

$$a_{00} + (a_{10}x + a_{01}y) + (a_{20}x^2 + \dots) + \dots + (a_{n0}x^n + \dots + a_{0n}y^n) = 0$$

is $1 + 2 + 3 + \dots + (n + 1) = \frac{1}{2}(n + 1)(n + 2)$, and therefore the number of arbitrary constants is $\frac{1}{2}(n + 1)(n + 2) - 1 = \frac{1}{2}n(n + 3)$.

Thus a conic can, in general, be drawn through 5 points, a cubic through 9 and a sextic through 27.

Note. This result is only true in *general*, since the equations determining the constants may be indeterminate.

Examples. (i) Find the equation of the conic through $(0, \pm a)$, $(\pm a, 0)$ $(2a, 2a)$.

When $x = 0$, $y = \pm a$, equation is $Ax^2 + x(By + C) + D(y^2 - a^2) = 0$

When $y = 0$, $x = \pm a$, equation is $Ax^2 + Bxy + Ay^2 = Aa^2$.

When $x = 2a$, $y = 2a$; therefore $A(7a^2) + 4Ba^2 = 0$, i.e. the required equation is $4x^2 - 7xy + 4y^2 = 4a^2$.

(ii) Find the conic through $(0, 0)$, $(0, -6)$, $(2, 0)$, $(1, -3)$, $(3, 3)$. This is *indeterminate*, since 4 of the points lie on the line $y = 3x - 6$; it may therefore be taken as $(y - mx)(y - 3x + 6) = 0$ where m is arbitrary.

(iii) Find the cubic through $(0, 2)$ for which $y = 0$, $y = 1$ are inflexional tangents at $x = 0$ and for which $x + y = 0$ is an asymptote. This is equivalent to 9 points; for $y = 0$, $y = 1$ must lead to $x^3 = 0$, and the substitution of $x + y = 0$ must lead to a simple equation in x. The result is $3xy(y - 1) + y(y - 1)(y - 2) - 2x^3 = 0$.

3.42. The Number of Intersections of two Algebraic Curves. Let the curves of degrees m, n respectively be given by

$$(1)\quad y^mP_0(x) + y^{m-1}P_1(x) + \dots + P_m(x) = 0,$$
$$(2)\quad y^nQ_0(x) + y^{n-1}Q_1(x) + \dots + Q_n(x) = 0$$

where P_r, Q_s are polynomials of degrees not higher than r, s respectively. Form the $(m + n)$ equations

$$y^{m+r}P_0(x) + \dots + y^rP_m(x) = 0, \quad (r = 0, 1, \dots, n - 1),$$
$$y^{n+s}Q_0(x) + \dots + y^sQ_n(x) = 0, \quad (s = 0, 1, \dots, m - 1).$$

These consist of $(m+n)$ homogeneous linear equations in the numbers $1, y, y^2, \ldots, y^{m+n-1}$. The eliminant is $\Delta = 0$, where Δ is the determinant of the coefficients. This determinant may easily be shown to be a polynomial in x of degree not higher than mn. (*Ref.* *Hilton, Plane Algebraic Curves, I, 7.*)

Notes. The actual degree of the determinant will often be less than mn, and we then say that the curves intersect at infinity. The introduction of intersections at infinity may be avoided by using homogeneous co-ordinates, this being equivalent to a projective transformation.

The equations from which $\Delta = 0$ is obtained also determine y as a rational function of x, and so to each value of x obtained from $\Delta = 0$ we can find a corresponding value of y. Thus, in general, two curves of degrees m, n intersect in not more than mn points. Some of these intersections may be multiple (corresponding to multiple roots), and some may be imaginary.

Example. $y^3 - y^2(3x-3) + y(3x+2) - x^2(2x+1) = 0$, $y^2 - y - x^2 = 0$. Multiply the second by y and subtract it from the first, thus giving

$$y^2(3x-2) - y(x^2+3x+2) + x^2(2x+1) = 0.$$

Solving for y^2, y we find $x^4(5x+3)^2 = x^2(x^2+5x-3)$ and $y = 5x+3$ $(x \neq 0)$. Thus the curves meet at $(0, 0)$, $(0, 1)$, $(-\frac{3}{5}, \frac{9}{5})$, $(-\frac{1}{5}, -\frac{1}{5})$ and there is double contact (common tangent) at the first two.

3.43. ***The Maximum Number of Double Points for a Curve of Degree n.*** Suppose that there are N double points and no other singular points. Let $\frac{1}{2}m(m+3) < N$; then a curve of degree m can be drawn through $\frac{1}{2}m(m+3)$ of the double points. The number of intersections with the given curve given by these double points is $m(m+3)$ which must be $< mn$, i.e. $m < n-3$. Thus $N < \frac{1}{2}(n+2)(n+1)$ but *may* be greater than $\frac{1}{2}n(n-3)$. Take therefore a curve of degree $n-2$ through the N double points and through $\{\frac{1}{2}(n-2)(n+1) - N\}$ other points of the curve. The number of intersections is $2N + \frac{1}{2}(n-2)(n+1) - N$ and this must be $< n(n-2)$, i.e. $N < \frac{1}{2}(n-1)(n-2)$.

We shall show by examples that a curve can possess this number $\frac{1}{2}(n-1)(n-2)$. The difference between the number and the actual number is called the *Deficiency* of the curve.

Notes. (i) The same result may be proved by taking a curve of degree $n-1$ through the N double points and through $\frac{1}{2}(n-1)(n+2) - N$ other points.

(ii) It does not appear to be a simple matter to obtain a correspondingly simple formula for the maximum number of singular points of $f(x, y) = 0$ where there are n_2 double points, n_3 triple points, etc. It is sometimes true that a multiple point of order q may be regarded as $\frac{1}{2}q(q-1)$ double points. Thus if a curve of degree n possess a multiple point A of order $n-1$, it can possess no other singular point B: for AB would then meet the curve in $(n+1)$ points at least. In this case one multiple point of order $(n-1)$ is equivalent to $\frac{1}{2}(n-1)(n-2)$ double points. That the equivalence is not always true is most simply shown by the fact that a quintic can possess 6 double points, but it cannot possess 2 triple points. However, our present object is to establish a relationship between the number of singular points and the problem of determining when a curve is unicursal. It will be sufficient for us to use the method adapted for double points to other particular cases that

arise. The problem is more adequately dealt with in the theory of algebraic functions, where conditions may be stated without reference to geometrical ideas.

Example. A sextic has one quadruple point A. If all the other singularities are double points, what is the maximum number m? Here $m < 8$, for a conic could be drawn through A and 4 double points (giving 12 intersections), but a cubic could *not* be taken through A and 8 double points (since there would then be 20 intersections). Take a cubic through A, m double points and $8 - m$ other points. Then $4 + 2m + 8 - m < 18$, i.e. $m < 6$. Thus a quadruple point in this example is equivalent to 4 double points.

3.44. A Curve of Deficiency Zero is Unicursal. Take a curve of degree $(n - 2)$ through the $\frac{1}{2}(n - 1)(n - 2)$ double points and through $(n - 3)$ other points of a given curve $f(x, y) = 0$ of degree n.

The number of *arbitrary* coefficients is

$$\tfrac{1}{2}(n - 2)(n + 1) - \tfrac{1}{2}(n - 1)(n - 2) - (n - 3)$$

i.e. 1. The new curve is therefore of the form

$$\phi(x, y) + t\psi(x, y) = 0 \ (t \text{ being the arbitrary coefficient}).$$

The number of *known* intersections is $(n - 1)(n - 2) + (n - 3)$ and therefore the number unknown is

$$n(n - 1) - (n - 1)(n - 2) - (n - 3) = 1.$$

The co-ordinates of this point of intersection must therefore be expressible as rational functions of t.

Note. The same result is obtained by taking a curve of degree $(n - 1)$ through the $\frac{1}{2}(n - 1)(n - 2)$ double points and through $(2n - 3)$ other points.

Examples. (i) A sextic with one quadruple point A and six double points B_r $(r = 1$ to $6)$ is unicursal. Take a cubic through these points and one other point of the curve. One coefficient is undetermined. The number of unknown intersections is $18 - (4 + 12 + 1) = 1$.

(ii) A curve of degree n with one multiple point A of order $(n - 1)$ is unicursal. Let A be the point (a, b) and take the variable line $y - b = t(x - a)$. The substitution of y in the equation of the curve leads to an equation in x having a root a of multiplicity $(n - 1)$. The remaining root is a rational function of t.

3.45. The Conic. A conic (assumed irreducible) cannot have a double point, and is therefore unicursal. If (a, b) is any point on it, the substitution $(y - b) = t(x - a)$ (*see* § 3.44, note) leads to a quadratic in x, one root of which is a and the other a rational function of t.

Note. It is assumed here that the reader is already familiar with the simple equations of the conic to which the general equation of the second degree may be reduced. The examples given below are to be regarded as illustrations of the general theory. It is useful to note, however, that the method of Newton's polygon enables us to obtain very quickly the nature, shape, and position of the conic given by a general equation.

Examples. (i) $x^2 + xy + y^2 = x + y$.

From Newton's polygon, $(0, 0)$, $y = -x$; $(1, 0)$, $x - 1 + y^2 = 0$; $(0, 1)$, $y - 1 + x^2 = 0$; (∞, ∞), $x^2 + xy + y^2 = 0$, imaginary. (*Ellipse.*)

$y = tx$ gives $x = (1 + t)/(1 + t + t^2)$, $y = t(1 + t)/(1 + t + t^2)$. (*Fig.* 21 (*a*).)

(ii) $2x^2 - 5xy + 2y^2 = 2x$.

(∞, ∞), asymptotes: $y = 2x + \frac{2}{3}$, $y = \frac{1}{2}x - \frac{2}{3}$; $(0, 0)$, $y^2 = x$, $(1, 0)$, $2(x - 1) = 5y$.

$y = tx$ gives $x(t - 2)(2t - 1) = 2$, $y(t - 2)(2t - 1) = 2t$. *Hyperbola.* (*Fig.* 21 (*b*).)

(iii) $(2x + y)^2 = x - y$.
$(0, 0)$, $y = x$; (∞, ∞), $2x + y = \pm\sqrt{(3x)}$; $(0, -1)$, $5x = -(y + 1)$, $(\frac{1}{4}, 0)$, $-2y = x - \frac{1}{4}$.
$y = tx$ gives $x(t + 2)^2 = 1 - t$, $y(t + 2)^2 = t(1 - t)$. *Parabola.* (*Fig. 21* (*c*).)

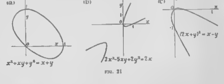


FIG. 21

3.46. *The Cubic.* A Cubic cannot have more than one double point, and if it has one double point, it is unicursal. If the origin is taken at the double point, the equation of the curve must be of the form

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 = Ex^2 + Fxy + Gy^2.$$

The line $y = tx$ gives

$$x : y : 1 = E + Ft + Gt^2 : t(E + Ft + Gt^2) : A + Bt + Ct^2 + Dt^3$$

Example. $x^3 + y^3 = 3axy$.
$y = tx$ gives $x(1 + t^3) = 3at$, $y(1 + t^3) = 3at^2$.
Also $(0, 0)$, $y^2 = 3ax$ and $x^2 = 3ay$; (∞, ∞) $x + y + a = 0$.
The curve is parallel to OY when $2t^3 = 1$ giving the point $(a\sqrt[3]{4}, a\sqrt[3]{2})$ and the curve is parallel to OX at $(a\sqrt[3]{2}, a\sqrt[3]{4})$. When $t \to -1$, the point tends to the point of contact of the asymptote. (*Fig. 22* (*a*).)

3.47. *Other Examples of the Unicursal Curves.*

(i) $x^4 - y^4 = x^2y$.
Triple point at $(0, 0)$ and therefore unicursal.
$(0, 0)$, $y = x^2$ and $x^2 = -y^3$; (∞, ∞), $y = \pm x - \frac{1}{4}$.
$y = tx$ gives $x(1 - t^4) = t$, $y(1 - t^4) = t^2$. Never parallel to OX or OY. (*Fig. 22* (*b*).)
(ii) $y^2(3y^2 + 2y - 9) + 4(x^2 - 1)^2 = 0$.
Double points at $(\pm 1, 0)$, $(0, 1)$ (found by solving $f = f_x = f_y = 0$). It is therefore unicursal. Take a variable conic through these points and $(0, -2)$. This will be found to be $y^2 + txy + 2x^2 + y - 2 = 0$, and the variable point of intersection is given by

$$x : y : 1 = 2t(24 - 5t^2) : -2(t^2 - 4)(t^2 - 16) : 3t^4 - 24t^2 + 64.$$

It is naturally simpler to draw the curve by our previous methods. In the diagram are shown the conics for $t = 1$ (an ellipse) and $t = 3$ (two straight lines). (*Fig. 22* (*c*).)
(iii) $x^4(y + 1) + 2x^2(y^2 - 1) = (y - 1)^3(y + 1)^2$.
It will be found that this quintic has a triple point at $(0, 1)$ and double points at $(0, -1)$, $(\pm 1, 0)$. It is therefore unicursal. A variable conic through these 4 points is obviously $x^2 + y^2 - txy = 1$, and the variable intersection will be found given by

$$x : y : 1 = t(t^2 - 2)(t^2 - 6) : (4 - 8t^2 + t^4) \cdot (4 + 4t^2 - t^4).$$

In the diagram is shown also the conic for $t = 1{\cdot}8$. (*Fig. 22* (*d*).)

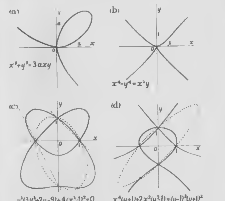


FIG. 22

3.5. Graphical Representation of Functions of Two Variables. An explicit function z of two variables (x, y) given by the relation $z = f(x, y)$ may be represented by drawing the system of curves $f(x, y) = z$ for different values of z.

If we regard the plane XOY as a *horizontal* plane and z the *height* of the point P vertically above the point (x, y), the curves $f(x, y) = z$ are appropriately called *level curves* or *contours*. Again, if $f(x_0, y_0) = z_0$ and $f(x, y) < f(x_0, y_0)$ near (x_0, y_0), the point (x_0, y_0) gives a *maximum* value to z and may be called a *summit*. Similarly we may have a *minimum* or *immit*. Other appropriate terms will arise in the examples we shall use as illustrations.

It is to be expected that the critical values of z are those that belong to contours having singular points. Thus if (a, b) is a point obtained by solving the equations $f_x = 0 = f_y$, this point is singular for the contour $f(x, y) = f(a, b)$. It is important however to realize that whilst (a, b) may be a singular point in the usual sense, the curve $f(x, y) = f(a, b)$

may sometimes be composite and (a, b) an intersection of the curves of which it is composed.

3.51. The Linear Function $z = ax + by + c$. The contours are parallel lines. Let P_1, P_2 be the points (x_1, y_1), (x_2, y_2) and let Q_1, Q_2 be the corresponding points of the surface of heights z_1, z_2 respectively, where $z_1 = ax_1 + by_1 + c$; $z_2 = ax_2 + by_2 + c$. (*Fig. 23.*)

Also let $x_2 - x_1 = r\cos\theta$, $y_2 - y_1 = r\sin\theta$ where r is P_1P_2 and θ the angle that P_1P_2 makes with OX. The inclination of Q_1Q_2 to the horizontal is given by

$$\tan\alpha = |z_1 - z_2|/r = |a\cos\theta + b\sin\theta|$$

and is independent of r. The surface therefore satisfies the Euclidean definition of a *plane*.

The *lines of greatest slope* are given by $\tan\theta = b/a$, and are, as we expect, *perpendicular* to the contours. The inclination of the plane to

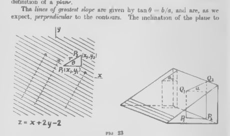


FIG 23

the horizontal (i.e. the *gradient*) is defined to be that of the lines of greatest slope and is given by $\tan\alpha_0 = \sqrt{(a^2 + b^2)}$.

More generally, the curves that cut the contours at right angles are called lines (or curves) of greatest slope (or simply, the lines of slope). If $f(x, y) = z$ gives the contours, the lines of slope must satisfy the differential equation $dy/dx = f_y/f_x$, since dy/dx for the contour through (x_0, y_0) is the value of $-f_x/f_y$ at (x_0, y_0). These curves are shown by dotted lines in the various illustrations.

The example $z = x + 2y - 2$ is illustrated in *Fig. 23.*

3.52. Quadratic Functions. The contours are similar conics but since, by a change of axes, the equations may be reduced to simpler forms, it is sufficient to consider the following cases.

(i) $z - z_0 = ax^2 + by^2$ $(a > 0,\ b > 0)$.

The contours are similar ellipses. $z > z_0$; $(0, 0, z_0)$ *minimum*

(ii) $z - z_0 = -ax^2 - by^2$ $(a > 0,\ b > 0)$.

The contours are similar ellipses. $z < z_0$; $(0, 0, z_0)$ *maximum*

(iii) $z - z_0 = ax^2 - by^2$ $(a > 0,\ b > 0)$.

$z = z_0$ gives two real straight lines which are asymptotic to all the contours. These lines divide the $x - y$ plane into two pairs of opposite sectors. In one pair $z > z_0$ and in the other $z < z_0$ so that $(0, 0, z_0)$ is a maximum for displacements into one pair of sectors and a minimum for displacements into the other. Such a point is called a *Saddle Point* (*minimax*, *pass*, or *col*).

(iv) $z = ax^2 + 2by$ $(a \neq 0,\ b \neq 0)$.

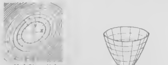
The contours are similar parabolas. *No* maxima nor minima.

(v) $z - z_0 = k(\alpha x + \beta y)^2$.

The contours are parallel straight lines, but $z - z_0$ and k must have the same sign. Every point on the line $\alpha x + \beta y = 0$ gives a minimum (maximum) if $k > 0$ $(k < 0)$ for all displacements except those actually along this line. Along the line z is constant (z_0). If $k > 0$, the surface is a *trough* and if $k < 0$ it is a *ridge*.

Examples. (i) $34x^2 - 24xy + 41y^2 = z$.

$(0, 0, 0)$ is a *minimum*; elliptic contours; this surface is called an *elliptic paraboloid*. (*Fig. 24.*)

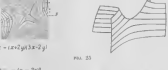


$z = 34x^2 - 24xy + 41y^2$

FIG. 24

(ii) $3x^2 + 4xy - 4y^2 = z$.

$(0, 0, 0)$ is a *saddle point*; hyperbolic contours with asymptotes $x + 2y = 0$, $3x - 2y = 0$; this surface is called a hyperbolic paraboloid. (*Fig. 25.*)



$z = (x+2y)(3x-2y)$

FIG. 25

(iii) $z = -(y - 2x)^2$.

z cannot be positive; contours are parallel lines $y = 2x \pm \sqrt{(-z)}$

The line $y = 2x$ is a maximum for z for all displacements except those along the line. For these z is stationary (with the value zero).

This surface is called a *parabolic cylinder*. (*Fig.* 26.)

$z = -(y-2x)^2$

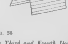


FIG. 26

3.53. *Examples of Functions of the Third and Fourth Degrees.*

(i) $z = 3x^2 - y^2 + x^3$.

The contours have singular points (a) when $z = 0$ at $x = 0$, $y = 0$ (*Node*), (b) when $z = 4$, at $x = -2$, $y = 0$ (*Isolated Point*), obtained by solving $z_x = 0 = z_y$. There is therefore a maximum at $(-2, 0)$ and a *saddle-point* at $(0, 0)$. (*Fig.* 27 (a).)

(ii) $z = x^2 + y^2 - \frac{1}{2}x^4$.

Minimum $z = 0$ at $(0, 0)$ and two *saddle-points* for $z = \frac{1}{2}$ at $(\pm 1, 0)$. (Lines of slope are $y^2(1 - x^2)/x^2 =$ constant.) (*Fig.* 27 (b).)

(iii) $z = -4x^2 - y^2 + x^2y$.

Maximum $z = 0$ at $(0, 0)$, two *saddle-points* for $z = -16$ at $(\pm 2\sqrt{2}, 4)$. (*Fig.* 27 (c).)

(a) $z = 3x^2 - y^2 + x^3$ (b) $z = x^2 + y^2 - \frac{1}{2}x^4$ (c) $z = -4x^2 - y^2 + x^2y$

FIG. 27

3.54 *Contours with Cusps.*

(i) $z = (y - x^2)^2 + x^4$. *Minimum* at $(0, 0)$. (*Fig.* 28 (a).)
(ii) $z = (y - x^2)^2 - x^3$. *Saddle Point* at $(0, 0)$. (*Fig.* 28 (b).)
(iii) $z = (y - x^2)^2 - x^4$. *Saddle-Point* at $(0, 0)$. (*Fig.* 28 (c).)

(a) $z = (y - x^2)^2 + x^4$ (b) $z = (y - x^2)^2 - x^3$ (c) $z = (y - x^2)^2 - x^4$

FIG. 28

3.55. *Examples of Rational Functions (illustrating Discontinuity).*

(i) $z(x^4 + y^2) = 2x^2y$, with $z(0, 0) = 0$; the function z does not tend to a definite limit when $(x, y) \to (0, 0)$, and is therefore discontinuous there. The contours are parabolas. (*Fig. 29* (*a*))

FIG. 29

(ii) $z(x - 2) = x^2y^2$, this has a simple infinite discontinuity along the line $x = 2$. The point $(0, 0)$ is a maximum (in the broad sense). (*Fig. 29* (*b*).)

(iii) $z(x^2 + y^2) = x^6 + y^6$, with $z(0, 0) = 0$. The function z is continuous for all finite (x, y). (Take polar co-ordinates to find points on the curve from $\theta = 0$ to $\theta = \pi/4$.) (*Fig. 29* (*c*).)

Examples III.

Draw the curves given in *Examples 1–14*.

1. $y^4 = x^5$ **2.** $x^3y^7 = 1$ **3.** $y + x^4 = 0$ **4.** $x^{10}y^9 + 1 = 0$
5. $y^{10} + x^9 = 0$ **6.** $x^2y^4 = 1$ **7.** $y^7 = x^4$ **8.** $y^7 + x^7 = 0$
9. $x^2y^3 = 1$ **10.** $y^4 = x^3$ **11.** $y^3 + x^4 = 0$ **12.** $y^3 + x^5 = 0$
13. $xy^4 + 1 = 0$ **14.** $y^2x^5 = 1$

15. Show that the function $y = \sqrt{(1 + x)} + \sqrt{(1 - x)}$ satisfies the equation $y^4 - 4y^2 + 4x^2 = 0$ and write down explicit forms for the other branches of y given by this equation. Illustrate by drawing the curve $4x^2 = y^2(4 - y^2)$.

Find the algebraic equations satisfied by the functions y given in *Examples 16–22*.

16. $y\sqrt{(x + 1)} = \sqrt{x}$ **17.** $y = \sqrt{x} + \sqrt[3]{x}$ **18.** $y\sqrt{x} = \sqrt{(x + 1)} + \sqrt{(x - 1)}$
19. $y\{\sqrt{(x + 2)} + \sqrt{(x - 2)}\} = \sqrt{(x - 1)} + \sqrt{(x + 1)}$
20. $y = \sqrt[3]{x} + \sqrt{(x - 1)}$ **21.** $y = \sqrt[3]{\{x^2(x - 1)\}} + \sqrt[3]{\{x(x - 1)^2\}}$
22. $y = \sqrt{x} + \sqrt[3]{x}$

23. Draw the graph of $y^4 - 4xy^2 + 4x^4 = 0$ and find explicit forms for the four branches of y.

24. Prove that if $y\sqrt{(x + 2)} = \sqrt{(1 + x)} - \sqrt{(1 - x)}$, then

$$y^4(x + 2)^2 - 4y^2(x + 2) + 4x^2 = 0$$

Obtain explicit forms of the four branches and state for what values of x the branches are real.

25. Show that the algebraic equation satisfied by

$$y = \sqrt{x} + \sqrt{(x - 1)} + \sqrt{(x + 1)}$$

is $y^8 - 12xy^6 + 2(15x^2 + 4)y^4 - 4xy^2(7x^2 - 4) + (3x^2 - 4)^2 = 0$. What are the 8 roots of this equation when (i) $x = 2$, (ii) $x = 1$?

26. Find the tangents to the curve $y(2y - 1) = x^2(x - 1)$ at the points $(0, 0)$, $(1, 0)$, $(0, \frac{1}{2})$, $(1, \frac{1}{2})$. Find also the closer approximations to the curve at these points.

27. Obtain the first non-linear approximations to the curve

$$y^2(y - 1) = x(x - 2)(x - 3)$$

at $(0, 0)$, $(0, 1)$, $(2, 1)$, (∞, ∞).

Obtain the first non-linear approximations at $(0, 0)$ to the curves given in *Examples 28–35*.

28. $(x^2 - y^2)(x + 2y) + x^4 + y^3 = 0$

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29. $x^3 - y^3 - (x - y)(x^2 + y^2) + x^3 + y^4 = 0$
30. $(x - y)^2(x + y) + (x^4 - y^4) + x^5 + y^7 = 0$ 31. $(x - y)^3 + x^5 - y^6 = 0$
32. $(x^2 - y^2)(x + 2y) + 2x(x + y)^2(x - y) + x^6 + y^6 = 0$
33. $(x - y)^3 + 2x^4 - 4x^3y + 4xy^3 - 2y^4 - x^5 - y^5 = 0$
34. $(y - x^2)^2 + y^3 + x^5 = 0$ 35. $(y - x^2)^2 - x(y - x^2)^2 + y^4 - x^5 = 0$

Obtain the first non-linear approximations to the curves given in *Examples 36–57*, when one at least of the variables is large.

36. $y^2(x - 2) = x^4$ 37. $xy^4 = (x - 1)^3$ 38. $y^2(x^2 - 1) = x^7$
39. $y^2(x^2 + 1) = x^3$ 40. $y(y - 1)(x - 1) = x^3$
41. $y^2(y - 1)(x^2 + 1) = x^4$ 42. $(y^2 - 1)^2(x^2 - 1) = 4x^3$
43. $2x - y^2 + 4x^3y^4 = 0$ 44. $x^4 - y^4 + 2xy^2 = 0$ 45. $y^5 - x^4y^2 - 4x^3 = 0$
46. $4y^5 - x^5 - x^7y^2 - 2x^7y^2 - 8x^2y^5 = 0$ 47. $y^2(x - 2) = 8x^2(x + 1)^2$
48. $y^2(x - 1)^2 = (2x + 1)(4x + 3)^2$ 49. $y^4(4x^2 - 1) = (4x^2 - 1)^3$
50. $4y^2(x - 2)^2(x^2 + 1) = (x^2 - 1)(x - 1)^2$ 51. $y^2(y - 1) = x(x - 1)(x - 2)$
52. $y^2(y - 1)^2(x^2 - 1)(x + 2)^2 = x^2(x - 1)^2(y^2 - 1)$
53. $y^2(y + 1)^2(x^2 + 1) = 8x^4(y - 1)$
54. $y^4(y - 1)^2(x^2 + 1) = 3(9x^2 - 1)(y - 1)$
55. $y(3y^2 - y + 3)(x^2 - 1)^2 = 2(y^2 + 1)(x^4 - x^2 + 1)$
56. $x^4y^2 - x^2y^4 + 4xy^4 - 2x^3y^2 + 2x - y^2 = 0$
57. $x^2y^2(y - 2x) + xy(y - 2x)^2(y + 4x) + 3x^3 - 2y = 0$

Obtain the first non-linear approximations at (0, 0) to the curves given in *Examples 58–62*.

58. $(x - y)^2 = x(x - y)^2 - x^4 + y^5$
59. $(2x - y)^2 - (2x - y)^3 + (2x - y)(x^2 + y^2) - x^5 = 0$
60. $(x - y)^2 - 2(x - y)^3 + x^5 = 0$ 61. $(y - 2x)^2 = 2y^4 - x^5$
62. $(x + y)^2 - 2(x - y)^3 + (x + y)(x^2 - y^4) + x^5 - x^6 = 0$

Draw the curves given in *Examples 63–122*.

63. $y^2 = x^2(x - 1)$ 64. $y^3 = x^2(x + 1)$ 65. $y^2 = x(x - 1)$
66. $y^2 = x^4(x - 5)^2$ 67. $y^4 = x^2(x^2 + x + 1)$ 68. $y^2(x + 2) = x^3$
69. $y^2(x + 1) = 2x^3$ 70. $y^2(x^2 + 1) = x^2 - 1$
71. $xy^4 = (x - 1)^2(x + 1)^2$ 72. $y^4(x - 1)^2 = x^7$
73. $y^2(x^2 + 1) = x^2$ 74. $y^2(x^2 + 1) = (x - 1)^2$ 75. $y^4x^2(x + 1) = 1$
76. $y^4 + (x - 1)(x^2 - 1)^2 = 0$ 77. $y^4(x + 1)^3$
78. $y^2(x + 1) = x(x - 1)^2$ 79. $y^2(x - 1)^2 = x^2(x - 1)$
80. $y^4(x + 1) = x^2(x - 1)^3$ 81. $y^2(x + 1)(x - 1)^2 = x^4$
82. $y^4(x + 2)^2 = x^4(x - 1)$ 83. $y^2(x + 1)^2(x - 1)^6 = x^9$
84. $y^2(x^2 - 4)(x^2 + 9) = x^2 - 1$ 85. $y^2(x^2 - 1)(x^2 - 4) = 7x^2 - 13$
86. $4x^2y^2 - 9x^2 - 4y^6 + 16 = 0$ 87. $y(y - 1)(x^2 - 1) = 2x^3$
88. $y^2(x - 1)^2 = x^4(y - 1)$ 89. $y^2(x + 1)^2 = x^4(y - 1)^2$
90. $y^2(y - 1)(x - 1)^2 = 4x^2$ 91. $y^2(x^2 + 1) = x(y - 1)$
92. $y^4(y - 1)^2(x + 1) = x^2(y - 1)^4$ 93. $(y - 1)(x - 1)^2 = x^2y^2$
94. $y^2(y - 1)^2(4y - 5) = x^2(x^2 + 3x + 1)$
95. $(y^2 + 1)(x^2 + 4) = (x^2 + 1)(y^2 - y + 1)$ 96. $x^3 + y^3 = 2xy$
97. $y + x^4 = 2xy^2$ 98. $x^4 + y^4 = 2xy$ 99. $x^3 - y^3 = x^2y^2$
100. $x^4 + y^4 = x^2y$ 101. $x^4 + y^4 = 2x^2y$ 102. $x^4 - y^4 = x^2y^2$
103. $x^4 - y^4 = x^2y^2$ 104. $x^5 - y^5 = 5xy$ 105. $x^5 - y^5 = 5x^2y$
106. $x^5 + y^5 = 5x^2y^2$ 107. $x^5 - y^5 = 5x^2y^2$ 108. $x(x + y)(x - y - 1) = 2$
109. $y(x - y)^2 = x + y$ 110. $y^2(x - y)^2 = x^2(y - 2)$
111. $(x + y)(x - y)^2 = y(2x - y)$ 112. $x^2 + xy = y^2(1 - y)$
113. $2x^4 - 2x^3y - 2x^2y - y^4 = 0$ 114. $x^4 - y^4 = x^2(x^2 - xy - y^2)$
115. $2(y - x^2)^2 = x^3 + x^5$ 116. $2x^5 - 3x^3y^2 - 4xy^4 - y^5 = 0$
117. $x^4 - 2x^2y + 3xy^2 - y^4 = 0$ 118. $x^5 - 3x^2y^2 + xy^4 - y^5 = 0$
119. $x^5 - 4x^4y + 2x^2y + y^4 = 0$ 120. $(y - x^2)^2 = x^5 + 2x^3y$
121. $y^4 - x^2 + 2xy(x^2y^2 - 1) = 0$ 122. $y^2 - x^2 - 2xy(xy - 1) = 0$

Find the singular points of the curves given in *Examples 123–8*.

123. $y^2(9 - y)^2 = x^2(2x + 9)^2$ 124. $(x^2 - 1)^2y^2 = x^2(y^2 - 1)^2$
125. $x^2y^2 + 144x + 48y + 432 = 0$ 126. $(x + 1)^4(y - 1)^2 + 64x^2y^2 = 0$
127. $(x + y + 1)^3 = 27xy$

128. $4y^4(x+16) - y^3(111x^3 - 128x^2 + 36x + 128) + 32(x^2 - 1)^2(x+2) = 0$

129. Prove that if a sextic possess one triple point, it cannot possess more than seven double points besides.

130. Prove that if a curve of degree $n(> 5)$ possess two triple points, it cannot possess more than $\frac{1}{2}(n^2 - 3n - 8)$ double points besides.

Express the co-ordinates of a point on the curves given in *Examples 131–3* rationally in terms of a parameter t and draw the curves.

131. $x^3 + y^3 = x^2$ **132.** $x(x^2 - 4y^2) = x^2 - y^2$

133. $4x^5 - 17x^2y^2 + 4xy^4 = y^5$

134. Show that the quartic $y^2(y-3) + 4(x^2 - 1)^2 = 0$ has three double points and prove that the variable conic $xy + t(y + 2x^2 - 2) = 0$ meets the quartic in the point given by $x = t(2t^2 - 3)$, $y = -1 + 8t^2 - 4t^4$.

135. Sketch the curve $(x^2 - a^2)^2 = ay^2(2y + 3a)$ showing that it has three double points. By taking a variable conic through these and the point $(0, \frac{1}{2}a)$ obtain the co-ordinates of a point (x, y) of the curve in the form $4x = at(6 - t^2)$, $8y = a(t^2 + 2t - 2)(t^2 - 2t - 2)$

136. Trace the curve given by $t(t+1)x = 1$, $t(t+3)y = 1$ and obtain the algebraic relation connecting x and y.

137. Show that the point given by $x(1 + 2t) = t^2(2 + t)$, $y(1 + 2t)^2 = t(2 - t)^2$ lies on the curve $(x^2 + 6xy - y^2)^2 = 16xy(4xy - 3x - 3y + 4)^2$

138. Sketch the curve given by $x : y : 1 = t(1 - t^2) : (1 + t^2)^2 : (1 - 3t^2 + t^4)$

Draw the contour lines of the surfaces given in *Examples 139–59*, and discuss the continuity of z.

139. $yz = y^2 - x^2$ **140.** $xyz = x^2 + y^2$ **141.** $xyz = x^2 + y^2$

142. $(y^2 + x^4)z = y^2 - x^4$ **143.** $z = xy + x^4$ **144.** $z = xy + x^3$

145. $x^2z = xy^2 + y^3 - x^4$ **146.** $z = x^2y + xy^2$ **147.** $zy^2 = x(1 - y^2)$

148. $z = xy + x^2 + x^2y$ **149.** $z = (x^2 + y^2)(x^2 - 4y^2)$

150. $z = (x^2 + y^2 - 1)(x^2 + y^2 - 4)$ **151.** $z(y^2 - x^2) = x(x^2 - y^4)$

152. $z = 2x^2 + y^2 - 3xy^2$ **153.** $zy^2 = x(x - y^2)$

154. $z(x - 1) = x^3 - y^3 - x^2 + y^2$ **155.** $z = (x^2 + y^2)(x^2 + 4y^2)$

156. $z = (x^2 - y^2)(x^2 - 4y^2)$. **157.** $zy^2 = x^2(y^2 - x^2 + 1)$

158. $(x - 1)z = x(y^2 - x^2) + (x^2 + y^2)$

159. $z = (4x^2 + y^2 - 4)(x^2 + 4y^2 - 4)$

Solutions.

15. $y = \sqrt{(1 + x)} \pm \sqrt{(1 - x)},\ -\sqrt{(1 - x)} \pm \sqrt{(1 - x)}$ **16.** $y^2(x + 1) = x$

17. $y^4 - 2xy^2 - 4xy + x(x - 1) = 0$ **18.** $x^2y^4 - 4x^2y^2 + 4 = 0$

19. $16y^6 - 16x^2y^4 + 4(5x^2 - 2)y^4 - 4x^2y^2 + 1 = 0$

20. $y^6 - 3y^4(x - 1) - 2xy^3 + 3y^2(x - 1)^2 - 6xy(x - 1) = x^3 - 4x^2 - 3x - 1$

21. $y^3 - 3xy(x - 1) = x(x - 1)(2x - 1)$

22. $y^5 - 3xy^4 - 2xy^3 + 3x^2y^2 - 6x^2y - x^3(1 - x) = 0$

23. $y = \pm\sqrt{(x - x^2)} \pm \sqrt{(x + x^2)}$

24. $y\sqrt{(x - 2)} = \pm\sqrt{(1 + x)} \pm \sqrt{(1 - x)}$. Real when $|x| < 1$

25. (i) $\pm\sqrt{2} \pm \sqrt{3} \pm 1$, (ii) ± 1, ± 2 each occurring twice.

26. $(0, 0)$, $y = x^3$; $(1, 0)$, $y = 1 - x - (x - 1)^3$; $(0, \frac{1}{2})$, $y = \frac{1}{2} - x^2$; $(1, \frac{1}{2})$, $y - \frac{1}{2} = x - 1 + (x - 1)^3$

27. $(0, 0)$, $y^2 = 6x$; $(0, 1)$, $y = 1 - 6x - 77x^2$.

$$(2, 1),\ y = 1 - 2(x - 2) - 7(x - 2)^2;\ (\infty, \infty),\ y = x - \frac{4}{3} - \frac{2}{3x}$$

28. $y = x + \frac{1}{6}x^3$, $y = -x + \frac{1}{3}x^3$, $y = -\frac{1}{2}x - \frac{3}{8}x^3$

29. $y = x + x^4$, $y = -x - 2x^3$

30. $y = x + 2x^3$, $y = x + \frac{1}{6}x^3$, $y = -x - \frac{5}{6}x^4$ **31.** $y = x \pm \sqrt{(-x^3)}$

32. $y + x = x^2$, $y = x + \frac{1}{4}x^3$, $x - 2y + x^3 = 0$

33. $y = x \pm x^3$ **34.** $y = x^2 \pm \sqrt{(-x^5)}$ **35.** $y = x^2 - x^3$

36. $9y = 9x + 6 + 8/x$, (∞, ∞). $y^2(x - 2) = 16$, $(2, \infty)$

37. $\pm y = x - \frac{5}{4} + \frac{5}{32x}$, (∞, ∞); $y^2x + 1 = 0$, $(0, \infty)$.

38. $y = 1 - \dfrac{1}{2x^2}, (1, \infty)$; $y = 1 + \dfrac{1}{2x^2}, (-1, \infty)$

39. $y = 1 - \dfrac{1}{3x^3}, (1, \infty)$; $y^3(x+1) = -\frac{1}{3}, (-1, \infty)$

40. $y = x + \dfrac{1}{2x}, (\infty, \infty)$, $y = -x - 1 - \dfrac{1}{2x}, (\infty, \infty)$; $y^2(x-1) = 1, (1, \infty)$

41. $y = x + \dfrac{1}{3} - \dfrac{2}{9x}, (\infty, \infty)$ 42. $y - \sqrt{3} = \sqrt{3}/(6x^2), (\infty, \sqrt{3})$;
$y + \sqrt{3} = -\sqrt{3}/(6x^2), (\infty, \sqrt{3})$, $y^4(x-1) = 3, (1, \infty)$; $y^4(x+1) = -3, (-1, \infty)$

43. $4x^3y^3 = 1, (0, \infty)$ 44. $\pm y = x + \dfrac{1}{2} + \dfrac{1}{8x}, (\infty, \infty)$

45. $xy = \pm 2, (\infty, 0)$; $\pm y = x - \dfrac{1}{x^2}, (\infty, \infty)$

46. $xy = \pm 1, (\infty, 0)$; $y - \dfrac{1}{2} = -\dfrac{3}{2x^2}, (\infty, \frac{1}{2})$;

$2y = \pm x - \frac{1}{2} \mp \dfrac{3}{8x}, (\infty, \infty)$, $2x^3y^3 = 1, (0, \infty)$

47. $y = 2x + \dfrac{2}{3x}, (\infty, \infty)$; $y^3(x+2) = 32, (-2, \infty)$

48. $y = 2 + \dfrac{2}{x}, (\infty, 2)$; $y^2(x-1)^3 = 147, (1, \infty)$

49. $\pm y = 2x - \dfrac{1}{2x}, (\infty, \infty)$

50. $\pm y = \dfrac{1}{2} + \dfrac{1}{3x}, (\infty, \pm\frac{1}{2})$; $20y^2(x-2)^2 = 3, (2, \infty)$

51. $y = x - \dfrac{2}{3} - \dfrac{2}{9x}, (\infty, \infty)$

52. $y = x - \dfrac{3}{4} + \dfrac{63}{32x}, (\infty, \infty)$; $y = -x + \dfrac{9}{4} - \dfrac{63}{32x}, (\infty, \infty)$;
$5y^4(x+2)^3 = 576, (-2, \infty)$

53. $y = 2x - 1 - \dfrac{1}{6x}, (\infty, \infty)$; $2x^2(y-1) = 1, (\infty, 1)$; $3y^3(x+1) = 8$,
$(-1, \infty)$ 54. $63x^4(y-3) = -40, (\infty, 3)$

55. $y = \frac{1}{2}x^2, (\infty, \infty)$; $y - 1 = -\dfrac{4}{x^3}, (\infty, 1)$; $x^2(y-3) = 18, (\infty, 3)$,
$2x^3y = 1, (0, \infty)$

56. $y = x + 1 + \dfrac{5}{2x}, (\infty, \infty)$; $y = -x - 3 - \dfrac{31}{2x}, (\infty, \infty)$; $(x-4)y = -8$,
$(4, \infty)$; $4xy^2 = 1, (0, \infty)$; $x^3y^3 = -2, (\infty, 0)$

57. $y = 2x - \dfrac{3}{8x^2}, (\infty, \infty)$; $y = \pm 2\sqrt{2} - \dfrac{1}{x}, (\infty, \pm 2\sqrt{2})$; $16x^2y + 3 = 0$,
$(\infty, 0)$; $xy^2 + 2 = 0, (0, \infty)$

58. $y = -x \pm \sqrt{2x^3}$ 59. $y = 2x \pm \sqrt{(-x^3)}$

60. Origin isolated with real tangent $x = y$.

61. Origin isolated with real tangent $y = -2x$.

62. $y + x + x^3 = \pm 2x^4$

123. $(0, 0), (0, 6), (-\frac{3}{2}, 0), (-\frac{3}{2}, 6), (-3, 3)$

124. $(0, 0), (\pm 1, \pm 1)$. 125. $(-3, -5)$

126. $(0, 1), (-1, 0), (3, -2)$. 127. $(1, 1)$

128. $(\pm 1, 0), (0, \pm 1), (2, \pm 2)$ 131. $x = 1 + t^2, y = t(1 + t^2)$

132. $x : y : 1 = (1 - t^2) : t(1 - t^2) : (1 - 4t^2)$

133. $t^4x = (4t^2 - 1)(t^2 - 4), y = tx$ 136. $(x - y)^3 = 2ay(3y - x)$

CHAPTER IV

FUNCTIONS DEFINED BY SEQUENCES. DISCONTINUOUS FUNCTIONS. CONVERGENCE OF SERIES. SINGLE AND DOUBLE POWER SERIES. EXPONENTIAL, LOGARITHMIC AND CIRCULAR FUNCTIONS.

4. Functions defined by Sequences. Functions that are not algebraic are called *Transcendental*, and it may be expected that a simple way of defining such functions is through the medium of convergent sequences of known functions. Thus a function may be defined as $\lim f(x, n)$ when $n \to \infty$ for those values of x for which the limit exists. In some cases, such a function may be expressible in terms of algebraic functions but this does not make it algebraic.

Example. If $F(x) = \lim x\left(\frac{x^n - 1 - n}{x^n + 1 + n}\right)$, then when $|x| > 1$, $F(x) = x$, but when $|x| < 1$, $f(x) = -x$. There are, therefore, discontinuities at $x = \pm 1$.

Since discontinuities are of frequent occurrence in functions defined by sequences, it is convenient here to classify the various types of discontinuities that arise.

4.01. Discontinuities. Let $f(x)$ be defined at all points near $x = a$, and let $x_1, x_2, \ldots$, be an *increasing* monotone tending to a. Let U_n, L_n be the upper and lower bounds of $f(x)$ in $x_n < x < a$.
Then $U_1 \geq U_2 \geq \ldots \geq U_n \geq \ldots \geq L_n \geq \ldots \geq L_2 \geq L_1$.

The sequences U_n, L_n therefore tend to limits U, L respectively (if $f(x)$ is bounded) and $U \geq L$. If $f(x)$ is unbounded, one at least of these sequences tends to $\pm\infty$, and therefore we shall include $+\infty$ or $-\infty$ as possible 'values' of U, L.

It may be shown that U, L are independent of the choice of monotone that tends to a.

The limits U, L are denoted by $\overline{f}(a-0)$, $\underline{f}(a-0)$ respectively. Similarly by considering a *decreasing* monotone tending to a, we may define $\overline{f}(a+0)$, $\underline{f}(a+0)$.

(i) If $\overline{f}(a-0) = \underline{f}(a-0) = f(a) = \underline{f}(a+0) = \overline{f}(a+0)$, $f(x)$ is obviously *continuous* at $x = a$. Otherwise it is *discontinuous*.

(ii) If all the limits are finite, the discontinuity is said to be finite (or bounded); if one, at least, is infinite, the discontinuity is said to be infinite.

(iii) If $\overline{f}(a-0) = \underline{f}(a-0)$, each is the same as $f(a-0)$; and if $\overline{f}(a+0) = \underline{f}(a+0)$, each is the same as $f(a+0)$.

(iv) If $f(a-0)$, $f(a+0)$ *both* exist ($f(x)$ not being continuous at a), the discontinuity is said to be of the *first kind*. Otherwise it is of the *second kind*.

The discontinuity is still said to be of first kind when one of the limits $f(a-0)$, $f(a+0)$ is infinite, or both are infinite.

(v) If $f(a-0) = f(a+0) \neq f(x)$, the discontinuity is said to be *removable*.

(vi) The greatest of the numbers $\overline{f(a+0)}$, $\overline{f(a-0)}$, $f(a)$ is sometimes called the *maximum* of the function at a; and the least of the numbers $\underline{f(a+0)}$, $\underline{f(a-0)}$, $f(a)$ is called the minimum of the function at a. The excess of the maximum over the minimum is called the *saltus* at a; whilst the *oscillation* at a is defined to be the excess of the greater of $\overline{f(a+0)}$, $\overline{f(a-0)}$ over the smaller of $\underline{f(a+0)}$, $\underline{f(a-0)}$.

(vii) If the set of points of discontinuity of the first kind is infinite, it is *enumerable*.

At such a discontinuity, the saltus is not zero, but the oscillation on each side is zero, since the function tends to a limit on each side. Let k be any positive number and let $E(k)$ be the set of points for which the saltus of $f(x)$ is $\geq k$. Let a be a limiting-point of $E(k)$, if this set be infinite. Then near a, an infinite number of points of $E(k)$ exist, so that on one side of a we can always find a point for which the oscillation is not zero. Thus a is a discontinuity of the second kind and does not belong to $E(k)$. The set $E(k)$ is therefore isolated and enumerable. Take a sequence of numbers $k_n (> 0)$ tending steadily to zero. The limiting set $E(k_1) + E(k_2) + \ldots$ contains only the points of discontinuity and being an enumerable (or finite) set of enumerable (or finite) sets must be enumerable.

Examples. (i) Let $f(x) = \lim_{n\to\infty} x\left(\frac{x^n - 1}{x^n + 1}\right)$ (§ 4, *Example.*)

Here, $f(-1-0) = -1$; $f(-1) = f(-1+0) = 1$; $f(1-0) = f(1) = -1$; $f(1+0) = 1$. (Both of first kind.)

(ii) Let $f(x) = \lim_{n\to\infty} \dfrac{n^2(x-1)(x-2)(x-3) + nx(x-1) + 2}{n^2(x-1)(x-2) - n(x-1)(x+2) + x^2 + 3}$

Here $f(x) = x - 3$, $(x \neq 1, x \neq 2)$. $f(1) = \frac{1}{2}$, $f(2) = -\frac{1}{2}$. (Removable.)

(iii) Let $f(x) = \sin(1/x)$, $(x \neq 0)$; $f(0) = 0$.

$\overline{f(+0)} = 1$, $\underline{f(+0)} = -1$; $\overline{f(-0)} = 1$, $\underline{f(-0)} = -1$.

Finite discontinuity of the second kind. Saltus at 0 is 2.

(iv) Let $f(x) = \{\sin(1/x)\}/x$; $f(0) = 0$.

$\overline{f(+0)} = \overline{f(-0)} = +\infty$; $\underline{f(+0)} = \underline{f(-0)} = -\infty$.

Infinite discontinuity of the second kind. Saltus at 0 is $+\infty$.

(v) Let $f(x) = x\sin(1/x)$; $f(0) = 0$. The function is continuous.

(vi) Let $f(x) =$ the greatest integer $\not> x$.

Here $f(n) = f(n+0) = n$; $f(n-0) = n - 1$. (n integral.)

$f(x)$ has finite discontinuities (first kind) at $x = n$, but is continuous on the right of $x = n$.

4.02. *Infinite Series.* From the sequence $u(x, n)$ we can form a second sequence $S(x, n)$ given by

$$S(x, n) = u(x, 1) + u(x, 2) + \ldots + u(x, n)$$

and if $S(x, n) \to S(x)$ when $n \to \infty$, we write

$$S(x) = u(x, 1) + u(x, 2) + \ldots = \sum_{1}^{\infty} u(x, n)$$

the right-hand side being called a convergent *infinite series*, and $S(x)$ its *sum*.

Since the variable n takes only integer values 1, 2, 3, . . . , it is usually more convenient to write it as a suffix and obtain

$$S(x) = u_1(x) + u_2(x) + \ldots = \sum_1^\infty u_n(x).$$

Similarly we may have infinite series that diverge to $+\infty$ or $-\infty$ or that oscillate (finitely or infinitely).

It is seldom possible to find a suitable expression for $S_n(x)$ from which the convergence of the series can be directly investigated, and for this reason tests have been devised that apply to the general term of the series $u_n(x)$. Before considering such tests in general, we note the following facts as a preliminary:

(i) It is *necessary* for convergence that $\lim u_n(x) = 0$.

For if $S_n(x) \to S(x)$, then $S_{n+1}(x) \to S(x)$ and therefore

$$u_n(x) = S_{n+1}(x) - S_n(x) \to 0.$$

(ii) It is *not sufficient* for convergence that $\lim u_n(x) = 0$.

It will be shown later that, for example, $\sum_1^\infty \frac{1}{n}$ diverges although $\frac{1}{n} \to 0$.

It is, of course, sufficient for non-convergence that $\lim u_n(x) \neq 0$. Thus the series $\frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \ldots$ diverges, since the nth term tends to $\frac{1}{1+0}$.

4.1. Series of Positive Terms. With the object of obtaining *comparison tests* for convergence, let us assume that all the terms are *positive*.

Note. If the only question that is being considered is that of convergence, a finite number of terms may be omitted without altering the character of the series. It is sufficient, therefore, that the conditions stated should hold from and after some fixed term; and this will always be implied in the course of our work.

Let us suppose also that the variable x that occurs in $u_n(x)$ has a particular value. The terms are therefore constants and it is simpler to use the notation $S_n = u_1 + u_2 + \ldots + u_n$. Since $u_n > 0$ (all n), S_n is an increasing monotone. It therefore tends to a limit, if bounded and to $+\infty$, if unbounded. Thus:

A series of positive terms either (i) *converges or* (ii) *diverges to* $+\infty$.

4.11. General Comparison Theorems (positive terms).

A. If $u_n < v_n$ and $\sum_1^\infty v_n$ converges, then $\sum_1^\infty u_n$ converges.

If $u_n > v_n$ and $\sum_1^\infty v_n$ diverges, then $\sum_1^\infty u_n$ diverges.

B. If $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ and $\sum_1^\infty v_n$ converges, then $\sum_1^\infty u_n$ converges.

If $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ and $\sum_1^\infty v_n$ diverges, then $\sum_1^\infty u_n$ diverges.

A. If $u_n < v_n$, $\sum_1^n u_n < \sum_1^n v_n$ and therefore $\sum_1^n u_n$ is bounded and convergent.

$$\text{B. } \sum_1^n u_n = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_2 u_3}{u_1 u_2} + \ldots + \frac{u_2 u_3 \ldots u_n}{u_1 u_2 \ldots u_{n-1}}\right) < \frac{u_1}{v_1} \sum_1^n v_n.$$

Therefore $\sum_1^n u_n$ is bounded and convergent.

The theorems for divergence follow by similar reasoning.

4.12. Comparison Series. The following series of positive terms are the most useful in practice for obtaining tests for convergence

$$\text{(i) } \sum_0^\infty c^n, \ (c > 0), \quad \text{(ii) } \sum_1^\infty \frac{1}{n^p}, \quad \text{(iii) } \sum_2^\infty \frac{1}{n (\log n)^p}.$$

Note. It will be shown that the exponential function may be defined by a series whose convergence may be established by means of the geometric series (i). It is convenient, however, at this stage to assume the properties of this function in dealing with the series (ii) and (iii).

$$\text{(i) } \sum_0^n c^n = \frac{1 - c^{n+1}}{1 - c}, \ (c \neq 1); \ \text{ and } = n + 1, \ (c = 1)$$

The series converges if $c < 1$ and diverges to $+\infty$ when $c \geq 1$.

Note. The series $\sum_0^\infty x^n$ (for any x), converges for $|x| < 1$, oscillates finitely if $x = -1$ and oscillates infinitely when $x < -1$.

$$\text{(ii) } \sum_1^\infty \frac{1}{n^p}; \ \text{ let } S_r = \sum_1^r \frac{1}{n^p}.$$

Since all the terms are > 0, it is sufficient for convergence that $S_1, S_3, S_7, \ldots, S_M \ldots$ should be convergent ($M = 2^m - 1$, $m = 1, 2, 3, \ldots$).

$$\text{But} \quad S_M < 1 + \frac{2}{2^p} + \frac{4}{4^p} + \ldots + \frac{2^{m-1}}{2^{(m-1)p}} \text{ (a geometric series)}$$

$$< \frac{1}{1 - 2^{1-p}} \ (p > 1).$$

The original series therefore *converges if $p > 1$.*

Again, the original series diverges if $S_1, S_2, S_4, \ldots, S_M, \ldots$ is a divergent sequence ($M = 2^m$, $m = 0, 1, 2, \ldots$).

But $S_M > 1 + \dfrac{1}{2^p} + \dfrac{2}{4^p} + \ldots + \dfrac{2^{m-1}}{2^{mp}}$, a geometric series which diverges if $p < 1$, i.e. the original series *diverges if $p < 1$.*

(iii) By a method of grouping the terms similar to that used in (ii), we may show that the series (iii) also *converges if $p > 1$* and *diverges if $p < 1$.*

Notes. (i) A similar proof may be obtained to show that the series whose general terms are

$$\frac{1}{n \log n (\log \log n)^p}, \quad \frac{1}{n \log n (\log \log n)(\log \log \log n)^p}, \ \&c. \ldots$$

are all convergent if $p > 1$ and divergent if $p < 1$.

(ii) The convergence of these series may be established by the *Maclaurin-Cauchy Integral Test*, which involves the use of an infinite definite integral. (*See Chapter XI, 11.02.*)

4.13. Tests for Convergence (positive terms). The two principal ways in which convergence may be established are (i) by *direct* comparison with known series, using Theorem A, § 4.11, and (ii) by *ratio-comparison* with known series, using Theorem B, § 4.11. It will be found that the second method leads to more practical results than the first and that in each case the most useful form of the result is one involving a limit.

4.14. Comparison Tests (positive terms). Theorem A relates the convergence (or divergence) of a series Σu_n *directly* with that of a known series Σv_n.

Example. $\sum \frac{1}{(\log n)^{\log n}}$. When n is large, $\log \log n > 2$, and therefore $(\log n)^{\log n} > n^2$, i.e. the series converges since $\sum \frac{1}{n^2}$ converges.

4.15. Limit Forms of the Comparison Test. (i) If $\lim \frac{u_n}{v_n}$ exist (and is not infinite), and if Σv_n is convergent so also is Σu_n: for a finite number K (indepeodent of n) can be found such $0 < u_n < Kv_n$ so that Σu_n converges since $K\Sigma v_n$ converges.

(ii) If $\lim \frac{u_n}{v_n}$ exist (and is not zero), then Σu_n is divergent if Σv_n is divergent; for a *positive* number k can be found, independent of n, such that $u_n > kv_n$ and therefore Σu_n diverges since $k\Sigma v_n$ diverges.

Examples. (i) If a is not an integer, $\sum \frac{1}{a+n}$ diverges because $\lim \frac{n}{a+n} = 1$ and $\sum \frac{1}{n}$ diverges.

(ii) Let $u_n = n^q/(n+1)^{p+q}$, and $v_n = n^{-p}$.

Then $(u_n/v_n) \to 1$ and so $\Sigma n^q/(n+1)^{p+q}$ converges if $p > 1$ and diverges if $p < 1$.

Notes. (i) These tests are *sufficient* tests but *not necessary*

(ii) When $\lim \frac{u_n}{v_n}$ does *not* exist, $\overline{\lim} \left(\frac{u_n}{v_n}\right)$ may be used in the *convergence* test and $\underline{\lim} \left(\frac{u_n}{v_n}\right)$ in the divergence test.

(iii) If u_n can be expressed asymptotically in the form $v_n + o(v_n)$, then Σu_n converges or diverges with Σv_n, for $\lim \frac{u_n}{v_n} = 1$.

Example. Let $u_n = \left(n \log \frac{2n+1}{2n-1} - 1\right)$: then $u_n = \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right)$, using the logarithmic expansion, i.e. Σu_n is convergent since $\sum \frac{1}{n^2}$ converges.

4.16. Cauchy's Test. If $\lim u_n^{\frac{1}{n}}$ exists and is equal to k, Σu_n converges if $k < 1$ and diverges if $k > 1$.

Let $k < 1$, and let k_1 be any number such that $k < k_1 < 1$, then ultimately $u_n^{\frac{1}{n}} < k_1$, i.e. $u_n < k_1^{\,n}$ and therefore Σu_n converges if $k < 1$.

The proof for divergence when $k > 1$ may be obtained from the above by reversing the signs of inequality.

Example. Let $u_n = \left(1 + \frac{1}{n}\right)^n x^n$, $(x > 0)$. Here $u_n^{\frac{1}{n}} \to x$. The series Σu_n therefore converges if $x < 1$ and diverges if $x > 1$.

When $x = 1$, $u_n \to e$, and therefore the series diverges if $x = 1$.

Notes. (i) The test fails if $u_n^{\frac{1}{n}} \to 1$.

(ii) When $\lim u_n^{\frac{1}{n}}$ does *not* exist, the series converges if $\overline{\lim}\, u_n^{\frac{1}{n}} < 1$, and diverges if $\overline{\lim}\, u_n^{\frac{1}{n}} > 1$, the *upper* limit being used in both cases.

4.17. Ratio-Tests for Convergence. Ratio-tests are obtained by applying Theorem B, § 4.11, and using the series Σc^n, Σn^{-p}, $\Sigma n^{-1}(\log n)^{-p}$ for comparison.

When (i) $v_n = c^n$, $\dfrac{v_n}{v_{n+1}} = \dfrac{1}{c}$.

(ii) $v_n = n^{-p}$, $\dfrac{v_n}{v_{n+1}} = \left(1 + \dfrac{1}{n}\right)^p = 1 + \dfrac{p}{n} + O\!\left(\dfrac{1}{n^2}\right)$.

(iii) $v_n = n^{-1}(\log n)^{-p}$, $\dfrac{v_n}{v_{n+1}} = 1 + \dfrac{1}{n} + \dfrac{p}{n \log n} + o\!\left(\dfrac{1}{n \log n}\right)$

for in (iii)
$$\frac{v_n}{v_{n+1}} = \left(1 + \frac{1}{n}\right)\left\{1 + \log\left(1 + \frac{1}{n}\right) \Big/ \log n\right\}^p$$
$$= \left(1 + \frac{1}{n}\right)\left\{1 + \frac{1}{n \log n} + o\!\left(\frac{1}{n \log n}\right)\right\}^p$$
$$= \left(1 + \frac{1}{n}\right)\left\{1 + \frac{p}{n \log n} + o\!\left(\frac{1}{n \log n}\right)\right\}.$$

Suppose therefore $\dfrac{u_n}{u_{n+1}}$ can be expressed in the form $A + \dfrac{B}{n} + O\!\left(\dfrac{1}{n^2}\right)$, since this covers a large number of cases occurring in practice. Then

(i) If $A > 1$, Σu_n is convergent, and if $A < 1$, Σu_n is divergent.

For when $A > 1$, we can find c such that $A > \dfrac{1}{c} > 1$ and such that ultimately $u_n/u_{n+1} > \dfrac{1}{c}$, i.e. $> v_n/v_{n+1}$ where $v_n = c^n$, i.e. Σu_n is convergent since Σc^n is convergent.

Similarly, divergence is established when $A < 1$.

(ii) If $A = 1$, $B > 1$, Σu_n is convergent, and if $A = 1$, $B < 1$, Σu_n is divergent.

For when $B > 1$, we can find p such that $B > p > 1$ and such that ultimately $u_n/u_{n+1} > 1 + p/n$, i.e. $> v_n/v_{n+1}$, where $v_n = n^{-p}$, i.e. Σu_n is convergent since Σn^{-p} is convergent when $p > 1$. Similarly may divergence be established when $B < 1$.

(iii) If $A = 1$, $B = 1$, Σu_n is divergent.

For $\log n = o(n)$ and therefore $\dfrac{1}{n^2} = o\left(\dfrac{1}{n \log n}\right)$ so that ultimately $u_n/u_{n+1} < v_n/v_{n+1}$ if $v_n = n^{-1} (\log n)^{-1}$.

4.18. The Limit Forms of the Ratio-Tests. By comparisons similar to those used in the previous paragraph we deduce that

(i) If $\lim \dfrac{u_n}{u_{n+1}}$ exists and is equal to A, Σu_n converges if $A > 1$ and diverges if $A < 1$. (*d'Alembert.*)

For $u_n/u_{n+1} = A + o(1)$.

(ii) If $\lim n\left(\dfrac{u_n}{u_{n+1}} - 1\right)$ exists and is equal to B, Σu_n converges if $B > 1$ and diverges if $B < 1$. (*Raabe.*)

For $u_n/u_{n+1} = 1 + B/n + o(1/n)$.

(iii) If $\lim \left\{n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n$ exists and is equal to K, Σu_n converges if $K > 1$ and diverges if $K < 1$. (*de Morgan and Bertrand.*)

For $u_n/u_{n+1} = 1 + 1/n + K/(n \log n) + o(1/n \log n)$.

Examples. (i) $1 + \dfrac{a.b}{1.c} + \dfrac{a(a+1).b(b+1)}{1.2.c(c+1)} + \ldots$, where c is not a negative integer. The terms are ultimately positive.

Here $\dfrac{u_n}{u_{n+1}} = \dfrac{(n+1)(c+n)}{(a+n)(b+n)}$ (taking $1 = u_0$)

$$= 1 + \frac{c - a - b + 1}{n} + O\left(\frac{1}{n^2}\right).$$

The series converges if $c > a + b$ and diverges if $c \leq a + b$.

(ii) Let $u_{n+1} = \left\{\dfrac{1.3.5. \ldots (2n-1)}{2.4.6. \ldots 2n}\right\}^p$.

Here $\dfrac{u_n}{u_{n+1}} = \left(\dfrac{2n}{2n-1}\right)^p = 1 + \dfrac{p}{2n} + O\left(\dfrac{1}{n^2}\right)$, so that Σu_n converges if $p > 2$, and diverges if $p < 2$.

Notes. (i) d'Alembert's test fails when $\lim \dfrac{u_n}{u_{n+1}} = 1$. When it fails, Raabe's test is applicable but fails also when $\lim n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = 1$. If Raabe's test fails, the test of Bertrand and Morgan may be applied, which also fails in the critical case. By continuing the set of comparison series

$\Sigma(n \log n)^{-1} (\log \log n)^{-p}$, $\Sigma(n \log n . \log \log n)^{-1} (\log \log \log n)^{-p}$, . . . tests of greater and greater precision may be obtained but no practical purpose is served by doing so. No comparison test can be universally effective.

(ii) When $\lim \dfrac{u_n}{u_{n+1}}$ does not exist, d'Alembert's test may be taken in the form

$$\Sigma u_n \text{ is convergent, if } \underline{\lim}\left(\frac{u_n}{u_{n+1}}\right) > 1 \text{ and divergent if } \overline{\lim}\left(\frac{u_n}{u_{n+1}}\right) < 1.$$

Corresponding modifications may be made to the other tests.

(iii) It is *not sufficient* for convergence that $u_n/u_{n+1} > 1$ for all values of n, since, for example, the limit of u_n/u_{n+1} may be 1. Thus if $u_n = 1/n$, $u_n > u_{n+1}$, but the series is divergent.

(iv) It is *sufficient* for divergence that $u_n < u_{n+1}$ even when $\lim (u_n/u_{n+1}) = 1$, for Σu_n is then $> nu_1$.

4.19. Summary of Convergence Tests (positive terms). The essential features of §§ 4.13–4.18 may be summarized as follows (where C denotes convergent and D divergent):

A. *Direct Comparison*

(*a*) If $u_n < v_n$ ($> v_n$) and Σv_n is $C(D)$, then Σu_n is $C(D)$ (*Theorem A.*) In particular

(b_1) If $u_n = v_n + o(v_n)$, and Σv_n is $C(D)$, then Σu_n is $C(D)$, i.e. *expand* u_n for n large

(b_2) If $\lim u_n^{\frac{1}{n}} = k$ and $k < 1$ (> 1), Σu_n is $C(D)$. (*Cauchy.*)

B. *Ratio-tests.*

(*a*) If $\dfrac{u_n}{u_{n+1}} > (<) \dfrac{v_n}{v_{n+1}}$ and Σv_n is $C(D)$, then Σu_n is $C(D)$ (*Theorem B.*) In particular

(b_1) If $\dfrac{u_n}{u_{n+1}} = A + \dfrac{B}{n} + O\left(\dfrac{1}{n^2}\right)$, and ($A > 1$) or ($A = 1$, $B > 1$), Σu_n is C, but if ($A < 1$) or ($A = 1$, $B \leq 1$), Σu_n is D. (*Bromwich.*)

i.e. *expand* $\dfrac{u_n}{u_{n+1}}$ when n is large.

(b_2) If $\lim \dfrac{u_n}{u_{n+1}} = A$, and $A > 1$ (< 1), Σu_n is $C(D)$. (*d'Alembert.*)

(b_3) If $\lim n\left\{\dfrac{u_n}{u_{n+1}} - 1\right\} = B$, and $B > 1$ (< 1), Σu_n is $C(D)$. (*Raabe.*)

(b_4) If $\lim \log n\left\{n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - 1\right\} = K$ and $K > 1$ (< 1) Σu_n is $C(D)$. (*de Morgan and Bertrand.*)

4.191. Notes on Cauchy's and d'Alembert's Tests. Cauchy's test is of greater theoretical importance than d'Alembert's, but the latter is more useful in practice. Cauchy's test, in its simpler form, fails when $\lim u_n^{\frac{1}{n}} = 1$, and a series of tests may be devised to deal with this case analogous to those obtained when d'Alembert's test fails. (*Ref. Bromwich, Infinite Series, II, 15.*)

It is to be expected that if both limits $\lim u_n^{\frac{1}{n}}$, $\lim \dfrac{u_n}{u_{n+1}}$ exist, they are equal, but we have given no reason to suppose that the existence of the one implies that of the other.

Consider, however, the series

$$u_0 + u_1c + u_2c^2 + \ldots + u_nc^n + \ldots \qquad (c > 0)$$

Let $G = \overline{\lim}\, \dfrac{u_{n+1}}{u_n}$ and $g = \underline{\lim}\, \dfrac{u_{n+1}}{u_n}$. We have already shown that the series converges when $Gc < 1$ and diverges when $gc > 1$, and no information is given by the general d'Alembert test when $1/G < c < 1/g$. Also let $H = \overline{\lim}\, u_n^{\frac{1}{n}}$ and $h = \underline{\lim}\, u_n^{\frac{1}{n}}$. The series is convergent when $HC < 1$ and divergent when $HC > 1$ (even when $hc > 1$). Thus the general Cauchy test is more exact than d'Alembert, and fails only when $cH = 1$. It follows from the above that $\dfrac{1}{G} < \dfrac{1}{H} < \dfrac{1}{g}$, i.e. $G > H > g$, and we deduce from the consideration of $\dfrac{1}{u_n}$ that also $G > h > g$. Thus

$$\underline{\lim}\, \frac{u_{n+1}}{u_n} < \underline{\lim}\, u_n^{\frac{1}{n}} < \overline{\lim}\, u_n^{\frac{1}{n}} < \overline{\lim}\, \frac{u_{n+1}}{u_n}.$$

The existence of $\lim \dfrac{u_{n+1}}{u_n}$ implies that of $\lim u_n^{\frac{1}{n}}$ and that the limits are equal; but the converse is not necessarily true.

4.192. Change in the Order of Summation (positive terms). Let Σu_n be convergent with a sum S. Let the order of the terms be changed so as to form a new series and let u_n' denote the new nth term and S_n' the sum of n terms of the new series. Every S_n' is contained in S_m for some value of m and therefore S_n' tends to a limit $S' \leq S$. Similarly $S \leq S'$. Therefore $S' = S$, or, an alteration in the order of the summation for a series of *positive* terms does not affect the sum.

4.2. Series in General. When a series contains an infinite number of terms of both signs, the comparison tests cannot be immediately applied.

4.21. Absolute Convergence. The comparison tests may, however, be applied to $\Sigma|u_n|$ and if $\Sigma|u_n|$ is convergent, so also is Σu_n.

Let $S_n = \sum\limits_1^n u_n$ and $T_n = \sum\limits_1^n |u_n|$; if T_n converges, $T_m - T_n$ is ultimately small (m, n both large); but

$$T_m - T_n = |u_{n+1}| + |u_{n+2}| + \ldots + |u_m| > |u_{n+1} + u_{n+2} + \ldots + u_m|$$

i.e. $\qquad > |S_m - S_n|$,

i.e. $|S_m - S_n|$ is small (m, n large); or S_n converges.

When $\Sigma|u_n|$ converges, Σu_n is said to be *absolutely convergent*.

Example. $1 - 2^{-p} + 3^{-p} - 4^{-p} + \ldots$ is absolutely convergent if $p > 1$, since $\sum\limits_1^\infty n^{-p}$ is convergent.

4.22. Non-absolute Convergence. Conversely, Σu_n may be convergent when $\Sigma|u_n|$ is divergent, and in this case Σu_n is said to be *non-absolutely convergent* (or *conditionally convergent* or *semi-convergent*).

4.23. Leibniz's Rule for Convergence. This is a test of frequent application to series that may not be absolutely convergent.

If (i) $a_n > a_{n+1} > 0$, (ii) $\lim a_n = 0$, then $\sum_1^\infty (-1)^{n-1} a_n$ converges.

If $S_r = \sum_1^r (-1)^{n-1} a_n$, then

$$S_{2r} = (a_1 - a_2) \ldots + (a_{2r-1} - a_{2r})$$
$$= a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2r-2} - a_{2r-1}) - a_{2r}$$

i.e. S_{2r} is a positive increasing monotone $< a_1$; also

$$S_{2r+1} = a_1 - (a_2 - a_3) - \ldots - (a_{2r} - a_{2r+1})$$
$$= (a_1 - a_2) + \ldots + (a_{2r-1} - a_{2r}) + a_{2r+1}$$

so that S_{2r+1} is a decreasing monotone > 0. Thus S_{2r}, S_{2r+1} both tend to limits as $r \to \infty$ and these limits are equal since

$$S_{2r+1} - S_{2r} = a_{2r+1}$$

which $\to 0$.

Thus the series $\sum_1^\infty (-1)^{n-1} a_n$ has a sum between 0 and a_1.

Notes. (i) The sum lies between S_{2n} and S_{2n+1} for all n.

(ii) It is, of course, sufficient that a_n should decrease *ultimately* provided that it tends to zero.

Examples. (i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges to a sum between $\frac{1}{2}$ and 1, the convergence not being absolute. (Actual value is $\log_e 2 = 0{\cdot}6931 \ldots$)

(ii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ converges to a sum between $\frac{3}{4}$ and 1, the convergence being absolute. (Actual value is $\frac{1}{12}\pi^2 = 0{\cdot}8225 \ldots$)

(iii) $a_n = \frac{n^3}{n^4 + a}$ (where no denominator vanishes). Here a_n is ultimately positive and tends to zero. Also, by finding the maximum value of the rational function $\frac{x^3}{x^4 + a}$ we deduce that a_n decreases steadily when $n^4 > 3a$. Therefore $\sum_1^\infty (-1)^{n-1} \frac{n^3}{n^4 + a}$ converges (not absolutely).

4.24. Fundamental Property of Absolutely Convergent Series. The sum of an absolutely convergent series is unaltered by a change in the order of the terms. Let Σu_n be absolutely convergent to S and denote $\sum_1^n u_n$ by S_n.

Let $p_n = u_n$ when $u_n > 0$ and $p_n = 0$ when $u_n < 0$.
Let $q_n = 0$ when $u_n > 0$ and $q_n = -u_n$ when $u_n < 0$.

Then $P_n = \sum_1^n p_n$, $Q_n = \sum_1^n q_n$ are series of *positive* terms where $S_n = P_n - Q_n$; $P_n + Q_n = T_n$ where $T_n = \sum_1^n |u_n|$, so that

$$P_n \to \tfrac{1}{2}(T + S),\ Q_n \to \tfrac{1}{2}(T - S) \text{ where } T = \sum_1^\infty |u_n|$$

Let the terms of the original series be deranged and let *accented* symbols be used for the corresponding series and terms. Then

$$\lim S_n' = \lim (P_n' - Q_n') = P - Q = S.$$

since $\Sigma p_n'$, $\Sigma q_n'$ are derangements of series of *positive terms*.

Notes. (i) It is *necessary* for absolute convergence that the series obtained by omitting all the positive or all the negative terms should be convergent.

(ii) If a series is *non-absolutely* convergent, the series described in Note (i) tend respectively to $+\infty$ and $-\infty$.

4.25. *Derangement of a Non-absolutely Convergent Series.* If the terms of such a series be deranged, the sum is in general altered. Riemann has shown that the sum can be made to have any *arbitrary* value by a suitable derangement. The series can also be made divergent or oscillatory.

Example. If $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = k$, find the sum of the series
$$1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{2} + \tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{15} - \tfrac{1}{4} + \dots$$
obtained from the first by taking 4 positive terms followed by 1 negative.

Denote the sum of n terms of the new series by S_n; then since the nth term tends to zero, S_n is convergent if S_{5n} is convergent and has the same limit.

Denote $\sum\limits_1^n (-1)^{n-1}/n$ by U_n and $\sum\limits_1^n 1/n$ by V_n.

Then $\quad V_{2n} = U_{2n} + V_n$, $V_{4n} - U_{4n} = V_{2n}$; $V_{8n} = U_{8n} + V_{4n}$

so that $\quad V_{4n} = U_{4n} + U_{2n} + V_n$; $V_{8n} - U_{8n} = U_{4n} + U_{2n} + V_n$

Now $\quad S_{5n} = 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{8n-1} - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$

$$= V_{8n} - \tfrac{1}{2}V_{4n} - \tfrac{1}{2}V_n = U_{8n} + \tfrac{1}{2}U_{4n} + \tfrac{1}{2}U_{2n}$$

i.e. $S_{5n} \to 2k$ since $U_n \to k$.

4.26. *Multiplication of Series.* Let $S_n = \sum\limits_1^n u_n$, $T_n = \sum\limits_1^n v_n$ be *absolutely* convergent to the sums S, T respectively, so that $S_nT_n \to ST$.

The series obtained by multiplying the terms of Σu_n by those of Σv_n is absolutely convergent with sum ST, since $(\sum\limits_1^n |u_n|) \times (\sum\limits_1^n |v_n|)$ converges. The terms can therefore be arranged in any order without effecting the sum. The simplest way of exhibiting the product is by means of the array

$$\begin{array}{l} u_1v_1 + u_1v_2 + u_1v_3 + \dots \\ u_2v_1 + u_2v_2 + u_2v_3 + \dots \\ u_3v_1 + u_3v_2 + u_3v_3 + \dots \\ \dots\dots\dots\dots\dots \end{array}$$

The sum 'by squares' is $\lim S_nT_n = ST$. This must be equal to the sum by any other method that includes each term once and once only. In particular, it is equal to the sum 'by diagonals'

$$u_1v_1 + (u_1v_2 + u_2v_1) + (u_1v_3 + u_2v_2 + u_3v_1) + \dots$$

Example. Prove that $(\sum\limits_0^\infty q^n) \times (\sum\limits_0^\infty (n+1)q^n) = \sum\limits_0^\infty \frac{(n+1)(n+2)}{1.2} q^n$ when $|q| < 1$.

By the ratio-test it is easily shown that the series are all absolutely convergent when $|q| < 1$ and otherwise are not convergent. Arrangement by diagonals of the product on the left gives the coefficient of q^n to be
$$1 + 2 + 3 + \dots + (n+1) = \tfrac{1}{2}(n+1)(n+2).$$

Notes. The equation $(\Sigma u_n) \times (\Sigma v_n) = (\Sigma w_n)$, where
$$w_n = u_1v_n + u_2v_{n-1} + \dots + u_nv_1$$
can be proved true under less restrictive conditions than those given above. In particular, it has been shown by

(i) *Abel*: that $(\Sigma u_n) \times (\Sigma v_n) = (\Sigma w_n)$ if $\Sigma u_n, \Sigma v_n, \Sigma w_n$ are *convergent* (see § *4.37* (ii).)

(ii) *Mertens*: that $(\Sigma u_n) \times (\Sigma v_n) = (\Sigma w_n)$ if *one of the series* $\Sigma u_n, \Sigma v_n$ *is absolutely* convergent and the other *convergent.*

(iii) *Pringsheim*: that $(\Sigma(-1)^{n-1}u_n) \times (\Sigma(-1)^{n-1}v_n) = (\Sigma(-1)^{n-1}w_n)$, when u_n, v_n *are monotones* decreasing to zero limit, provided $\Sigma u_n v_n$ *is convergent.*

4.3. Functions defined by Power Series. The series $\sum_0^n a_n x^n$ is called a *Power Series* and may be regarded as defining a function $F(x)$ for those values of x for which the series converges.

Note. Although $F(x)$ may initially be defined in this way, it is often possible to *continue* the meaning of $F(x)$ beyond the domain of convergence of the series. For example, $\sum_0^\infty x^n$ defines a function only for $|x| < 1$, but we can prove it equal to $(1 - x)^{-1}$ which is defined for all values of x, except $x = 1$.

4.31. Domain of Convergence of a Power Series. By d'Alembert's test the series is absolutely convergent if $\lim \left|\dfrac{a_n}{a_{n+1}x}\right|$ exists and is greater than 1. Let $\lim \left|\dfrac{a_n}{a_{n+1}}\right| = R$. Then $\Sigma a_n x^n$ is absolutely convergent when $|x| < R$

It is *not* convergent for $|x| > R$. For if $\lim \dfrac{a_n}{a_{n+1}} = R$, then (§ 4.191) $\lim |a_n{}^{\frac{1}{n}}| = 1/R$ and $|a_n x^n|^{\frac{1}{n}} \to R_1/R$ when $|x_1| = R_1$, i.e. there is an infinity of terms > 1 when $R_1 > R$.

The series may or may not converge when $x = \pm R$ and more exact tests must be applied. The number R is called the *Radius of Convergence*; and the domain of convergence consists of the interval $-R < x < R$ and *possibly* $x = R$, $x = -R$.

Note. When $\lim |a_n/a_{n+1}|$ does not exist, *d'Alembert's* test does not give the radius of convergence. However, Cauchy's test, in its general form, shows that $R = \overline{\lim} |a_n|^{\frac{1}{n}}$.

4.32. Substitution of a Polynomial in a Power Series. Let

$$x = b_0 + b_1\xi + b_2\xi^2 + \dots + b_m\xi^m$$

be substituted in $\Sigma a_n x^n$ (of radius of convergence R). It is legitimate to arrange this as a power series in ξ at least when this series, written out *at length*, is *absolutely* convergent. The rearrangement is therefore correct at least if

$$|b_0| + |b_1|\,|\xi| + \dots + |b_m|\,|\xi|^m < R$$

and for this it is *necessary* that $|b_0| < R$. The inequality is then certainly satisfied when $|\xi| < k$ where k is some positive number. It may be expected, however, that the greatest value k obtained in this way is less than what is actually necessary for the correctness of the rearrangement.

Note. It may be proved by the principle of analytic continuation that if both

$\sum_0^\infty a_n(b_0 + b_1\xi + \dots + b_m\xi^m)^n$ and the rearranged series are convergent, then the rearrangement is legitimate. (*Chap. X, 10.72.*)

Example. $1/(1 + x + x^2) = \sum_0^\infty (-1)^n(x + x^2)^n$.

When rearranged the series is $1 - x + x^2 - x^4 + x^5 - x^7 \dots$ and the expansion is legitimate at least when $|x| + |x|^2 < 1$, i.e. when $|x| < 0.62$ (approx.). Actually, however, $1/(1 + x + x^2) = (1 - x)/(1 - x^3)$ when $x \neq 1$ and therefore $1/(1 + x + x^2) = (1 - x)(1 + x^3 + x^6 + \dots) = 1 - x + x^3 - x^4 + x^6 - x^7 \dots$ for $|x| < 1$.

4.33. Power Series obtained by Term-by-Term Differentiation. If we differentiate term-by-term we obtain new functions $F_1, F_2, F_3 \dots$ defined by the power-series

$$F_1(x) = \sum_0^\infty (n + 1)a_{n+1}x^n; \quad F_2(x) = \sum_0^\infty (n + 1)(n + 2)a_{n+2}x^n; \quad \dots$$

$$F_r(x) = \sum_0^\infty (n + 1)(n + 2) \dots (n + r)a_{n+r}x^n.$$

The radius of convergence of $F_r(x)$ is $\lim\left(\frac{n+1}{n+r+1}\left|\frac{a_n}{a_{n+1}}\right|\right) = R$, (since r is fixed).

Notes. (i) The radii of convergence of $F, F_1, F_2, \dots$ are all equal even when $\lim\left|\frac{a_n}{a_{n+1}}\right|$ does not exist.

(ii) The series for $F_1, F_2, \dots$ need not be convergent at $x = \pm R$, even when the series for F is convergent at $x = R$ or $-R$.

4.34. The Continuity of a Power Series. If $F(x) = \sum_0^\infty a_nx^n$, then

$$F(x) = a_0 + xG(x)$$

where $G(x)$ is *bounded* when $|x| < R$. $F(x)$ is therefore *continuous* at $x = 0$, $(R \neq 0)$ and tends to the value a_0.

Let $x = x_0 + h$, where $|x_0| < R$ and $|x_0 + h| < R$ and let $|h|$ be less than $R - |x_0|$ (> 0).

Now $F(x_0 + h) = \sum_0^\infty a_n(x_0 + h)^n$ and the series when written out at length is *absolutely* convergent since $\sum_0^\infty |a_n|(|x_0| + |h|)^n$ is convergent. It may therefore be arranged in powers of h without altering its value. The coefficient of h^n in the rearrangement is

$$a_n + a_{n+1}(n + 1)x_0 + a_{n+2}\frac{(n + 1)(n + 2)}{1 \cdot 2}x_0^2 + \dots, \text{ i.e. } = \frac{1}{n!}F_n(x_0).$$

Thus $F(x_0 + h) = F(x_0) + hF_1(x_0) + \dots + \frac{h^n}{n!}F_n(x_0) + \dots$ for at least the interval $|h| < R - |x_0|$. This power series in h is continuous at $h = 0$, since $R - |x_0| > 0$, i.e. $F(x)$ is continuous at $x = x_0$ and has the value $F(x_0)$.

4.35. Abel's Theorem on the Continuity of a Power Series. The previous paragraph has established the continuity of $F(x)$ *within* the interval

of convergence. Abel's Theorem gives the condition for continuity at the *ends* of the interval.

If $F(R)$ is convergent, then $F(x) \to F(R)$ when $x \to R$. Similarly $F(x) \to F(-R)$ when $x \to -R$ if $F(-R)$ is convergent. Since we may write Rx' for x, it is sufficient to take *unity* for the radius of convergence.

Let therefore *unity* be the radius of convergence of $F(x) = \sum\limits_0^\infty a_n x^n$ and let $\sum\limits_0^\infty a_n$ be convergent (not necessarily absolutely). It is required to prove that $\lim\limits_{x \to 1\,0} \sum\limits_0^\infty a_n x^n = \sum\limits_0^\infty a_n$.

Let $F(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + \rho_n(x)$
and $\quad F(1) = a_0 + a_1 + a_2 + \ldots + a_{n-1} + r_n$,
where $\rho_n(x) = a_n x^n + a_{n+1} x^{n+1} + \ldots$; $r_n = a_n + a_{n+1} + \ldots$.

Since Σa_n converges, we can find n_0 such that $|r_n| < \varepsilon$ (*all* $n > n_0$).

Since $a_n = r_n - r_{n+1}$,

$$\rho_n(x) = r_n x^n - (1 - x)(r_{n+1} x^n + r_{n+2} x^{n+1} + \ldots)$$

and therefore $|\rho_n(x)| < \varepsilon + (1 - x)\varepsilon(x^n + x^{n+1} + \ldots)$ for $n > n_0$ and $0 < x < 1$, i.e.

$$|\rho_n(x)| < \varepsilon + \varepsilon x^n < 2\varepsilon.$$

Now $F(x) - F(1)$

$$= a_1(x - 1) + a_2(x^2 - 1) + \ldots + a_{n-1}(x^{n-1} - 1) + \rho_n(x) - r_n.$$

But given ε, we can find δ, such that $\left|\sum\limits_1^{n-1} a_r(x^r - 1)\right| < \varepsilon$ for all x such that $1 - \delta < x < 1$, since $\sum\limits_1^{n-1} a_r(x^r - 1)$ is a polynomial vanishing at $x = 1$. Also $|\rho_n(x)| < 2\varepsilon$ and $|r_n| < \varepsilon$,

i.e. $\qquad |F(x) - F(1)| < 4\varepsilon$ for $1 - \delta < x < 1$,

i.e. $\qquad F(x) \to F(1)$ when $x \to 1$ from the *left*

Similarly $F(x) \to F(-1)$ when $x \to -1$ from the *right* if $F(-1)$ converges.

4.36. The Derivatives of a Power Series. If $F(x) = \sum\limits_0^\infty a_n x^n$ and $-R < x_0 < R$,

$$\left\{\frac{F(x_0 + h) - F(x_0)}{h}\right\} = F_1(x_0) + \frac{h}{2!}F_2(x_0) + \frac{h^2}{3!}F_3(x_0) + \ldots, \qquad (\S 4.34)$$

where the series on the right is a power series in h with a *non-zero* interval of convergence equal at least to $R - |x_0|$. It is therefore continuous at $h = 0$, i.e. $F'(x) = F_1(x)$, or the first derivative (and similarly any higher derivative) is obtained by term-by-term differentiation.

Note. If $F_1(R)$ converges it is the derivative of $F(x)$ on the *left* of $x = R$, and $F_1(-R)$, if it converges, is the derivative on the *right* of $x = -R$. (By Abel's Theorem.)

4.37. Multiplication of Power Series. If $F(x) = \sum\limits_0^\infty a_n x^n$, $(|x| < R_1)$, $G(x) = \sum\limits_0^\infty b_n x^n$, $(|x| < R_2)$, then $F(x)G(x) = \sum\limits_0^\infty c_n x^n$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

and $|x|$ is less than the smaller of R_1, R_2. This follows from the fact that $F(x)$, $G(x)$ are *absolutely* convergent *within* their intervals.

Notes. (i) If $F(x)$ converges for R_1 (the *smaller* of R_1, R_2), and the product series converges for R_1, then the result is true for R_1, by Abel's Theorem.

(ii) If unity is the common radius of convergence, we have Abel's Theorem on the Multiplication of Series: $(\Sigma a_n) \times (\Sigma b_n) = (\Sigma c_n)$ if all three series converge.

(iii) If unity is the radius of convergence of $\Sigma a_n x^n$ and $R(=1)$ is the radius of convergence of $\Sigma b_n x^n$, then *Merten's Theorem* (§ 4.26) shows that $(\Sigma a_n) \times (\Sigma b_n) = (\Sigma c_n)$ for then (Σb_n) is *absolutely* convergent.

Example. The series $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$ is convergent for $-1 < x \leq 1$. The series obtained by squaring is $\sum\limits_2^\infty (-1)^n a_n x^n$ where

$$a_n = \frac{1}{1.(n-1)} + \frac{1}{2(n-2)} + \ldots + \frac{1}{(n-1).1}$$

$$= \frac{1}{n}\left(1 + \frac{1}{n-1} + \frac{1}{2} + \frac{1}{n-2} + \ldots + \frac{1}{n-1} + 1\right)$$

$$= \frac{2}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\right)$$

Thus $(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots)^2 = x^2 - \frac{2}{3}x^3(1 + \frac{1}{2}) + \frac{2}{4}x^4(1 + \frac{1}{2} + \frac{1}{3}) - \ldots$ for $-1 < x \leq 1$, since the series on the right is convergent for $x = 1$, (*by Leibniz's rule*).

4.38. Identity of Two Power Series. If $F(x) = \sum\limits_0^\infty a_n x^n$ is identically zero for all values of x in a *non-zero* interval, then all the coefficients must vanish; for $F(x)$ and all its derivatives must vanish at $x = 0$,

i.e. $$a_0 = a_1 = a_2 = \ldots = 0.$$

Similarly, if it is known that $\sum\limits_0^\infty a_n x^n = \sum\limits_0^\infty b_n x^n$ for a non-zero interval then $a_n = b_n$ for all values of n.

4.39. Taylor's Expansion for a Power Series. If x_0, $(x_0 + h)$ are within the interval of convergence of $\sum\limits_0^\infty a_n x^n = F(x)$, then

$$F(x_0 + h) = \sum\limits_0^\infty a_n (x_0 + h)^n$$

and it has already been shown that the coefficient of h^n in the rearrangement is $F^{(n)}(x_0)/n!$.

i.e. $$F(x_0 + h) = F(x_0) + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \ldots + \frac{h^n}{n!}F^{(n)}(x_0) + \ldots$$

so that the *infinite* Taylor expansion is valid for $F(x_0 + h)$ for at least the interval $|h| < R - |x_0|$.

4.391. Entire (or Integral) Functions. If $\lim \left|\dfrac{a_n}{a_{n+1}}\right|$ is infinite, the

function $\sum_{0}^{\infty} a_n x^n$ is defined for *all* finite values of x and is called an *Entire* (or *Integral*) Function. It possess all derivatives and its derivatives are entire functions.

Example. $1 - \dfrac{x^2}{(1!)^2} + \dfrac{x^4}{(2!)^2} - \dfrac{x^6}{(3!)^2} + \ldots$

Regard this as a power series in x^2; then $\lim \left|\dfrac{a_n}{a_{n+1}}\right| = \lim \left(\dfrac{(n+1)!}{n!}\right)^2$ which is infinite. This function is therefore entire.

4.4. The Elementary Transcendental Functions. The elementary transcendental functions are usually taken to be

(i) The *Exponential Function* and its inverse, i.e. the *Logarithmic Function*, together with the related *Hyperbolic Functions* and their inverses.

(ii) The *Circular* Functions and their inverses. It is not our intention here to develop all the well-known properties of these functions, as it will be presumed that these are known to the reader. However, it is necessary to show how these functions may be adequately *defined* in terms of the ideas that have been introduced in the previous paragraphs and to indicate how their properties may be established.

4.41. The Exponential Function $E(x)$. The *exponential function* may be defined by the power series

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

Since $\lim \dfrac{(n+1)!}{n!}$ is infinite, it is defined for *all* finite x.

(i) Its characteristic property $E(x) \cdot E(y) = E(x+y)$ follows by the rule for multiplication of series, and by repeated applications of this result we find that $E(n) = e^n$ where $e = E(1)$ ($= 2{\cdot}718 \ldots$) (n integral). The number $e^{p/q}$ may then be identified with $E(p/q)$ so that $E(x) = e^x$ for x *rational*.

The number e^x for x irrational is naturally defined to be $\lim\limits_{n\to\infty} e^{x_n}$ (if this exists) where x_n is any sequence of rational numbers tending to x.

But $\lim\limits_{n\to\infty} e^{x_n} = \lim\limits_{n\to\infty} E(x_n) = E(\lim x_n) = E(x)$ since $E(x)$ is *continuous*.

Thus it is consistent to write $E(x) = e^x$ for *all* x.

Similarly $\{E(x)\}^a$ when a is irrational is defined to be $\lim \{E(x)\}^{a_n}$ where a_n is any sequence of rational numbers tending to a, and this is easily shown to be $E(ax)$ or e^{ax}.

(ii) Differentiation term-by-term shows that $\dfrac{d^n}{dx^n}(e^x) = e^x$ for all values of n.

(iii) The function e^x is obviously > 0 for $x > 0$, and since $e^{-x} = \dfrac{1}{e^x}$,

then $e^x > 0$ for *all* x. Also since $\frac{d}{dx}(e^x) = e^x$, the function increases steadily for *all* x; when $x \to +\infty$, $e^x \to +\infty$, and when $x \to -\infty$, $e^x \to 0$. (*Fig. 1.*)

4.42. The Logarithmic Function $\log x$. If $e^y = x$, as y increases steadily from $-\infty$ to $+\infty$, x increases steadily from 0 to $+\infty$; the

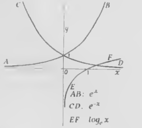


FIG. 1

relation therefore determines y as a single valued continuous function of x for $x > 0$. The function y is denoted by $\log x$, and since $\frac{dx}{dy} = x$, it follows that $\frac{d}{dx}(\log x) = \frac{1}{x}$ (*Fig. 1.*)

4.43. The Function a^x. Let $a > 0$, and let $b = \log a$, then

$$a^x = e^{bx} = e^{x \log a}$$

thus defining a^x for $a > 0$ and all x. Also

(i) The derivative of a^x is $a^x \log a$.

(ii) The derivative of x^n is $\frac{d}{dx}(e^{n \log x}) = nx^{n-1}$ for all those values of a, x, n for which the functions have been defined.

4.44. The Logarithmic Scale Since $e^x > x^{m+1}/(m+1)!$, $(x > 0)$ when m is a fixed positive integer, however large, $e^x/x^\alpha \to +\infty$ when $x \to +\infty$ (α any real number).

Also $e^{-x}x^\alpha \to 0$ when $x \to +\infty$, α being any real number.

It follows that $u/(\log u)^\beta \to +\infty$ when $u \to +\infty$, (any β).

Taking $\beta > 0$, and writing α for $1/\beta$ and x for u, we deduce that $(\log x)/x^\alpha \to 0$ when $x \to +\infty$ however small α (> 0) may be

Writing now $1/x$ for x, we obtain finally that $x^\alpha \log x \to 0$ when $x \to 0$ from the right, however small $\alpha\ (> 0)$ may be.

Summarizing:

(i) $e^x/x^\alpha \to +\infty$; $e^{-x}x^\alpha \to 0$, when $x \to +\infty$, (*all* α)
(ii) $(\log x)/x^\alpha \to 0$ when $x \to +\infty$, $(\alpha > 0)$.
(iii) $x^\alpha (\log x) \to 0$ when $x \to +0$, $(\alpha > 0)$.

Thus e^x, and therefore e^{kx}, $(k > 0)$ increases to $+\infty$ more rapidly than *any* power of x, and therefore more rapidly than any polynomial; and $\log x$ increases to infinity more slowly than any positive power of x and therefore more slowly than any polynomial.

A set of functions, called the *logarithmic scale*, may therefore be constructed as indicated in the following scheme

$$\ldots \prec \log\log\log x \prec \log\log x \prec \log x \prec x \prec e^x \prec e^{e^x} \ldots$$

which all tend to $+\infty$ when $x \to +\infty$; each function is the log of the one that follows it and tends to infinity more slowly than the one that follows it, i.e. such that the ratio tends to zero.

Note. We use the notation $f(x) \prec \phi(x)$ when $f(x) = o\{\phi(x)\}$ for x large.

Other Functions may be inserted in the logarithmic scale. Thus

$$\log x \prec (\log x)^2 \prec (\log x)^3 \ldots \prec x \prec x^2 \prec x^3 \ldots$$

Example. Determine the relative positions of the functions $x^{\log x}$, $(\log x)^x$, $(\log x)^{\log\log x}$ in the logarithmic scale.

$\log x \prec (\log x)^2 \prec x$. Therefore $x \prec x^{\log x} \prec e^x$.

Also $e^x \prec (\log x)^x$, but since $x \log\log x \prec x^2 \prec e^x$, it follows that $(\log x)^x \prec e^{e^x}$.

Again $(\log\log x)^2 \prec \log x$ and therefore $(\log x)^{\log\log x} \prec x$.

Thus $\log x \prec (\log x)^{\log\log x} \prec x \prec x^{\log x} \prec e^x \prec (\log x)^x \prec e^{e^x}$.

§ 45. *The Expansion of* $\log(1 + x)$, (x *small*). If $f(x) = \log(1 + x)$, $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ and Maclaurin's expansion gives

$$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots + (-1)^{n-1}\frac{x^n}{n} + R_n, \text{ where}$$

$$|R_n| = \frac{|x|^{n+1}}{(n+1)|(1 + \theta x)|^{n+1}}, \ (0 < \theta < 1).$$

If n is *fixed* and x is small,

$$R_n = O(x^{n+1}) \text{ and therefore}$$

$$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots + (-1)^{n-1}\frac{x^n}{n} + O(x^{n+1}), \ (x \text{ small}).$$

In particular,

$$\lim_{x\to 0} \frac{1}{x^{n+1}}\left(\log(1 + x) - x + \frac{1}{2}x^2 - \ldots + (-1)^n \frac{x^n}{n}\right) = \frac{(-1)^n}{(n+1)}.$$

Notes. (i) It is easy to see that if $|x| < 1$, $R_n \to 0$ when $n \to \infty$, and therefore the *infinite* series for $\log(1 + x)$, (which is convergent for $|x| \leq 1$) is valid for $|x| < 1$. It may also be shown valid for $x = 1$, but it is simpler to obtain the *infinite* series by integration. (*See Chap. V, § 5.71.*)

(ii) Since $\frac{1}{x}\log(1 + x) \to 1$ when $x \to 0$, it follows that $(1 + x)^{\frac{1}{x}} \to e$ when

$x \to 0$. In particular $\left(1 + \frac{1}{n}\right)^n \to e$ when $n \to \infty$, and it is by means of this (*the exponential limit*), that the exponential function is sometimes defined and its properties developed. It should be noted also that

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e;\ \lim_{x\to 0}(1+ax)^{\frac{1}{x}} = \lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^x = e^a.$$

4.46. The Hyperbolic Functions and their Inverses. (a) The *hyperbolic functions* $\cosh x$, $\sinh x$ are most simply defined by the equations: $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$, and the other hyperbolic functions by the equations $\tanh x = S/C$, $\coth x = C/S$, $\operatorname{sech} x = 1/C$, $\operatorname{cosech} = 1/S$, ($S = \sinh x$, $C = \cosh x$).

Notes. (i) The functions $\cosh x$, $\operatorname{sech} x$ are *even*, the others are *odd* and they have a relation to the rectangular hyperbola analogous to that of the circular functions to the circle. Many of their properties are analogous to those of the circular functions, but the analytical relationship between the hyperbolic and the circular functions requires the use of the complex variable for its expression. Their graphs are easily obtained from a knowledge of those of e^x and e^{-x} (*Fig. 2*).

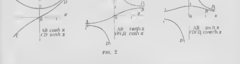


FIG. 2

(ii) The most useful of their simpler properties are
$\cosh^2 x - \sinh^2 x = 1$, $1 - \tanh^2 x = \operatorname{sech}^2 x$; $\sinh 2x = 2\sinh x\cosh x$;
$\cosh 2x = \cosh^2 x + \sinh^2 x$; $\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$,
$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x\sinh y$, $\cosh x \geq 1$, $|\sinh x| < \cosh x$,
$\lim_{x\to\infty}(\tanh x) = 1$, $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$, $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$
(the infinite series being convergent for *all* x), $\cosh 0 = 1$; $\sinh 0 = 0$,

$$\frac{d}{dx}(\sinh x) = \cosh x,\ \frac{d}{dx}(\cosh x) = \sinh x;\ y = A\cosh mx + B\sinh mx$$

satisfies the equation $y'' = m^2 y$.

(b) If $x = \cosh y$, x cannot be less than 1, and for $x > 1$, there are *two* values of y for a given x, as may be seen from the graph. The *positive* value y_1 is taken to be $\cosh^{-1} x$ (or arg $\cosh x$). The other value is therefore $-\cosh^{-1} x$.

By solving the equation $2x = e^y + e^{-y}$ for e^y we find that

$$y_1, y_2 = \log(x \pm \sqrt{(x^2-1)})$$

so that $\cosh^{-1} x = \log(x + \sqrt{(x^2-1)})$ and $\frac{d}{dx}(\cosh^{-1} x)$ is equal to
$1/\sqrt{(x^2-1)}$

Similarly, $x = \sinh y$, determines a *single*-valued function

$$y = \sinh^{-1} x = \log(x + \sqrt{(x^2+1)})$$

where

$$\frac{dy}{dx} = \frac{1}{\sqrt{(x^2+1)}}.$$

And $x = \tanh y$, (for $|x| < 1$), determines a *single*-valued function $y = \tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}$ where $\frac{dy}{dx} = \frac{1}{1-x^2}$.

4·47. The Circular (or Trigonometric) Functions and their Inverses. The elementary method of establishing the properties of $\sin x$, $\cos x$ involve the assumption that a circular arc has a *length* which provides the *measure* of the angle subtended. Various ways may be suggested for using these properties to provide *new* definitions that are strictly arithmetical in character. From our present point of view, the most suitable method consists in using *infinite power series* as definitions, and Maclaurin's expansion enables us to obtain what these series must be. Thus, assuming that $d^n(\cos x)/dx^n = \cos(x + \frac{1}{2}n\pi)$, we easily obtain

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + R, \text{ where}$$

$$R = \left(\frac{x^{2n+1}}{(2n+1)!}\right)\cos(\theta x + \tfrac{1}{2}(n+1)\pi), \ (0 < \theta < 1).$$

But $|\cos(\theta x + \frac{1}{2}(n+1)\pi)| \le 1$, and $\frac{x^{2n+1}}{(2n+1)!} \to 0$ when $n \to \infty$, therefore $R \to 0$ when $n \to \infty$ for all finite x,

i.e.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

and similarly $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ for all finite x. These power series (which, like e^x, are *entire* functions) may be taken as *new* definitions of $\cos x$, $\sin x$.

Notes. (i) For the problem of re-establishing the properties of these functions, read *Whittaker and Watson, Modern Analysis, Appendix.*

(ii) Another effective method of defining $\sin x$ ($= S(x)$), $\cos x$ ($= C(x)$) consists in using the relations $S' = C$, $C' = -S$, $S(0) = 0$, $C(0) = 1$. (*Ref. Bromwich, Infinite Series*, § 60.)

The *graphs* of the circular functions should be familiar to the reader. They are useful, for example, when dealing with the inverse functions *arc sin x, arc cos x, arc tan x.*

Thus (*a*) $y = \arcsin x$, ($|x| < 1$), is defined to be that value of y satisfying the equation $x = \sin y$ and also the inequality $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$. The other values are $n\pi + (-1)^n \arcsin x$, (n being a positive or negative integer) and $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{(1-x^2)}}$.

(*b*) $y = \arccos x$, ($|x| < 1$), is that value y satisfying the relation

$x = \cos y$ and also the inequality $0 \leq y \leq \pi$; the other values are $2n\pi \pm \text{arc cos } x$ and $\frac{d}{dx}(\text{arc cos } x) = -\frac{1}{\sqrt{(1-x^2)}}$ so that

$$\text{arc sin } x + \text{arc cos } x = \tfrac{1}{2}\pi.$$

(c) $y = \text{arc tan } x$ is that value y that satisfies the relation $x = \tan y$, and also the inequality $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$; the other values are

$$n\pi + \text{arc tan } x$$

and
$$\frac{d}{dx}(\text{arc tan } x) = \frac{1}{1+x^2}.$$

Also
$$\lim_{x \to +\infty} \text{arc tan } x = \tfrac{1}{2}\pi; \quad \lim_{x \to -\infty} \text{arc tan } x = -\tfrac{1}{2}\pi.$$

Note arc sec x = arc cos $(1/x)$, arc cosec x = arc sin $(1/x)$, arc cot x = arc tan $(1/x)$

4.5. Functions defined by Multiple Sequences. Functions of several variables $x_1, x_2, \ldots, x_r$ may also be defined by the limits of sequences of the sth order for those values $x_1, x_2, \ldots, x_r$ for which the multiple limits exist Thus $F(x_1, x_2, \ldots, x_r)$ may be defined as $\lim f(x_1, x_2, \ldots x_r, n_1, n_2, \ldots, n_s)$ where $n_1, n_2, \ldots, n_s$ are integers that tend to infinity independently. Definitions could also be made by limits of functions of the type $f(x_1, \ldots, x_r, \xi_1, \ldots, \xi_s)$ where ξ_m tends to infinity by continuous real variation. In many cases, however, such a definition is not more general than that obtained by integral variation, and in any case the function f may have significance only for integral values ξ_m.

4.51. Functions defined by Double Series. From the terms of a double sequence $u(m, n, x, y, z, \ldots)$ we can form the double sequence

$$S_{mn} = S(m, n, x, y, z, \ldots) = \sum_{r=1}^{n}\sum_{s=1}^{m} u(r, s, x, y, z, \ldots)$$

and if S_{mn} tends to a limit $F(x, y, z, \ldots)$ when m, n tend independently to infinity, $F(x, y, z, \ldots)$ is called the sum (Pringsheim) of the infinite double series.

Writing m, n as suffixes, we may exhibit the double series as follows:

$$\begin{array}{l} u_{11} + u_{12} + u_{13} + \ldots + u_{1n} + \ldots \\ + u_{21} + u_{22} + u_{23} + \ldots + u_{2n} + \ldots \\ + u_{31} + \ldots \qquad \ldots \ldots \\ + \ldots \qquad \ldots \qquad \ldots \ldots \\ + u_{m1} + u_{m2} + \ldots \quad \ldots + u_{mn} + \\ + \ldots \qquad \ldots \qquad \ldots \ldots \end{array}$$

It is obviously *necessary* (but not sufficient) for convergence that $\lim u_{mn} = 0$.

The *necessary and sufficient condition* for convergence is that $|S_{pq} - S_{mn}|$ should be ultimately small

4.52. Repeated Series. If the mth row is summed, $(m = 1, 2, 3, \ldots)$ and if the sum of these sums is taken, the result may be written

$\sum\limits_{m}\sum\limits_{n} u_{mn}$ and is called the *sum by rows*. Similarly $\sum\limits_{n}\sum\limits_{m} u_{mn}$ denotes the *sum by columns*. It is not true in general that $\sum\limits_{m}\sum\limits_{n} u_{mn} = \sum\limits_{n}\sum\limits_{m} u_{mn}$ even when the double series converges in the Pringsheim sense.

Note. It is often convenient to use the symbol $\sum\limits_{mn} u_{mn}$ for the sum of the infinite double series.

Example.

$$\begin{array}{l} 1 - 2 + 3 - 4 + 5 + \cdots \\ + 2 - 5 + 3 - 4 - 5 - \cdots \\ + 3 - 3 + 0 - 0 + 0 + \cdots \\ + 4 - 4 + 0 - 0 + 0 + \cdots \\ \cdots\cdots\cdots\cdots \end{array}$$

Here $S_{mn} \to 0$, but the sum by rows and the sum by columns are not convergent, since, for example, $\sum\limits_{n} u_{1n}$ and $\sum\limits_{m} u_{m1}$ tend to $+\infty$.

Again, $\sum\limits_{m}\sum\limits_{n} u_{mn}$ may be equal to $\sum\limits_{n}\sum\limits_{m} u_{mn}$ when the double series is not convergent.

Example.

$$\begin{array}{l} 1 + 1 + 0 + 0 + 0 + 0 + \cdots \\ 1 - 1 + 0 + 0 + 0 - 0 + \cdots \\ + 0 + 0 + 1 - 1 + 0 - 0 + \cdots \\ + 0 + 0 - 1 + 1 + 0 - 0 + \cdots \\ 0 + 0 + 0 + 0 + 1 - 1 + \cdots \\ 0 + 0 - 0 + 0 - 1 + 1 + \cdots \end{array}$$

Here $\sum\limits_{m}\sum\limits_{n} u_{mn} = 2 = \sum\limits_{n}\sum\limits_{m} u_{mn}$, but S_{mn} is either 3 or 2 and is therefore not convergent.

4.53. Pringsheim's Theorem on Double Series. If the rows and the columns converge and if the double series converges to S, then

$$\sum_{m}\sum_{n} u_{mn} = \sum_{n}\sum_{m} u_{mn} = S.$$

For, since $\lim\limits_{n} S_{mn}$ exists, $|\lim\limits_{n} S_{mn} - S_{mn}|$ is ultimately small, (all m and n large); and since $\lim\limits_{mn} S_{mn} = S$, then $|S - S_{mn}|$ is small, (m, n large): i.e. $|\lim\limits_{n} S_{mn} - S|$ is small, (m large), i.e. $\lim\limits_{m}\lim\limits_{n} S_{mn} = S$. Similarly $\lim\limits_{n}\lim\limits_{m} S_{mn} = S$.

4.54. Double Series of Positive Terms. By a method similar to that used for simple series of positive terms, we can show that a double series of positive terms, if bounded, must converge and if unbounded, must diverge to $+\infty$. Also its sum, when convergent, is independent of the mode of summation, provided every term is included once and once only. In particular, the sum may be effected by *rows*, by *columns*, by *diagonals*, or by *rectangles*.

A sum by rectangles is $\sum\limits_{n} S_{mn}$ where $m = \phi(n)$ and $\phi(n)$ tends steadily to $+\infty$ when $n \to +\infty$ for example, if $m = kn$, (k fixed).

Also if the series converges by any such method, it converges to the same sum by any other appropriate method.

4.55. Tests for Convergence for a Double Series of Positive Terms. Tests for convergence (or divergence) may be established by direct comparison with a known series of positive terms. For

(i) If $0 < u_{mn} \leq v_{mn}$ and $\sum\limits_{mn} v_{mn}$ converges, then $\sum\limits_{mn} u_{mn}$ converges, and

(ii) If $u_{mn} \geq v_{mn} \geq 0$ and $\sum\limits_{mn} v_{mn}$ diverges, then $\sum\limits_{mn} u_{mn}$ diverges. In particular (all the terms being positive),

(i) If $\sum\limits_{n} C_n$ converges and $u_{mn} < C_m C_n$, then $\sum\limits_{mn} u_{mn}$ converges; for $\sum\limits_{mn} u_{mn} < \sum\limits_{mn} C_m C_n$ which converges since $(\sum\limits_{m} C_m)(\sum\limits_{n} C_n) = (\sum\limits_{n} C_n)^2$.

(ii) If $\sum\limits_{n} C_n$ converges and $u_{mn} < \dfrac{C_{m+n}}{m+n}$, then $\sum\limits_{mn} u_{mn}$ converges; for

$\sum\limits_{mn} u_{mn} = \sum\limits_{n} (u_{n1} + u_{n-1,2} + \ldots + u_{1n}) < \sum\limits_{n} \dfrac{n}{n+1} C_{n+1} < \sum\limits_{n} C_{n+1}$.

(iii) If $\sum\limits_{n} A_n$ converges (or diverges) and $\sum\limits_{n} D_n$ diverges and

$$u_{mn} > A_m D_n$$

then $\sum\limits_{mn} u_{mn}$ diverges; for $\sum\limits_{mn} u_{mn} > \sum\limits_{mn} A_m D_n > (\sum\limits_{m} A_m)(\sum\limits_{n} D_n)$ which diverges.

(iv) If $\sum\limits_{n} D_n$ diverges and $u_{mn} > \dfrac{D_{m+n}}{m+n}$, then $\sum\limits_{mn} u_{mn}$ diverges, for the sum by diagonals of $\sum\limits_{mn} u_{mn}$ is $> \sum\limits_{n} \dfrac{n}{n+1} D_{n+1} \geq \dfrac{1}{2} \sum D_{n+1}$.

Examples. (i) The simple series $\sum\limits_{m} \dfrac{1}{m^\alpha}$, $\sum\limits_{n} \dfrac{1}{n^\beta}$ converge when $\alpha > 1$, $\beta > 1$ and diverge when $\alpha \leq 1$, $\beta \leq 1$. Therefore $\sum\limits_{mn} \dfrac{1}{m^\alpha n^\beta}$ converges when $\alpha > 1$, $\beta > 1$ but diverges when one at least of the numbers α, β is less than or equal to 1.

(ii) The series $\sum\limits_{mn} \dfrac{1}{m^\alpha + n^\alpha}$. Since $m^\alpha + n^\alpha \geq 2m^{\alpha/2}n^{\alpha/2}$, the series converges when $\alpha > 2$.

And since $m^\alpha + n^\alpha \leq (m + n)^\alpha$ (when $\alpha \geq 1$), the sum by diagonals is greater than $\sum\limits_{n} \dfrac{1}{n^{\alpha-1}}$ which diverges if $\alpha < 2$.

Also when $\alpha \leq 1$, the sum by rows (or columns) obviously diverges.

4.56. Example of a Double Series. An example of a fairly comprehensive type is given by $\displaystyle\sum_{mn} \frac{1}{p_{mn}}$ where $p_{mn} = \displaystyle\sum_{r=1}^{r=s} a_r m^{\alpha_r} n^{\beta_r}$, $\alpha_r > 0$, $\beta_r > 0$, and $p_{mn} > 0$ (all large m, n).

Draw Newton's polygon for p_{mn}, the axes of reference being $O\xi$, $O\eta$ and suppose for simplicity that the coefficients a_r that correspond to those sides giving approximations to p_{mn} for m or n large, are positive. Denote by ω, the region determined by $\xi > 1$, $\eta > 1$. Then if the poly-

gon overlaps ω (in the strict sense), the double series is convergent; otherwise, it is divergent.

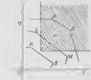


FIG. 3.

It is obvious that $\sum \frac{1}{p_{mn}}$ converges or diverges with $\sum \frac{1}{p'_{mn}}$ where p'_{mn} consists of the terms that lie on the sides giving the approximations to p_{mn} at (c, ∞), (∞, ∞), (∞, c).

(1) Suppose that the polygon overlaps ω. (*Fig. 3.*)

Then (i) there is at least one vertex A in ω,

or (ii) there is *no* vertex in ω but there is *one* side crossing ω (like LM in the figure).

(i) Let the term corresponding to A be $a_1 m^{\alpha_1} n^{\beta_1}$, where $\alpha_1 > 1$, $\beta_1 > 1$.

But $\sum_{mn} \frac{1}{p'_{mn}} < \frac{1}{a_1 m^{\alpha_1} n^{\beta_1}}$ which is convergent. (*Example* (i), § 4.55.)

(ii) Let the terms corresponding to LM be

$$a_1 m^{\alpha_1} n^{\beta_1}, \quad a_2 m^{\alpha_2} n^{\beta_2}.$$

From the well-known inequality $\left(\frac{p_1\rho_1 + p_2\rho_2}{p_1 + p_2}\right)^{p_1+p_2} > \rho_1^{p_1}\rho_2^{p_2}$ (all the numbers being positive), we deduce, by putting

$$p_1\rho_1 = a_1 m^{\alpha_1} n^{\beta_1}, \; p_2\rho_2 = a_2 m^{\alpha_2} n^{\beta_2}$$

that
$$a_1 m^{\alpha_1} n^{\beta_1} + a_2 m^{\alpha_2} n^{\beta_2} > K m^{\lambda} n^{\mu}$$
where
$$K = (p_1 + p_2)\left(\frac{a_1}{p_1}\right)^{\frac{p_1}{p_1+p_2}}\left(\frac{a_2}{p_2}\right)^{\frac{p_2}{p_1+p_2}}, \; \lambda = \frac{p_1\alpha_1 + p_2\alpha_2}{p_1 + p_2}, \; \mu = \frac{p_1\beta_1 + p_2\beta_2}{p_1 + p_2}.$$

But since LM *crosses* ω, numbers p_1, p_2 (> 0) can be found such that $\lambda, \mu > 1$.

Thus the double series is convergent.

(2) Suppose that the polygon does not overlap ω (but that it may be in contact with the boundary of ω).

Then (i) the whole of the polygon lies between $\xi = 0$ and $\xi = 1$ inclusive,

or (ii) the whole of the polygon lies between $\eta = 0$ and $\eta = 1$ inclusive,

or (iii) there is at least one side RS lying on a line passing through the point (1, 1) or passing between this point and (0, 0). (*Fig. 3.*)

In case (i) every index α is < 1 and therefore the sum of every column diverges, and in case (ii) every index β is < 1 and the sum of every row diverges. These cases are of course not mutually exclusive.

In case (iii), let R, S be respectively the points (α_1, β_1), (α_2, β_2) and let the equation of the line joining be $\mu\xi + \eta = \nu$ $(\mu, \nu > 0)$, so that $\nu = \mu\alpha_1 + \beta_1 = \mu\alpha_2 + \beta_2$.

Since the line passes through (1, 1) or passes between this point and (0, 0), we must have $\nu \leq \mu < 1$. Also, there is no loss in generality if we assume $\mu > 1$ (since, if necessary, m, n can be interchanged).

Let $I(x)$ denote the greatest integer $< x$; then $I(x) = kx$, where $\frac{1}{2} \leq k < 1$ $(x > 1)$.

If $m = I(n^{\mu})$, then $a_1 m^{\alpha_1} n^{\beta_1} + a_2 m^{\alpha_2} n^{\beta_2} = \lambda n^{\nu}$, where $\lambda\ (> 0)$ is bounded, (all n); whilst if (α_r, β_r) is another point of the polygon not collinear with R, S, the corresponding term $a_r m^{\alpha_r} n^{\beta_r} = \lambda' n^{\nu'}$, where $\nu' < \nu$ and $\lambda'\ (> 0)$ is bounded. The double series $\sum_{mn} \frac{1}{p'_{mn}}$ therefore

diverges with $\sum_{mn} \frac{1}{a_1 m^{\alpha_1} n^{\beta_1} + a_2 m^{\alpha_2} n^{\beta_2}}$. But $\sum_{m=1}^{I(n^{\mu})} \frac{1}{a_1 m^{\alpha_1} n^{\beta_1} + a_2 m^{\alpha_2} n^{\beta_2}} > \frac{kn^{\mu}}{\lambda n^{\nu}}$

and therefore $\sum_{mn} \frac{1}{a_1 m^{\alpha_1} n^{\beta_1} + a_2 m^{\alpha_2} n^{\beta_2}}$ is greater than $\sum_{n} \frac{k}{\lambda n^{\nu - \mu}}$ which

diverges since $\nu - \mu < 1$.

Notes. (i) The equation of the line joining (α_1, β_1) to (α_2, β_2) is of the form $p\xi + q\eta = 1$ where $p = (\beta_2 - \beta_1)/\Delta$, $q = (\alpha_1 - \alpha_2)/\Delta$, $\Delta = \alpha_1\beta_2 - \alpha_2\beta_1$. And (0, 0) is on the opposite side of the line from (1, 1) if $p + q > 1$ and on the same side if $p + q < 1$.

(ii) If $p\xi + q\eta = 1$ is the equation of a significant side of the polygon it is necessary (but not sufficient) for convergence that *every* $p + q$ should be > 1; whilst for divergence it is neither necessary nor sufficient that $p + q > 1$.

(iii) The conditions for the convergence of a double series of positive decreasing terms may also be related to that of a double (or simple) integral. (*See Chap. XI, § 11.03.*)

Examples. (i) $\Sigma \frac{1}{p_{mn}}$ where p_{mn} = [illegible].

$p + q$ for the first two terms is $20/21 < 1$. $p + q$ for the second and third is $12/11 > 1$. The series is divergent. Note, however, that when p_{mn} = [illegible], then double series is still divergent although $p + q < 1$, since the indices for m are both < 1.

(ii) p_{mn} = [illegible]. The first two terms are the same as in *Example* (i); $p + q$ for the last two is $116/117 < 1$. Series is convergent.

(iii) The series $\Sigma p_{mn}^{-\lambda}$ where $\lambda > 0$ is similarly convergent or divergent according to whether the polygon of p_{mn} enters or does not enter the region specified by $\xi\lambda > 1$, $\eta\lambda > 1$.

For example, if $am^2 + 2bmn + cn^2$ is of constant sign for $m, n > 0$, then the series $\Sigma \frac{1}{(am^2 + 2bmn + cn^2)^{\lambda}}$ converges if $\lambda > 1$ and diverges if $\lambda < 1$.

4.57. Absolutely Convergent Double Series. The series Σu_{mn} is said to be *absolutely* convergent if $\underset{mn}{\Sigma}|u_{mn}|$ is convergent; and as for simple series it may be proved that (i) the convergence of $\Sigma|u_{mn}|$ implies that of Σu_{mn} (ii) a derangement of the terms of an absolutely convergent double series does not alter the sum.

4.58. Substitution of one Power Series in another. Let $z = \sum_0^\infty b_n y^n$ have a radius of convergence q and let $y = \sum_0^\infty a_n x^n$ have a radius of convergence p. If y is substituted in z we can arrange the result as a double series thus

$$\begin{array}{lllll} & b_0 & & & \\ + & b_1 a_0 & + \; b_1 a_1 x & + \; b_1 a_2 x^2 + \; \ldots & \\ + & b_2 a_0^2 & + \; 2b_2 a_0 a_1 x + & \ldots \quad \ldots & \text{where } z \text{ is the} \\ + & b_3 a_0^3 + & \ldots \quad \ldots & \ldots \quad \ldots & \text{sum by } \textit{rows } (|y| < q) \\ + & \ldots & \ldots\ldots\ldots\ldots & \ldots & \end{array}$$

If the *double* series is absolutely convergent, the sum by columns is equal to the sum by rows, i.e. it is legitimate to arrange the expression as a power series in x. It is obviously necessary for the series to be convergent when $x = 0$, i.e. when $y = a_0$ and therefore $|a_0| < q$ is a *necessary* condition. When this condition is satisfied there must be some non-zero interval of x for which the double series is absolutely convergent, i.e. the rearrangement is certainly legitimate if x is *small*

(*Ref. Bromwich, Infinite Series, V, 36*, where the determination of such an interval is discussed.)

4.59. The Reversion of a Power Series. Let $y = a_1 x + a_2 x^2 + \ldots$ have a radius of convergence $p (\neq 0)$; then we may expect that when $a_1 \neq 0$, it is possible to express x as a power series in y having a non-zero radius of convergence. If we assume this series to be $b_1 y + b_2 y^2 + \ldots$, we can find formally, by substitution and by equating coefficients, the values of $b_1, b_2, b_3 \ldots$ in terms of $a_1, a_2, a_3, \ldots$ From these equations it is possible to determine an interval within which the series in y is absolutely convergent. The problem is in this way reduced to one of the same type as that in the previous paragraph.

(*Ref. Bromwich, Infinite Series, VIII, 55, and VIII, Examples B, 30–33*, where the determination of the interval is discussed, where the case $a_1 = 0$ is considered and also the relationship with Lagrange's expansion. *See also Chap. XI, § 11.32.*)

Examples. (i) Let $u_{mn} = 1/(m^2 - n^2)$, $(m \neq n)$; $u_{mn} = 0$, $(m = n)$. Find the sum by rows, by columns and by diagonals of $\underset{mn}{\Sigma} u_{mn}$

$$\sum_1^n u_{mn} = \frac{1}{2m}\left\{\left(\frac{1}{m-1} + \frac{1}{m+1}\right) + \ldots + \left(1 + \frac{1}{2m-1}\right) + \left(-1 + \frac{1}{2m+1}\right)\right.$$
$$\left. + \ldots + \left(-\frac{1}{n-m} + \frac{1}{m+n}\right)\right\}$$
$$= \frac{1}{2m}\left(S_{m+n} - \frac{3}{2m} - S_{n-m}\right) \text{ where } S_r = \sum_{s=1}^{r}\left(\frac{1}{s}\right).$$

If m is fixed $\lim\limits_{n} \Sigma u_{mn} = -3/4m^2$ since $S_{n,m} - S_{n-1,m} \to 0$

i.e.
$$\sum_m \sum_n u_{mn} = -\tfrac{1}{8}\pi^2 \left(\text{using the result } \sum_1^\infty \frac{1}{m^2} = \tfrac{1}{6}\pi^2\right)$$

Similarly $\sum\limits_n \sum\limits_m u_{mn} = \frac{1}{8}\pi^2$; the sum by diagonals is zero.

(ii) Prove that if $|x| < 1$, $\sum\limits_1^\infty \dfrac{x^n}{1-x^n} = \sum\limits_1^\infty \dfrac{x^{n^2}(1+x^n)}{(1-x^n)} = \sum\limits_1^\infty \lambda(n) x^n$ where $\lambda(n)$ is the number of divisors of n, (1, n included).

The double series:
$$\begin{array}{l} x + x^2 + x^3 + \dots + x^n + \dots \\ x^2 + x^4 + x^6 + \dots + x^{2n} + \dots \\ x^3 + x^6 + x^9 + \dots + x^{3n} + \dots \\ \dots\dots\dots\dots\dots\dots \\ x^m + x^{2m} + x^{3m} + \dots + x^{mn} + \dots \\ + \dots\dots\dots\dots\dots \end{array}$$

is obviously absolutely convergent when $|x| < 1$, being part of the series $\sum\limits_{mn} x^{m+n}$.

The sum by rows is $\sum\limits_m \dfrac{x^m}{1-x^m} = \sum\limits_1^\infty \dfrac{x^n}{1-x^n}$.

Take the first row and remainder of the first column, then the remainder of the second row and the remainder of the second column, and let the process be continued.

This gives $\left(\dfrac{x}{1-x} + \dfrac{x^2}{1-x}\right) + \left\{\dfrac{x^4}{1-x^2} + \dfrac{x^6}{1-x^2}\right\} + \dots$, i.e. $\sum\limits_1^\infty \dfrac{x^{n^2}(1+x^n)}{(1-x^n)}$.

Now arrange in powers of x. There is one term for each divisor of the index of the power, i.e. the series is equal to $\sum\limits_1^\infty \lambda(n) x^n$

4.6. Functions defined by Double Power Series. A function $F(x, y)$ may be defined by a double power series of the form $\sum\limits_{mn} a_{mn} x^m y^n$ for those values of x, y for which the series converges.

If all the series $\sum\limits_m a_{mn} x^m$ and $\sum\limits_n a_{mn} y^n$ are convergent and the double series is convergent, then
$$F(x, y) = \sum_{mn} a_{mn} x^m y^n = \sum_m \sum_n a_{mn} x^m y^n = \sum_n \sum_m a_{mn} x^m y^n.$$

If the double series is absolutely convergent, the summation may be effected in any order of the terms. In particular it is equal to
$$a_{00} + (a_{10}x + a_{01}y) + \dots (a_{n0}x^n + \dots + a_{0n}y^n) + \dots,$$
the sum by diagonals.

4.61. The Region of Convergence of a Double Power Series. This is not of such a simple character as that of a power series in one variable. However, if the double series is absolutely convergent for $x = x_0$, $y = y_0$, then it must obviously be absolutely convergent for $|x| < |x_0|$, $|y| < |y_0|$. Also in this case $F(x, y)$ is a continuous function at least for $|x| < |x_0|$, $|y| < |y_0|$, possessing continuous derivatives obtained by term-by-term differentiation. The region of convergence of the differential series may, however, have quite a different boundary from that of the original series.

Example. If
$$F(x, y) = 1 + 2x + 4x^2 + 8x^3 + \dots$$
$$+ 3y + 9y^2 + 27y^3 + \dots,$$

the region of convergence of $F(x, y)$ is given by the rectangle $|x| < \frac{1}{2}$, $|y| < \frac{1}{3}$; but that of F_a is given by the band $|x| < \frac{1}{2}$.

In the general case, let $x = r\cos\theta$, $y = r\sin\theta$; then

$F(x, y)$

$$= \sum_n (a_{n0}\cos^n\theta + a_{n-1,1}\cos^{n-1}\theta\sin\theta + \ldots + a_{0n}\sin^n\theta)r^n + \ldots$$

when it is summed by diagonals. This is absolutely convergent when $l|r| < 1$ where, by Cauchy's Test,

$$l = \lim_{n\to\infty} |(a_{n0}\cos^n\theta + \ldots + a_{0n}\sin^n\theta)|^{\frac{1}{n}}, \text{ (if this exists).}$$

This limit l is a function of θ and therefore the polar equation of the boundary is given by $rl(\theta) = 1$.

Note. *Lemaire* gives the boundary of absolute convergence in the form $|x|\lambda(|y|/|x|) = 1$, where $\lambda(k) = \lim\limits_{n\to\infty}\left\{\begin{matrix}\max & |a_{pq}k^q| \\ p + q = n\end{matrix}\right\}^{\frac{1}{n}}$. (*See Example 32, Chap. XI.*)

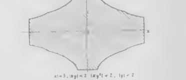
Examples. (i) $\sum\limits_0^\infty(2^{n+1}3^n x^{n+1}y^n + 2^n 3^{n+1}x^n y^{n+1})$.

Here $l = \lim\limits_{n\to\infty}|\{2^n 3^n\cos^n\theta\sin^n\theta(2\cos\theta + 3\sin\theta)\}|^{\frac{1}{2n+1}} = \sqrt{|6\cos\theta\sin\theta|}$ so that the boundary is $r\sqrt{|6\cos\theta\sin\theta|} = 1$, i.e. $|xy| = \frac{1}{6}$.

(ii) $\sum\limits_{m}\sum\limits_{n}\frac{(m+n)!}{m!n!}x^m y^n$.

$l = \lim[|\cos^n\theta| + {}^nC_1|\cos^{n-1}\theta\sin\theta| + \ldots + |\sin^n\theta|]^{\frac{1}{n}} = |\cos\theta| + |\sin\theta|$ so that the boundary is $|x| + |y| = 1$.

(iii) The region of absolute convergence for the expansion near $(0, 0)$ of the function $[(3 - x)(2 - xy)(2 - xy^2)(2 - y)]^{-1}$ is bounded by the curves $|x| = 3$, $|xy| = 2$, $|xy^2| = 2$, $|y| = 2$. (*Fig. 4.*)



$|x| < 3$, $|xy| < 2$, $|xy^2| < 2$, $|y| < 2$

FIG. 4

(iv) The series

$$\begin{array}{llllll}
0 & + 0 & + x^2 & + 2x^3 & + 4x^4 & + 8x^5 \\
+ 0 & + 0 & + 0 & + 0 & + 0 & + 0 \\
+ 0 & + 0 & - x^2y^2 & - 2x^3y^2 & - 4x^4y^2 & - 8x^5y^2 \\
+ y^2 & - xy^2 & + 0 & + 0 & + 0 & + 0 \\
+ 2y^3 & - 2xy^3 & + 0 & + 0 & + 0 & + 0 \\
+ 4y^4 & - 4xy^4 & + 0 & + 0 & + 0 & + 0 \\
+ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

The formula for a finite number of terms is

$$(1-y^2)(x^2+2x^3+\dots+2^{m-2}x^{m-1})+(1-x)(y^2+2y^3+\dots+2^{n-2}y^{n-1})$$

which tends to $\dfrac{x^2(1-y^2)}{1-2x}+\dfrac{y^2(1-x)}{1-2y}$ when $|x|<\frac{1}{2}$, $|y|<\frac{1}{2}$, the series being absolutely convergent.

Note, however, that the series converges (in the Pringsheim sense) to zero when $x = 1$ and $y = \pm 1$.

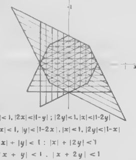


FIG. 5

(v) The region of absolute convergence does not in general coincide with that of the sum by rows or columns or diagonals.

Thus the series $\sum_{mn}(m+n)!x^my^n(2^m+2^n)/(m!n!)$ which is the expansion of $(1-2x-y)^{-1}+(1-x-2y)^{-1}$ is absolutely convergent when both inequalities $|2x|+|y|<1$, $|x|+|2y|<1$ are satisfied.

The sum by rows is absolutely convergent for

$$|2x|<1,\ |y|<|1-2x| \text{ with } |x|<1,\ 2|y|<|1-x|.$$

The sum by columns is absolutely convergent for

$$|y|<1,\ 2|x|<|1-y| \text{ with } |2y|<1,\ |x|<|1-2y|.$$

The sum by diagonals is absolutely convergent for

$$|2x+y|<1 \text{ with } |x+2y|<1 \quad (Fig.\ 5.)$$

Examples IV

Find the derivatives of the functions given in *Examples 1–15.*

1. $\tan(\frac{1}{2}\arctan\frac{1}{2}x)$ 2. $\frac{1}{3}\tan^3 x-\tan x+x$ 3. $\log\tan(\frac{1}{2}x+\frac{1}{4}\pi)$

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4. $\arctan\left(\frac{x\sqrt{3}}{1-x^2}\right) \quad 2\arctan\left(\frac{\sqrt{3}}{1+x^2}\right)$ 5. $\arccos\left(\frac{a + b\cos x}{b + a\cos x}\right)$

6. $\arccos\left\{\frac{\sqrt{(3x^2+6x+27)}}{2\sqrt{(2x^2+4x+8)}}\right\}$ 7. $x^{\log x}$ 8. $(\log x)^x$ 9. x^{x^x}

10. $(\log x)^{\log x}$ 11. $\log_{10}\log_{10} x$ 12. $(\log\log\log x)^{\log\log x}$

13. $e^{x\log x}$ 14. $e^{(\log x)^2}$ 15. e^{x^x}

16. If $\sin y = x\sin(y - \frac{1}{2}\pi)$, prove that $\frac{dy}{dx} = 2\sin^2(y + \frac{1}{2}\pi)$

17. If $y = x\{c_1(\log x)^3 + c_2(\log x)^2 + c_3(\log x) + c_4\}$, prove that
$$x^4y^{iv} + 2x^3y''' + x^2y'' - xy' + y = 0.$$
18. If $\sqrt{(1-e^2)}\tan x\tan y = 1$, prove that
$$\sqrt{(1-e^2\sin^2 x)}dy + \sqrt{(1-e^2\sin^2 y)}dx = 0.$$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the functions given in *Examples 19–21*

19. $\cos(2x+y) + \cos(x+2y) = 1$ 20. $xy = \log(x - y)$
21. $e^{x+y} + e^{x-y} = 2$

Obtain the nth derivatives of the functions given in *Examples 22–31*

22. $e^x\sin 2x$ 23. $\cos^3 x\sin x$ 24. x^2e^{3x} 25. $x^n\log x$
26. $\cos x\cos 2x\cos 3x$ 27. $x^4\sin x$ 28. $(\log x)^2$
29. $x^2(x-1)^{-1}$ 30. $x^2e^x\sin x$ 31. $x\sin^3 x$

32. If $z = \arctan\left(\frac{x^2-y^2}{x^2+y^2}\right)$, find z_x, z_y and verify that $xz_x + yz_y = 0$.

33. If $V = c_1\log(x^2+y^2) + c_2\arctan\left(\frac{y}{x}\right)$, show that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.

34. If $x = e^u\cos v$, $y = e^u\sin v$, prove that $(V_{xx} + V_{yy})e^{2u} = (V_{uu} + V_{vv})$.

35. Change the independent variables from x, y to u, v in the equation
$$ax^2z_{xx} + 2hxyz_{xy} + by^2z_{yy} + 2gxz_x + 2fyz_y + cz = 0$$
when $x = e^u$, $y = e^v$.

36. If $x = r\cos\theta$, $y = r\sin\theta$, show that $V_{xx} + V_{yy} = V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\theta\theta}$,
where V is a function of (x, y).

37. Show that the nth derivative of $x^{n-1}e^{1/x}$ is $(-1)^n e^{1/x}x^{-n-1}$.

38. If $(x-1)$ is small, show that
$$\frac{e^x}{(x-1)^2(x+2)} = e\left\{\frac{1}{3(x-1)^2} + \frac{2}{9(x-1)} + \frac{5}{54}\right\} + O(x-1)$$

39. Show that near $x = \pi$,
$$\frac{\sin x + x\cos x}{(x-\pi)^2} = -\frac{\pi}{(x-\pi)^2} - \frac{2}{(x-\pi)} + \frac{\pi}{2} + O(x-\pi)$$

40. Show that the coefficient of $(t-a)^{-1}$ in the expansion of $\frac{e^{xt}}{(t-a)^2(t-b)}$ near $t = a$ is $\frac{e^{ax}\{(a-b)^2x^2 - 2(a-b)x + 2\}}{2(a-b)^3}$ if $a \neq b$.

41. Find $\lim_{x\to 0}\left(\frac{e^x}{e^x-1} - \frac{1}{x}\right)$

42. Show that
$$\sum_0^\infty\frac{(c_0 + c_1n + c_2n^2 + c_3n^3 + c_4n^4 + c_5n^5)}{n!} = (c_0 + c_1 + 2c_2 + 5c_3 + 15c_4 + 52c_5)e.$$

43. Show that
(i) $\sin x < \tan x < \sec x < \cot x < \operatorname{cosec} x$ if $0 < x < \alpha$,
(ii) $\sin x < \cos x < \tan x < \cot x < \sec x < \operatorname{cosec} x$ if $\alpha < x < \frac{1}{4}\pi$
(iii) $\cos x < \sin x < \cot x < \tan x < \operatorname{cosec} x < \sec x$ if $\frac{1}{4}\pi < x < \frac{1}{2}\pi - \alpha$
(iv) $\cos x < \cot x < \sin x < \operatorname{cosec} x < \tan x < \sec x$ if $\frac{1}{2}\pi - \alpha < x < \frac{1}{2}\pi$

where $\alpha = \arcsin\left(\frac{\sqrt{5}-1}{2}\right)$, $(= 38° \, 11'$ approx.).

44. Show that
(i) $\tanh x < \sinh x < \operatorname{sech} x < \cosh x < \operatorname{cosech} x < \coth x$, if $0 < x < a$
(ii) $\tanh x < \operatorname{sech} x < \sinh x < \operatorname{cosech} x < \cosh x < \coth x$, if $a < x < b$
(iii) $\operatorname{sech} x < \tanh x < \operatorname{cosech} x < \sinh x < \coth x < \cosh x$, if $b < x < c$
(iv) $\operatorname{sech} x < \operatorname{cosech} x < \tanh x < \coth x < \sinh x < \cosh x$, if $c < x$

where $a = \frac{1}{2}\log(2 + \sqrt{5})$, $(= 0.72$ approx.); $b = \log(1 + \sqrt{2})$, $(0.88$ approx.); $c = \log \frac{1}{2}\{\sqrt{5} + 1 + \sqrt{(2\sqrt{5} + 2)}\}$, $(= 1.06$ approx.).

Prove the inequalities given in *Examples 45–55*.

45. $\dfrac{1}{x+1} < \log\left(1 + \dfrac{1}{x}\right) < \dfrac{1}{x}$, $(x > 0)$

46. $\left(1 + \dfrac{1}{x}\right)^x > \left(1 + \dfrac{1}{y}\right)^y$ if $x > y > 0$

47. $\left(1 - \dfrac{1}{x}\right)^{-x} < \left(1 - \dfrac{1}{y}\right)^{-y}$ if $x > y > 1$

48. $e^x > 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^n}{n!}$ if $x > 0$

49. e^{-x} lies between $1 - x + \dfrac{x^2}{2!} - \ldots + (-1)^n \dfrac{x^n}{n!}$ and $1 - x + \dfrac{x^2}{2!} - \ldots$

$$+ (-1)^{n+1}\frac{x^{n+1}}{(n+1)!}, \ (x > 0).$$

50. $e^x < \dfrac{1}{1-x}$, $(x < 1)$ **51.** $ax < x\log x + e^{a-1}$ $(x > 0)$

52. $\dfrac{x}{1+x} < 1 - e^{-x} < x$, $(x > -1)$ **53.** $(1 + x)\log(1 + x) > x$ $(x > -1)$

54. $|e^x - 1| \leq (e^{|x|} - 1) \leq |x|e^{|x|}$

55. If $S_n(x) = x - \dfrac{x^3}{3!} + \ldots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$

and $$C_n(x) = 1 - \frac{x^2}{2!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!}$$

then $\sin x$ lies between $S_n(x)$ and $S_{n+1}(x)$, and $\cos x$ lies between $C_n(x)$ and $C_{n+1}(x)$.

56. Arrange the following functions in order of greatness when x is large and also determine their places in the logarithmic scale:

$$(\log\log x)^{\log x},\ (\log x)^{x\log x},\ x^{x^2},\ x^{(\log x)^2},\ (\log x)^{(\log x)^2},\ (\log x)^{(\log x)\log x},$$
$$(\log x)^{x\log x}.$$

57. In a sequence a_n it is given that $a_{n+1} = a_1^{a_n}$; prove that (i) if $a_1 = 2$, $a_n \to +\infty$, (ii) if $a_1 = 1.2$, a_n tends to a definite limit, (iii) if $a_1 = 0.02$, a_n oscillates finitely with two limiting values.

Draw the graphs of the functions given in *Examples 58–60*, where $f(x)$ is the greatest integer $< x$ and point out any discontinuities that occur.

58. $f(x) - x$ **59.** $\sqrt{(x - f(x))}$ **60.** $\dfrac{1 - f(x)}{2x + f(x)}$

Discuss the discontinuities, (if any), of the functions given in *Examples 61–7*.

61. $x!$ **62.** x, $(x < 1)$; $(2 - x)$, $(1 < x < 3)$; 0, $(x > 3)$ **63.** $\frac{x-1}{x}$

64. $\frac{\sin x}{x}$, $(x \neq 0)$; 1, $(x = 0)$ **65.** $\frac{\sin x}{x^2}$, $(x \neq 0)$; 0, $(x = 0)$

66. $x^2 \sin\left(\frac{1}{x}\right)$, $(x \neq 0)$; 0, $(x = 0)$ **67.** $x \sin\left(\frac{1}{x^2}\right)$, $(x \neq 0)$; 0, $(x = 0)$

68. Prove that the function given by $y = 4x^3$, $(x < 1)$; $y = (x - 3)^2(4x - 3)$, $(1 < x < 2)$; $y = 2x^3 - (x^2 - 3)^{\frac{1}{2}}(2x^2 + 3)$, $(x > 2)$, and its first derivative but not its higher derivatives, are continuous functions for all values of x.

Determine the functions that are defined as the limits when n tends to infinity of the functions given in *Examples 69-76*. Discuss their discontinuities.

69. $\frac{x^n}{1 + x^{2n}}$ **70.** $\frac{x^n}{1 + x^{n+1}}$ **71.** $\frac{x}{1 - x} + \frac{1}{1 - x^n}$ **72.** $\left(\frac{n + 1 + x}{n}\right)^{\frac{1}{n}}$

73. $\frac{n^2x(x-1)(x-2) + nx(x-1) - 1}{n^2x(x-1) + nx - 2}$ **74.** $\frac{x^n(x - 1) - n(x + 1)}{x^{n+1}(x + 1) - n(x + 2)}$

75. $\frac{(x - 1)^n + (x - \frac{1}{2})^n}{(x - 1)^{n+1} + (x - \frac{1}{2})^{n+1}}$ **76.** $\frac{x^n + (x - 1)^n}{x^n + (x + 1)^n}$

Discuss the convergence of the series whose general terms are given in *Examples 77-100*.

77. $\frac{2^n}{(n - 1)(n + 2)}$ **78.** $\frac{1}{2^n(n + 1)(n + 2)}$ **79.** $\frac{1}{(n - 1)(n + 2)}$

80. $\frac{3n + 2}{2.6 \ldots (3n - 2)}$ **81.** $\frac{3^n}{5^n 2^{n+1}}$ **82.** $\left(\frac{1.3.5 \ldots 2n + 1}{2.4.6 \ldots 2n + 2}\right)^2$

83. $\left(\frac{2.4.6 \ldots 2n + 2}{1.3.5 \ldots 2n + 1}\right)^2$ **84.** $(-1)^n \cdot \frac{1.3.5 \ldots 2n + 1}{2.4.6 \ldots 2n + 2} \cdot \frac{1}{2n + 7}$

85. $\frac{1 + 2n^2}{1 + n^2}$ **86.** $\frac{(n!)^2}{(3n)!}$ **87.** $\frac{1 + n + n^2}{n!}$ **88.** $\frac{1}{n^{1 + \frac{1}{n}}}$

89. $\left(\frac{1.3.5 \ldots 2n - 1}{2.4.6 \ldots 2n}\right)^2 x^n$ **90.** $\frac{3.4.5 \ldots 2n + 2}{(2.4 \ldots 2n)(7.9 \ldots 2n + 6)} x^{2n}$

91. $\frac{(n - 1)^2(n + 2)}{(n - 1)^2(n + 3)} x^n$ **92.** $\frac{n^{100}x^n}{n!}$ **93.** $\frac{1.3.5 \ldots 2n - 1}{2.4.6 \ldots 2n} \cdot \frac{x^{2n+1}}{2n + 1}$

94. $(-1)^n \frac{x^{2n+1}}{2n + 1}$ **95.** $\frac{x^n}{(1 - x^{n+1})(1 - x^{n+2})}$ **96.** $\frac{x^n}{(\log n)^2}$

97. $\frac{n^2x^n}{(n + 1)^4}$ **98.** $(-1)^n \frac{(4n - 1)^{n-1}}{2.4 \ldots 2n} x^n$ **99.** $\frac{(2n - 1)^{n-1} x^n}{n!}$

100. $\frac{(p - 2q)(p - 4q) \ldots (p - 2nq)}{(n + 1)!} x^{n+1}$

Find the radii of convergence of the series given in *Examples 101-4*, and verify that the sums satisfy the associated differential equations.

101. $y = 1 - \frac{x^2}{(1!)^2} + \frac{x^4}{(2!)^2} - \frac{x^6}{(3!)^2} + \ldots;\ xy'' + y' + 4xy = 0$

102. $y = 1 - \frac{x^3}{3!} + \frac{1.4x^6}{6!} - \frac{1.4.7}{9!}x^9 + \ldots;\ y'' + xy = 0$

103. $y = 1 - \frac{2x}{(1!)^2} + \frac{2^2x^2}{(2!)^2} - \frac{2^3x^3}{(3!)^2} + \ldots;\ xy'' + y' + 2y = 0$

104. $y = 1 + \frac{3.10}{1.4}x + \frac{3.4.10.11}{1.2.4.5}x^2 + \ldots;\ x(1 - x)y'' + (4 - 14x)y' - 30y = 0$

105. $y = x^4 - \dfrac{x^6}{2.10} + \dfrac{x^8}{2.4.10.12} - \ldots$; $x^2y'' + xy' + (x^2 - 16)y = 0$

106. $y = 1 + \dfrac{2x^3}{(1!)3.1.2} + \dfrac{2x^6}{(2!)3^2.4.5} + \dfrac{2x^9}{(3!)3^3.7.8} + \ldots$,

$$y''' - 2y = x(xy'' - 2y')$$

107. Find the radii of convergence of the series

$$S_1(x) = \sum_0^\infty \frac{m^{2n}x^{2n}}{2^{2n}(n!)^2};\ S_2(x) = \sum_1^\infty \frac{m^{2n}x^{2n}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

and show that $y = S_1(x)$ or $y = S_1(x)\log x - S_2(x)$ satisfies the equation

$$x(y'' - m^2y) + y' = 0$$

108. Find the radius of convergence of the series

$$f(x) = \sum_2^\infty (-1)^n.\frac{x^n}{n!}\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

and show that $y = x^2e^{-x}\log x - 1 - x + x^2 - x^2f(x)$ satisfies the equation

$$xy'' + (x - 1)y' + y = 0$$

The series given in *Examples 109–11* are derangements of the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots\ (= \log 2)$$

Find their sums.

109. $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots$ (Two positive terms followed by one negative.)

110. $1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} + \frac{1}{7} - \frac{1}{10} \ldots$ (Two positive terms followed by four negative.)

111. $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} \ldots$ (One positive term followed by four negative.)

112. Show that

$$\left(\sum_0^\infty (-1)^n.\frac{x^{n+1}}{n+1}\right)\left(\sum_0^\infty \frac{x^{n+1}}{n+1}\right) = \sum_1^\infty\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1}\right)\frac{x^{2n}}{n}, (|x| < 1).$$

Establish the results given in *Examples 113–20*.

113. $$\left(\sum_0^\infty \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n}\cdot\frac{x^n}{2n+1}\right)\left(\sum_0^\infty \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n}.x^n\right) = \sum_0^\infty \frac{2.4 \ldots 2n}{3.5 \ldots (2n+1)}x^n, (|x| < 1)$$

114. $$\left(1 + \frac{x}{4} + \frac{1.3}{4.6}x^2 + \frac{1.3.5}{4.6.8}x^3 + \ldots\right)^2 = 1 + \frac{3x}{6} + \frac{3.5}{6.8}x^2 + \frac{3.5.7}{6.8.10}x^3 + \ldots\ (|x| < 1)$$

115. $(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots)(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots) = \frac{2}{2}(1 + \frac{1}{2}) - \frac{2}{3}(1 + \frac{1}{2} + \frac{1}{3}) + \ldots$

116. $\sum_1^\infty \dfrac{2n^2 + 1}{n^2(n+1)^2} = 1$ **117.** $\sum_1^\infty \dfrac{1}{n(2n-1)(2n+1)} = \sum_1^\infty \dfrac{1}{(n+1)(2n+1)}$

118. $\sum_1^\infty \dfrac{n}{2^n + (-1)^n} = 2\sum_1^\infty \dfrac{2^n - 1}{(2^n + (-1)^n)^2}$

119. $\sum_1^\infty \dfrac{x^n}{1 + x^{2n}} = \sum_0^\infty (-1)^n \dfrac{x^{2n+1}}{1 - x^{2n+1}}, (|x| < 1)$

120. $\sum_1^\infty (-1)^{n-1}\dfrac{x^n}{1 + x^{2n}} = \sum_0^\infty (-1)^n.\dfrac{x^{2n+1}}{1 + x^{2n+1}}, (|x| < 1)$

121. If $u_{mn} = \dfrac{1}{2^m} + \dfrac{2n+1}{2n+2} - \dfrac{(3n+1)^m}{(3n+2)^m}$, find $\lim_m \lim_n u_{mn}$ and $\lim_n \lim_m u_{mn}$.

122. Find the value of $\lim_{m}\lim_{n}\{\cos(m!\pi x)\}^{2n}$

123. Find the sum by rows, by columns, by diagonals and by squares of the double series

$$\begin{array}{l} -2+1+0+0+0+\dots \\ +1-2+1+0+0+\dots \\ +0+1-2+1+0+\dots \\ +0+0+1-2+1+\dots \\ \dots\dots\dots\dots\dots\dots \end{array}$$

124. If $f(n) > 0$, and $\sum_n f(n)$ is divergent show that $\sum_{mn} u_{mn}$ is divergent when $u_{mn} > f(m^2+n^2)$.

Determine whether the series Σp_{mn}^{-1} is convergent or divergent when p_{mn} has the values given in *Examples 125–8*

125. $m^2n^2 + m^3n^2 + m^2n^3 + m^2n^2$ **126.** $m^2n^2 + m^2n + mn + m$

127. $m^3n^2 + m^2n^2 + m^2n^3 + m^2$ **128.** $m^2n^2 + m^2n^3 + m^3n^3$

Determine the boundary of the region of convergence of the series given in *Examples 129–31*

129. $\sum_{mn}(2^m x^m + 3^n y^n)$

130. $\sum_{mn} a_{mn}x^m y^n$ where $a_{m0} = 2^{-m}$, $a_{nn} = 1$, $a_{0n} = 3^{-n}$, $a_{mn} = 0$ for other values.

131. $\sum_{mn} a_{mn}x^m y^n$, where $a_{2n,n} = \lambda^n$, $a_{m,2m} = \mu^m$, $a_{mn} = 0$ for other values.

Represent in a diagram the regions of convergence of the double series obtained by expanding the functions (near 0, 0) given in *Examples 132–5*. Also give the regions of convergence of the sum by rows, columns and diagonals.

132. $\dfrac{3-y}{1-2x} + \dfrac{2-x}{1-3y}$ **133.** $\dfrac{1}{2-x-2y}$

134. $\dfrac{1}{1-x-y} - \dfrac{1}{1-2x-y}$

135. $\dfrac{1}{1-2x-2y} - \dfrac{1}{1-2x-y} + \dfrac{1}{1-x-2y}$

Represent in a diagram the region of absolute convergence of the double series obtained by expanding (near 0, 0) the functions given in *Examples 136–8*

136. $\dfrac{1}{(1-x)(2-y)(4-x^2-y^2)}$ **137.** $\dfrac{1}{1-x^2-xy-y^2}$ **138.** $\dfrac{1}{1-y^2-x}$

Solutions

1. $\dfrac{2}{x^2}\left(1-\dfrac{2}{\sqrt{(x^2-4)}}\right)$ **2.** $\tan^6 x$ **3.** $\dfrac{\sqrt{2}}{\sin x+\cos x}$

4. $\dfrac{\sqrt{3}(1-x^2)}{x^4+x^2+1} + \dfrac{4x\sqrt{3}}{x^4+2x^2+4}$ **5.** $\dfrac{\sqrt{(b^2-a^2)}}{b+a\cos x}$

6. $\dfrac{\sqrt{15}}{(x^2+2x+4)\sqrt{(x^2-2x+9)}}$ **7.** $(2\log x).x^{\log x-1}$

8. $(\log x)^x\left(\log\log x+\dfrac{1}{\log x}\right)$ **9.** $x^{x^x}.x^x\left((\log x)^2+\log x+\dfrac{1}{x}\right)$

10. $\dfrac{1}{x}(\log x)^{\log x}(1+\log\log x)$ **11.** $\dfrac{1}{x\log x.\log 10}$

12. $\dfrac{(\log\log\log x)^{\log\log x}}{x\log x}\left(\log\log\log\log x+\dfrac{1}{\log\log\log x}\right)$

13. $\dfrac{2e^{x^{\log x}}(\log x).x^{\log x}}{x}$ **14.** $e^{(\log x)^x}\{(\log x)^x(\log\log x)+x(\log x)^{x-1}\}$

18. $e^{x^x}.x^x(1+\log x)$

19. $\{\sin(2x+y)+2\sin(x+2y)\}y' + \{2\sin(2x+y)+\sin(x+2y)\} = 0$; $\{\sin(2x+y)+2\sin(x+2y)\}^3y'' + 9\{1-\cos(2x+y)\cos(x+2y)\} = 0$

20. $-\frac{1-xy-y^2}{1-xy-x^2}$, $\frac{(x-y)\{3x^2+2xy+3y^2-2xy(x+y)^2-2\}}{(1-xy-x^2)^3}$

21. $-\coth y$; $-\coth y \operatorname{cosech}^2 y$ 22. $5^{\frac{n}{2}}e^x\sin(2x+n\tan^{-1}2)$

23. $-\frac{1}{4}3^n\sin(3x+\frac{1}{2}n\pi)+\frac{3}{4}\sin(x+\frac{1}{2}n\pi)$

24. $2^{n-3}e^{2x}\{8x^3+12nx^2+6n(n-1)x+n(n-1)(n-2)\}$

25. $n!\left(\log x + n - \frac{1}{2}{}^nc_2 + \frac{1}{3}{}^nc_3 - \dots + (-1)^{n-1}\frac{1}{n}\right)$

26. $2^{n-3}\{\cos(2x+\frac{1}{2}n\pi) + 2^n\cos(4x+\frac{1}{2}n\pi) - 3^n\cos(6x+\frac{1}{2}n\pi)\}$

27. $\{x^4-6n(n-1)x^2+n(n-1)(n-2)(n-3)\}\sin(x+\frac{1}{2}n\pi)$
$$-\{4nx^3-4n(n-1)(n-2)x\}\cos(x+\frac{1}{2}n\pi)$$

28. $2(-1)^{n-1}\frac{(n-1)!}{x^n}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n-1}\right)$

29. $\frac{(-1)^n.(2n-4)!(3x^2-4nx+4n(n-1))}{2^{2n-2}(n-2)!(x-1)^{n+1}}$

30. $e^{-x}2^{\frac{1}{2}n-1}[(2x^2+2nx)\sin(x+\frac{1}{4}n\pi)-\{2nx+n(n-1)\}\cos(x+\frac{1}{4}n\pi)]$

31. $\frac{3x}{4}\left\{\sin\left(x+\frac{1}{2}n\pi\right)-3^{n-1}\sin\left(3x+\frac{1}{2}n\pi\right)\right\}$
$$+\frac{3n}{4}\left\{\sin\left(x+\frac{1}{2}(n-1)\pi\right)-3^{n-2}\sin\left(3x+\frac{1}{2}(n-1)\pi\right)\right\}$$

32. $\frac{3x^3y^3}{x^6+y^6}$, $\frac{3x^3y^2}{x^6-y^6}$

35. $az_{uu}+2hz_{uv}+bz_{vv}+(2g-a)z_u+(2f-b)z_v+cz=0$

37. Use the infinite series for $e^{\frac{1}{x}}$ 41. $\frac{1}{2}$

45–54. May be proved by calculating the minimum (or maximum) values of the appropriate functions

56. $x < (\log\log x)^{\log x} < (\log x)^{(\log x)^2} < e^x < (\log x)^{x\log x} < x^{e^x}$
$$(\log x)^{(\log x)^{\log x}} < (\log x)^{x\log x} < e^{x^2} < x^{(\log x)^3}$$

57. (ii) The limit is the smaller root of $\log x = x\log(1\cdot2)$, i.e. 1·258 (approx.). (iii) The sequence oscillates between u, v determined from $(0\cdot02)^u = v$, $(0\cdot02)^v = u$, i.e. between 0·0314 and 0·884 (approx.).

58. $F(n) = 0 = F(n+0)$; $F(n-0) = -1$; points for which $n < x < n+1$ lie on the line joining $(n, 0)$ to $(n+1, -1)$.

59. $F(n) = 0 = F(n+0)$; $F(n-0) = 1$; for $0 < x < 1$, $F(x) = \sqrt{x}$ and for other points $F(n+x) = F(x)$, $(0 < x < 1)$.

60. $F(0)$ undefined; $F(+0) = +\infty$; $F(-0) = 0$; $F(n) = (1+n)/3n = F(n+0)$; $F(n-0) = n/(3n-1)$; $F(n+x) = \frac{1+n}{2x+3n}(0 < x < 1)$.

61. $f(+0) = f(0) = 0$; $f(x)$ undetermined for $x < 0$

62. Finite discontinuity of the first kind at $x = 3$ with $f(3-0) = f(3) = -1$, $f(3+0) = 0$.

63. Infinite discontinuity of the first kind at $x = 0$. $f(+0) = -\infty$, $f(-0) = +\infty$.

64. Continuous.

65. Infinite discontinuity of the second kind at $x = 0$. $\overline{f(+0)} = +\infty$, $\underline{f(+0)} = -\infty$

66. Continuous. **67.** Continuous.

69. $F(x) = 0$, $|x| \neq 1$; $F(1) = \frac{1}{2}$; $F(-1)$ undetermined, removable discontinuity of the first kind at $x = 1$.

70. $F(x) = \frac{1}{x}$, $|x| > 1$; $F(x) = 0$, $|x| < 1$; $F(1) = \frac{1}{2}$; $F(-1)$ undetermined; a finite discontinuity of the first kind at $x = 1$, with $F(1+0) = 1$, $F(1) = \frac{1}{2}$, $F(1-0) = 0$; also $F(-1+0) = 0$, $F(-1-0) = -1$.

71. $F(x) = \frac{1}{1-x}$, $|x| < 1$; $F(x) = \frac{x}{1-x}$, $|x| > 1$; $F(-1)$ undetermined, $F(1+0) = -\infty = -F(1-0)$, $F(-1+0) = \frac{1}{2}$, $F(-1-0) = -\frac{1}{2}$.

72. Continuous all x, $F(x) = e^{1-x}$.

73. $F(x) = x - 2$, $(x \neq 1, x \neq 0)$; $F(0) = \frac{1}{2}$, $F(1) = 0$; two removable discontinuities at $x = 1$, $x = 0$ with $F(+0) = -2$, $F(1+0) = -1$.

74. $F(x) = \frac{x-1}{x(x+1)}$, $|x| > 1$; $F(x) = \frac{x+1}{x+2}$, $|x| < 1$; $F(1+0) = 0$, $F(1) = F(1-0) = \frac{2}{3}$; $F(-1-0) = \infty$, $F(-1+0) = F(-1) = 0$.

75. $F(x) = \frac{1}{x-1}$, $x > \frac{3}{4}$, $F(x) = x - \frac{1}{2}$, $x < \frac{3}{4}$; $F\left(\frac{3}{4}\right)$ undefined.

76. $F(x) = 0$, $x > 0$; $F(x)$ not determined for $x < 0$.

77. D. **78.** C. **79.** C. **80.** C. **81.** C. **82.** C.

83. D. **84.** C. **85.** D. **86.** C. **87.** C. **88.** D.

89. C, $|x| < 1$, $x = 1$. **90.** C, $|x| < 1$. **91.** C, $|x| < 1$.

92. C. **93.** C, $|x| < 1$. **94.** C, $|x| < 1$.

95. $S_n = \frac{1}{1-x}\left(\frac{1}{1-x} - \frac{x^n}{1-x^{n+1}}\right)$, $(x \neq 1)$. C to $\frac{1}{(1-x)^2}$ if $|x| < 1$; C to $\frac{1}{x(1-x)^2}$ if $|x| > 1$.

96. C, $|x| < 1$ and $x = 1$. **97.** C, $|x| < 1$, $x = -1$.

98. C, $|x| < \frac{1}{2e}$. **99.** C, $|x| < \frac{1}{2e}$.

100. C, $|x| < \frac{1}{2q}$; $x = \frac{1}{2q}$, $q(p+2q) > 0$; $x = -\frac{1}{2q}$, $(pq > 0)$.

101. ∞ **102.** ∞ **103.** ∞ **104.** 1 **105.** ∞ **106.** ∞

107. ∞, ∞ **108.** ∞ **109.** $\frac{1}{2}\log 2$ **110.** $\frac{1}{2}\log 2$ **111.** 0

116. Take $\frac{2n^3 - 1}{n^3(n+1)^2} = \frac{2}{n} - \frac{2}{n+1} + \frac{1}{(n+1)^2} - \frac{1}{n^3}$.

117. Each series is equal to $2\log 2 - 1$.

118, 119, 120. Arrange as a double series after expanding the general term

121. $\lim_{m}\lim_{n} = 0$, $\lim_{n}\lim_{m} = 1$ **122.** 1, (x rational), 0, (x irrational).

123. Rows, -1, columns, 1, diagonals, oscillatory between 0 and -2; squares, -2; double series oscillates between -1 and -2.

125. D. **126.** C. **127.** D. **128.** C.

129. Rectangle bounded by $|x| = \frac{1}{2}$, $|y| = \frac{1}{2}$.

130. Area bounded by $|x| < 2$, $|y| < 2$, $|xy| < 1$.

131. The area bounded by $|x^2y| = 1/\lambda$, $|xy^2| = 1/\mu$.

132. All the series absolutely convergent when $|x| < \frac{1}{2}$, $|y| < \frac{1}{2}$. The double series, but not the others, is convergent also when $x = 2$, $y = 3$

133. Double series, $|x| + 2|y| < 2$, columns, $|y| < 1$ with $|x| < 2|1 - y|$; rows, $|x| < 2$ with $2|y| < |2 - x|$; diagonals, $|x + 2y| < 3$

134. Double series, $|x| + |y| < 1$ with $|2x| + |y| < 1$; rows, $|x| < 1$ with $|y| < |1 - x|$, and $|x| < \frac{1}{2}$ with $|y| < |1 - 2x|$; columns, $|y| < 1$ with

$|x| < |1 - y|$ and $|y| < 1$ with $2|x| < |1 - y|$; diagonals, $|x + y| < 1$ with $|2x + y| < 1$.

135. Double series within the square determined by $|x| + |y| < 1$; diagonals, within the hexagon $(\pm \frac{1}{2}, 0), (0, \pm \frac{1}{2}), (1, -1), (-1, 1)$; rows, within the pentagon $(\frac{1}{2}, 0), (0, -\frac{1}{2}), (-\frac{1}{2}, [illegible]\frac{1}{2})$; columns, within the pentagon $(0, \frac{1}{2}), ([illegible]\frac{1}{2}, 0), (-\frac{1}{2}, -\frac{1}{2})$.

136. Area within the segment of the circle $x^2 + y^2 = 4$ cut off between the lines $x = \pm 1$.

137. Within the area common to $x^2 \pm xy + y^2 = 1$.

138. Within the area bounded by $y^2 = 1 - x$, $(x \geq 0)$ and $y^2 = 1 + x$ $(x < 0)$.

CHAPTER V

INTEGRATION OF FUNCTIONS OF ONE VARIABLE.

5. The Indefinite Integral. A function $F(x)$ whose derivative is $f(x)$ is called an *integral* of $f(x)$ and is written $\int f(x)\,dx$. If $F(x)$, $G(x)$ are two integrals of $f(x)$, the derivative of $F(x) - G(x)$ must be zero. But the only continuous function possessing a zero derivative at all points of an interval, is, by the mean value theorem, a constant. Since the value of this constant is *arbitrary*, the general value of the integral of $f(x)$ is $\int f(x)dx + C$, where C is the arbitrary constant. This general value is known as the *indefinite integral* of $f(x)$.

When considering *methods* of integration, we shall often, for convenience, *omit* this constant.

5.01. Methods of Integration. From the above point of view, integration is a process inverse to differentiation, and it may therefore be expected that the process will not always be possible in terms of functions or operations that have hitherto been considered. We can, however, from our previous knowledge of derivatives obtain at the outset a list of integrals of certain simple functions. In order to obtain the most useful expressions for these simpler results (or standard forms, as they are usually called), it is better at this stage to consider the effect of a change of variable.

5.02. Change of Variable. Let $F(x) = \int f(x)\,dx$, i.e. $f(x) = F'(x)$, and let $x = \phi(u)$ be a continuous function of u possessing a derivative $\phi'(u)$, then $F\{\phi(u)\}$ is, for an appropriate interval, a continuous function of u, possessing the derivative $f\{\phi(u)\}\phi'(u)$ with respect to u,

i.e. $$\int f(x)\,dx = F(x) = F\{\phi(u)\} = \int f\{\phi(u)\}.\phi'(u)du + C.$$

Example. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{1}{a^2}\int \sec^2\theta d\theta$ (if $x = a\sin\theta$) $= \frac{\tan\theta}{a^2} = \frac{x}{a^2(a^2-x^2)^{\frac{1}{2}}}$.

5.03. Standard Forms. Directly from the results of differentiation with the use of a suitable change of variable, we obtain the list

I. (i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$, $(a(n+1) \neq 0)$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b|$, $(a \neq 0)$

(iii) $\int e^{ax}\,dx = \dfrac{1}{a}e^{ax}$, $(a \neq 0)$

(iv) $\int \cos(ax+b)\,dx = \dfrac{1}{a}\sin(ax+b)$, $(a \neq 0)$

(v) $\int \sin(ax+b)\,dx = -\dfrac{1}{a}\cos(ax+b)$, $(a \neq 0)$

(vi) $\int \tan x\,dx = -\log|\cos x|$
(vii) $\int \cot x\,dx = \log|\sin x|$
(viii) $\int \sec^2 x\,dx = \tan x$
(ix) $\int \operatorname{cosec}^2 x\,dx = -\cot x$

(x) $\int \sinh mx\,dx = \dfrac{1}{m}\cosh mx$, $(m \neq 0)$

(xi) $\int \cosh mx\,dx = \dfrac{1}{m}\sinh mx$, $(m \neq 0)$

(xii) $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arc\,tan}\left(\frac{x}{a}\right)$, $(a \neq 0)$

(xiii) $\displaystyle\int \frac{dx}{\sqrt{(a^2-x^2)}} = \operatorname{arc\,sin}\frac{x}{a}$, $(a > 0,\ |x| < a)$

(xiv) $\displaystyle\int \frac{dx}{\sqrt{(x^2+A)}} = \log\{x + \sqrt{(x^2+A)}\}$

A useful appendix to this list is:

II (i) $\int \{\phi(x)\}^n \phi'(x)\,dx = \dfrac{\{\phi(x)\}^{n+1}}{n+1}$, $(n \neq -1)$

(ii) $\displaystyle\int \frac{\phi'(x)\,dx}{\phi(x)} = \log|\phi(x)|$

(iii) $\int c^{ax}\,dx = \dfrac{c^{ax}}{a\log c}$, $(a\log c \neq 0)$

(iv) $\int \log x.\,dx = x(\log x - 1)$
(v) $\int \operatorname{cosec} x\,dx = \log|\tan\tfrac{1}{2}x|$

(vi) $\int \sec x\,dx = \log\left|\tan\left(\tfrac{1}{2}x + \tfrac{1}{4}\pi\right)\right| = \log\left|\dfrac{1+\sin x}{\cos x}\right|$

(vii) $\int \sin^2 x\,dx = \tfrac{1}{2}(x - \sin x\cos x)$
(viii) $\int \cos^2 x\,dx = \tfrac{1}{2}(x + \sin x\cos x)$
(ix) $\int \tan^2 x\,dx = \tan x - x$
(x) $\int \cot^2 x\,dx = -x - \cot x$

(xi) $\displaystyle\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{a^2(a^2-x^2)^{\frac{1}{2}}}$, $(a \neq 0)$

(xii) $\displaystyle\int \frac{dx}{(x^2+A)^{\frac{3}{2}}} = \frac{x}{A(x^2+A)^{\frac{1}{2}}}$, $(A \neq 0)$

(xiii) $\int \sqrt{(a^2-x^2)}\,dx = \tfrac{1}{2}x\sqrt{(a^2-x^2)} + \tfrac{1}{2}a^2\operatorname{arc\,sin}\left(\dfrac{x}{a}\right)$, $(a > 0)$

(xiv) $\int \sqrt{(x^2+A)}\,dx = \tfrac{1}{2}x\sqrt{(x^2+A)} + \tfrac{1}{2}A\log\{x + \sqrt{(x^2+A)}\}$

5.04. Integration by Parts. The formula for *Integration by Parts* is an adaptation of the formula for the derivative of a product, i.e. of the result

$$\frac{d}{dx}(uV) = uv + u'V, \text{ where } V = \int v\,dx.$$

Thus $\int uv\,dx = u\int v\,dx - \int\{u'\int v\,dx\}\,dx$, or

$\int$(1st Function) $\times$ (2nd Function) dx = (1st Function)

$\times$ (Integral of 2nd) $-\int$(Derivative of 1st) $\times$ (Integral of 2nd) dx.

This formula is often effective if u is an *inverse* function such as arc sin x or is a *positive* power of x or $\log x$, whilst v is e^x or a circular function or a power of x.

Examples. (i) $\int x^4 \log x\,dx = (\log x).\frac{x^5}{5} - \int \frac{1}{x}.\frac{x^5}{5}\,dx = \frac{x^5}{5}\left(\log x - \frac{1}{5}\right)$.

(ii) $\int x \text{ arc tan } x.dx$

$$= \frac{1}{2}x^2 \text{ arc tan } x - \frac{1}{2}\int\frac{x^2\,dx}{1+x^2}$$

$$= \frac{1}{2}x^2 \text{ arc tan } x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx = \frac{1}{2}(1+x^2) \text{ arc tan } x - \frac{1}{2}x.$$

5.05. Reduction Formulae. In some cases an integral may be evaluated by repeated applications of the formula for Integration by Parts. This is one way in which *Reduction Formulae* arise.

Thus if $I_m = \int \frac{x^m\,dx}{\sqrt{(1+x^2)}}$ where m is a positive integer

$$I_m = x^{m-1}\sqrt{(1+x^2)} - (m-1)\int x^{m-2}\sqrt{(1+x^2)}dx$$

$$= x^{m-1}\sqrt{(1+x^2)} - (m-1)(I_{m-2} + I_m) \text{ or}$$

$$I_m = \frac{x^{m-1}\sqrt{(1+x^2)}}{m} - \frac{m-1}{m}I_{m-2}.$$

By repeated applications of this formula, I_m is expressed in terms of $I_1[=\sqrt{(1+x^2)}]$ or $I_0[=\log\{x+\sqrt{(1+x^2)}\}]$.

5.06. Integration of the Rational Function. Let $P(x)/Q(x)$ denote a rational function where $P(x)$, $Q(x)$ are polynomials with no common factor. The denominator $Q(x)$ may, theoretically, be expressed in the form

$$Q(x) = k(x-\alpha)^p(x-\beta)^q \ldots (x^2+2bx+c)^s(x^2+2ex+f)^t \ldots$$

where $p, q, \ldots, s, t \ldots$ are positive integers, $k, \alpha, \beta, \ldots, b, c, e, f, \ldots$ are real numbers and $b^2 < c$, $e^2 < f, \ldots$

Then $P(x)/Q(x)$ may be expressed in *partial fractions*, as follows

$$\frac{P(x)}{Q(x)} = T(x) + \sum_{m=1}^{m=p}\frac{A_m}{(x-\alpha)^m}$$

$$+ \sum_{m=1}^{m=q}\frac{B_m}{(x-\beta)^m} + \ldots + \sum_{m=1}^{m=s}\frac{(L_m x + M_m)}{(x^2+2bx+c)^m} +$$

where $T(x)$ is the quotient when $P(x)$ is divided by $Q(x)$ (and may therefore be zero).

Note. The reader may verify by multiplying up by $Q(x)$ and equating coefficients that there is exactly the correct number of linear equations for the determination of the unknowns. (*Ref. Goursat, Cours d'Analyse, I, 5*, where a justification of this method of decomposition will be found.)

The theoretical integration of $P(x)/Q(x)$ resolves itself into two parts (i) the determination of the unknown constants in the partial fractions, (ii) the integration of the fractions.

(i) *The Determination of the Constants.* These constants may usually be determined most simply by finding the *approximations* to $P(x)/Q(x)$ near the *infinities* of $P(x)/Q(x)$. Thus

(*a*) Near a multiple real root α of $Q(x) = 0$, we may take $x = \alpha + \xi$ and expand near $\xi = 0$.

(*b*) The terms of the form $\sum_{1}^{s} \frac{(L_m x + M_m)}{(x^2 + 2bx + c)^m}$ may be found by developing the expansion in the form

$$(a_1 x + b_1) + (a_2 x + b_2)\xi + (a_3 x + b_3)\xi^2 + \dots$$

where $\xi = x^2 + 2bx + c$.

But it may be sometimes simpler to use the method of equating coefficients in the case of quadratic factors.

(*c*) The function $T(x)$ is simply the *asymptotic polynomial*.

(ii) *The Integration.* (*a*) The integrals of $T(x)$ and $\frac{A_m}{(x - \alpha)^m}$ are obvious.

(*b*) By reduction-formulae it is possible to express the integral of $\frac{L_m x + M_m}{(x^2 + 2bx + c)^m}$ in terms of $\int \frac{x\,dx}{x^2 + 2bx + c}$ and $\int \frac{dx}{x^2 + 2bx + c}$. However, it simplifies the analysis to write $u = x + b$ and the integrand becomes $\frac{qu + r}{(u^2 + k^2)^m}$, where $q = L_m$, $r = M_m - bL_m$, $k = \sqrt{(c - b^2)}$.

The integral of $\frac{u}{(u^2 + k^2)^m}$ is $-\frac{1}{2(m - 1)(u^2 + k^2)^{m-1}}$, whilst if

$$I_m = \int \frac{du}{(u^2 + k^2)^m}$$

we easily obtain the reduction-formula

$$2k^2(m - 1)I_m = \frac{u}{(u^2 + k^2)^{m-1}} + (2m - 3)I_{m-1}.$$

Note. I_m may also be expressed as $k^{1-2m}\int \cos^{2m-2}\theta\,d\theta$, where $u = k\tan\theta$.

These various points are illustrated in the following examples:

Examples. (i) $\int \frac{dx}{(x^2 + 4)^3} = \frac{x}{16(x^2 + 4)^2} + \frac{3}{16}\int \frac{dx}{(x^2 + 4)^2}$ by the above formula.

Also $\int \frac{dx}{(x^2+4)^2} = \frac{x}{8(x^2+4)} + \frac{1}{8}\int \frac{dx}{x^2+4}$ so that the given integral is equal to

$$\frac{x}{16(x^2+4)^2} + \frac{3x}{128(x^2+4)} + \frac{3}{256}\arctan \tfrac{1}{2}x.$$

(ii) $\int \frac{x^7 + 2x^3 - x + 1}{x^2(x-1)(x+1)(x-2)}\,dx.$ Denote the integrand by $F(x)$.

At ∞, $F(x) = x^2\left(1 + \frac{2}{x^4} - \frac{1}{x^6} + \frac{1}{x^7}\right)\left(1 - \frac{1}{x}\right)^{-1}\left(1 + \frac{1}{x}\right)^{-1}\left(1 - \frac{2}{x}\right)^{-1}$

$$= x^2 + 2x + 5 + O\left(\frac{1}{x}\right)$$

Near $x = 0$, $F(x) = \frac{1}{x^2}(1 - x \ldots)(1 + x \ldots)(1 - x \ldots)\frac{1}{2}\left(1 + \frac{1}{2}x \ldots\right)$

$$= \frac{1}{2x^2} - \frac{1}{4x} + O(1).$$

Near $x = 1$, $F(x) = \frac{3}{1^2(x-1)(2)(-1)} + O(1) = -\frac{3}{2(x-1)} + O(1).$

Near $x = -1$, $F(x) = -\frac{1}{6(x+1)} + O(1)$,

near $x = 2$, $F(x) = \frac{143}{12(x-2)} + O(1).$

Thus $F(x) = x^2 + 2x + 5 + \frac{1}{2x^2} - \frac{1}{4x} - \frac{3}{2(x-1)} - \frac{1}{6(x+1)} + \frac{143}{12(x-2)}$ and

$$\int F(x)\,dx = \frac{1}{3}x^3 + x^2 + 5x - \frac{1}{2x} - \frac{1}{4}\log|x| - \frac{3}{2}\log|x-1| - \frac{1}{6}\log|x+1|$$
$$+ \frac{143}{12}\log|x-2|.$$

(iii) $\int F(x)\,dx$ where $F(x) = \frac{x^8 - x + 1}{(x-1)^3(x+1)^3(x-2)}$

Near $x = 1$, take $x = 1 + \xi$ and expand.

Then $F(x) = \frac{(1 + 7\xi + 28\xi^2)(1 - \xi + \frac{3}{4}\xi^2)(1 + \xi + \xi^2)}{4\xi^3} + O(1)$

$$= \frac{1}{4(x-1)^3} - \frac{7}{4(x-1)^2} + \frac{115}{16(x-1)} + O(1) \text{ after simplification.}$$

Similarly the terms required near $x = -1$ are $\frac{1}{8(x+1)^3} - \frac{7}{48(x+1)}$.

Near $x = 2$, the term is $\frac{85}{3(x-2)}$, and near $x = \infty$, the terms are $x^2 + 3x + 9$.

The integral is

$$\frac{1}{3}x^3 + \frac{3}{2}x^2 + 9x - \frac{1}{8(x-1)^2} + \frac{7}{4(x-1)} + \frac{115}{116}\log|x-1| - \frac{1}{8(x+1)}$$
$$- \frac{7}{48}\log|x+1| + \frac{85}{3}\log|x-2|.$$

(iv) $\int F(x)\,dx$ where $F(x) = \frac{(x+1)}{(x-1)(x^2+x+1)(x^2+x+2)}$.

For the part corresponding to $x^2 + x + 1$, develop $F(x)$ as follows

$$F(x) = \frac{(x+1)(x+2)}{(x^2+x-2)(x^2+x+1)(x^2+x+2)} = \frac{-x-1+3x+2}{(-3)(x^2+x+1)(1)} = \frac{-\frac{2}{3}x-\frac{1}{3}}{(x^2+x+1)}$$

(retaining the significant part)

Similarly for $x^2 + x - 2$ we obtain $\dfrac{x - 2 + 3x + 2}{(\ \ 4)(-1)(x^2 + x + 2)} = \dfrac{\frac{1}{2}x}{(x^2 + x + 2)}$

Thus $F(x) = \dfrac{1}{6(x-1)} - \dfrac{2x+1}{3(x^2+x+1)} + \dfrac{x}{2(x^2+x+2)}$ and

$$\int F(x)\,dx = \tfrac{1}{6}\log|x-1| - \tfrac{1}{3}\log(x^2+x+1) + \tfrac{1}{4}\log(x^2+x+2) - \frac{1}{2\sqrt{7}}\arc\tan\left(\frac{2x+1}{\sqrt{7}}\right)$$

(v) $\int F(x)\,dx$ where $F(x) = \dfrac{x^3 - x + 2}{(x-1)(x-2)(x^2+2x+3)}$

$F(x) = -\dfrac{1}{3(x-1)} + \dfrac{8}{11(x-2)} + \dfrac{Lx+M}{x^2+2x+3}$; let $x = 0$, then

$$\tfrac{1}{3} = \tfrac{1}{3} - \tfrac{4}{11} + \tfrac{1}{3}M$$

and if $x \to \infty$, $\dfrac{1}{x} \ldots = \dfrac{1}{x}\left(-\dfrac{1}{3} + \dfrac{8}{11} + L\right)$, i.e. $L = \dfrac{20}{33}$, $M = \dfrac{12}{11}$.

$$\int F(x)dx = -\tfrac{1}{3}\log|x-1| + \tfrac{8}{11}\log|x-2| + \tfrac{10}{33}\log(x^2+2x+3) + \frac{8\sqrt{2}}{33}\arc\tan\left(\frac{x+1}{\sqrt{2}}\right)$$

(vi) $\int F(x)\,dx$ where $F(x) = \dfrac{x^3 - x^2 + 2}{(x-1)(x^2+x+1)^2}$

For the factor, $(x^2+x+1)^2$, take $x^2 = -1 - x + \theta$, so that $x^3 = 1 + \theta(x-1)$, $x^4 = x - \theta(2x+1)$, $x^5 = -1 - x + (x+3)\theta$, (ignoring θ^2 and higher powers).

Then

$$\frac{(x^3 - x^2 + 2)(x-2)}{(\theta-3)\theta^2} = \frac{(-x+4\theta)(x-2)}{(\theta-3)\theta^2} = -\frac{\{(1-x)+(4x+7)\theta\}\{3+\theta\}}{9\theta^2}$$

gives the required terms $\dfrac{\frac{1}{3}(x-1)}{\theta^2} - \dfrac{\frac{11}{9}(x+2)}{\theta}$ so that

$$F(x) = 1 + \frac{2}{9(x-1)} - \frac{(x-1)}{3(x^2+x+1)^2} - \frac{11(x+2)}{9(x^2+x+1)}$$

By noting that

$$\frac{d}{dx}\left\{\frac{x}{x^2+x+1}\right\} = -\frac{1}{x^2+x+1} + \frac{x+2}{(x^2+x+1)^2};$$

$$\frac{d}{dx}\left(\frac{1}{x^2+x+1}\right) = -\frac{2x+1}{(x^2+x+1)^2}$$

we find that

$$3\int\frac{x\,dx}{(x^2+x+1)^2} = -\frac{x+2}{x^2+x+1} - \frac{2}{\sqrt{3}}\arc\tan\left(\frac{2x+1}{\sqrt{3}}\right).$$

$$3\int\frac{dx}{(x^2+x+1)^2} = \frac{2x+1}{x^2+x+1} + \frac{4}{\sqrt{3}}\arc\tan\left(\frac{2x+1}{\sqrt{3}}\right)$$

The integral is $x + \dfrac{2}{9}\log|x-1| - \dfrac{11}{18}\log(x^2+x+1) + \dfrac{x+1}{3(x^2+x+1)}$

$$- \frac{13\sqrt{3}}{27}\arc\tan\left(\frac{2x+1}{\sqrt{3}}\right).$$

5.07. Notes on the Integral of the Rational Function. (a) The above examples indicate that the above method of integration is of no practical value when the rational function has many multiple factors in the denominator (especially if those be quadratic) or when (as in the general case) the factors of the denominator can only be determined approxi-

mately. The theoretical value of the method (apart from its use in the simpler cases) lies in the fact that it gives the *form* of the integral. It shows that the integral is expressible in terms of the rational function, the logarithmic function and the inverse tangent.

Note. The inverse tangent is expressible in terms of the logarithmic function by means of the complex variable.

(*b*) Even when the denominator cannot be factorized (except approximately) it is always possible to determine the non-logarithmic part of the integral.

(*Ref.* Goursat, *Cours d'Analyse, I*, 5, where Hermite's method of establishing this result is given.)

(*c*) Simplifications may be made in the method of decomposition into partial fractions for certain types of rational functions.

Examples.

(i) $F(x) = \dfrac{x^7 - 2x^4 + x^3 + 3}{(x^2+1)(x^2+4)} = x\dfrac{x^6 + x^2}{(x^2+1)(x^2+4)} - \dfrac{2x^4 - 3}{(x^2+1)(x^2+4)}$

$$= x\left(x^2 - 5 - \frac{2}{3(x^2+1)} + \frac{68}{3(x^2+4)}\right) - \left(2 - \frac{1}{3(x^2+1)} - \frac{29}{3(x^2+4)}\right).$$

(ii) $\dfrac{x^2 + x - 1}{(x^2+x+1)(x^2+x+2)} = -\dfrac{2}{x^2+x+1} + \dfrac{3}{x^2+x+2}$, (being a rational function of $x^2 + x$).

(iii) $\dfrac{1}{x^8 - 16} = \dfrac{1}{8(x^4-4)} - \dfrac{1}{8(x^4+4)} = \dfrac{1}{32(x^2-2)} - \dfrac{1}{32(x^2+2)} - \dfrac{1}{8(x^2-2x+2)(x^2+2x+2)}$

$$= \frac{1}{64\sqrt{2}}\left(\frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\right) - \frac{1}{32(x^2+2)} + \frac{x-2}{64(x^2-2x+2)} - \frac{x+2}{64(x^2+2x+2)}.$$

(For a general denominator of the type $x^{2n} - a^{2n}$ it is better to use complex numbers.)

(iv) Let $F(x) = \dfrac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_m)(x-c)^n}$ where $a_1, a_2, \ldots, a_m, c$ are all different and where (for simplicity) the degree of $f(x)$ is less than m.

Then $F(x) = \left\{\sum_{r=1}^{m}\dfrac{A_r}{x-a_r}\right\}\dfrac{1}{(x-c)^n}$ where $A_r = \dfrac{f(a_r)}{\phi'(a_r)}$, $\left(\phi(x) = \prod_1^m (x - a_r)\right)$

i.e. $F(x) = \sum_{r=1}^{m}\dfrac{A_r}{c-a_r}\left(1 + \dfrac{\theta}{c-a_r}\right)^{-1}\dfrac{1}{\theta^n}$ near $\theta = 0$ where $x = c + \theta$

$$= \sum_{r=1}^{m}\sum_{s=0}^{n-1}\frac{A_r}{(c-a_r)^{s+1}}\frac{(-1)^s}{\theta^{n-s}}$$

so that $F(x) = \sum_{r=1}^{r=m}\left\{\dfrac{f(a_r)}{\phi'(a_r)}\cdot\dfrac{1}{(a_r-c)^n(x-a_r)} + \sum_{s=0}^{n-1}\dfrac{f(a_r)}{\phi'(a_r)(c-a_r)^{s+1}}\cdot\dfrac{(-1)^s}{(x-c)^{n-s}}\right\}.$

Example. $\dfrac{x+1}{(x-1)(x-3)(x-2)^n} = \left(\dfrac{2}{x-3} - \dfrac{1}{x-1}\right)\dfrac{1}{(x-2)^n}$

$$= \{-2(1+\theta+\theta^2+\dots) - (1-\theta+\theta^2)\dots\}\frac{1}{\theta^n} \text{ near } \theta = 0$$

i.e. $\dfrac{x+1}{(x-1)(x-2)(x-2)^n} = \dfrac{2}{x-3} - \dfrac{(-1)^n}{(x-1)} - \dfrac{3}{(x-2)^n} + \dfrac{1}{(x-2)^{n-1}}$

$$- \frac{3}{(x-2)^{n-2}} \dots - \frac{2+(-1)^{n-1}}{(x-2)}$$

(v) Let $F(x) = \dfrac{P(x)}{(Q_n(x))^r}$ where the degree of $Q_n(x)$ is n, and that of $P(x)$ is $(nr-1)$ at most. Assume $F(x) = \dfrac{d}{dx}\left\{\dfrac{A(x)}{Q_n^{r-1}}\right\} + \dfrac{B(x)}{Q_n}$ where the degree of $A(x)$ is $n(r-1)-1$ and that of $B(x)$ is $(n-1)$.

Then $P = Q_nA_n' - (r-1)AQ_n' + BQ_n^{r-1}$ giving nr relations for the determination of the nr unknown coefficients in A, B. On integrating, we find

$$\int F(x)\,dx = \frac{A}{Q_n^{r-1}} + \int\frac{B\,dx}{Q_n}.$$

Example. Let $F(x) = \dfrac{x^5+11x^4+48x^3+115x^2+143x+64}{(x^2+4x+5)^3}$.

Take $F(x) = \dfrac{d}{dx}\left\{\dfrac{ax^3+bx^2+cx+d}{(x^2+4x+5)^2}\right\} + \dfrac{ex+f}{(x^2+4x+5)}$ so that

$$x^5+11x^4+48x^3+115x^2+143x+64 = (x^2+4x+5)(3ax^2+2bx+c) - 2(2x+4)(ax^3+bx^2+cx+d) + (ex+f)(x^2+4x+5)^2.$$

It will be found that $a = 0$, $b = 1$, $c = 1$, $d = 2$, $e = 1$, $f = 3$, so that the integral is $\dfrac{x^2+x+2}{(x^2+4x+5)^2} + \dfrac{1}{2}\log(x^2+4x+5) + \text{arc tan}\,(x+2)$.

5.08. *Differentiation under the Sign of Integration.* Let $F(x, \alpha)$ be a continuous function possessing continuous derivatives $F_{x\alpha}$, $F_{\alpha x}$ which are therefore equal.

Then $F_\alpha = \int\dfrac{\partial}{\partial x}(F_\alpha)\,dx = \int\dfrac{\partial}{\partial\alpha}(F_x)\,dx$; but $F(x, \alpha) = \int F_x\,dx$, i.e. if we write $f(x)$ for F_x, we have

$$\frac{\partial}{\partial\alpha}\left\{\int f(x)\,dx\right\} = \int\frac{\partial f}{\partial\alpha}\,dx.$$

Examples. If by integration of the rational function $\dfrac{P(x)}{Q(x)(x-p)}$ where $P(p) \neq 0$, $Q(p) \neq 0$, we find that $\displaystyle\int\frac{P(x)\,dx}{Q(x)(x-p)} = F(x, p)$, then

$$\int\frac{P(x)\,dx}{Q(x)(x-p)^m} = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}F(x, p).$$

Thus (i) $\displaystyle\int\frac{dx}{(x-1)(x-2)(x-\alpha)} = \int\left\{\frac{1}{(2-\alpha)(x-2)} - \frac{1}{(1-\alpha)(x-1)} + \frac{1}{(\alpha-1)(\alpha-2)(x-\alpha)}\right\}dx \quad (\alpha \neq 1, 2)$

$$= \frac{1}{(2-\alpha)}\log|x-2| - \frac{1}{(1-\alpha)}\log|x-1| + \frac{1}{(\alpha-1)(\alpha-2)}\log|x-\alpha|.$$

Differentiating $(n-1)$ times we find

$$\int \frac{dx}{(x-1)(x-2)(x-a)^n} = \frac{1}{(2-a)^n}\log|x-2| - \frac{1}{(1-a)^n}\log|x-1|$$
$$+ \left(\frac{1}{(1-a)^n} - \frac{1}{(2-a)^n}\right)\log|x-a| + \dots + \frac{1}{(n-1)(a-1)(a-2)(x-a)^{n-1}},$$

(ii) Since $\int \frac{dx}{x^2+2px+q} = \frac{1}{\sqrt{(q-p^2)}}\arctan\left\{\frac{x+p}{\sqrt{(q-p^2)}}\right\}$, $(q > p^2)$,

we deduce that

$$\int \frac{dx}{(x^2+2px+q)^m} = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dq^{m-1}}\left\{\frac{1}{\sqrt{(q-p^2)}}\arctan\left(\frac{x+p}{\sqrt{(q-p^2)}}\right)\right\},$$

$$\int \frac{2x\,dx}{(x^2+2px+q)^m} = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dp\,dq^{m-2}}\left\{\frac{1}{\sqrt{(q-p^2)}}\arctan\left(\frac{x+p}{\sqrt{(q-p^2)}}\right)\right\}.$$

Thus since $\int \frac{dx}{(x^2+a^2)} = \frac{1}{a}\arctan\frac{x}{a}$

$$\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\arctan\frac{x}{a} + \frac{x}{2a^2(x^2+a^2)},$$

$$\int \frac{dx}{(x^2+a^2)^3} = \frac{3}{8a^5}\arctan\frac{x}{a} + \frac{3x}{8a^4(x^2+a^2)} + \frac{x}{4a^2(x^2+a^2)^2}$$

(after differentiation and simplification).

5.1. Integrals associated with Algebraic Curves. The integral $\int R(x, y)dx$ when x, y are connected by an algebraic relation $f(x, y) = 0$ is called an *Abelian Integral*. It is not, in general, expressible in terms of elementary functions. It is, however, so expressible when the corresponding algebraic curve is *unicursal* (i.e. has zero deficiency, for the co-ordinates of a point on such a curve can be expressed rationally in terms of a parameter t).

Example. $\int y\,dx$ when $4y^3 - 12y^2 = x^4 - 8x^2$.

The curve is a quartic with 3 double points $(0, 0)$, $(\pm 2, 2)$ and is therefore unicursal. The co-ordinates of a point on it can be expressed in the form

$$x = 2t(2t^2 - 3),\ y = 4t^4 - 8t^2 + 3$$

and so the integral becomes $6\int(4t^4 - 8t^2 + 3)(2t^2 - 1)dt$.

Notes. (i) When the deficiency is not zero, the integral may be expressible in terms of elementary functions in *particular* cases.

Thus $$\int \frac{(2-3x^2)\,dx}{(1-x^2+x^4)\sqrt{(1+x^2-x^4)}} = \frac{2x}{\sqrt{(1+x^2-x^4)}}.$$

(ii) When the deficiency is *unity*, it can be proved that the co-ordinates of a point on it can be expressed as a *rational* function of t, u, where u^2 is a *quartic* (or *cubic*) in t. (*Ref. Clebsch, Crelle*, 64, 210.) Abelian Integrals of this type are called *Elliptic* and they can be expressed in terms of elementary functions and Elliptic Functions.

5.11. Integrals connected with Conics. The conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

being unicursal, the integration of $\int R(x, y)dx$ can be reduced to that of

a rational function of one variable t (although this general theoretical method is not necessarily the simplest).

Example. Find $\int \frac{dx}{y}$ when $x^2 + xy + y^2 = x - y$.

Let $y = tx$, then $x = \frac{1 - t}{1 + t + t^2}$, $y = \frac{t(1 - t)}{1 + t + t^2}$

and $$\int \frac{dx}{y} = \int \frac{(t^2 - 2t - 2)\,dt}{t(1 - t)(1 + t + t^2)} = \log|1 - t| - 2\log|t|$$
$$- \frac{1}{2}\log(1 + t + t^2) - \sqrt{3}\arctan\left(\frac{2t + 1}{\sqrt{3}}\right)$$
$$= \frac{1}{2}\log\left|\frac{(x - y)^3}{y^4}\right| - \sqrt{3}\arctan\left(\frac{x + 2y}{x\sqrt{3}}\right), \ (x \neq y \neq 0).$$

5.12. The Conic $y^2 = ax^2 + 2bx + c$ $(ac \neq b^2)$. The conic of most frequent occurrence in this connection is that given by
$$y^2 = ax^2 + 2bx + c$$
which is a hyperbola, a parabola or an ellipse according as $a >, =$ or < 0. The *general* equation of the conic may in fact be reduced to this form by a suitable change of the axes of reference.

(i) $a = 0$, $y^2 = 2bx + c$. Then $R(x, y) = R\{(y^2 - c)/2b, y\}$ and $b = y\frac{dy}{dx}$ so that the integral may be written $\int R_1(y)\,dy$, where R_1 is rational.

Example. $$\int \frac{(2x - 1)^{\frac{3}{2}}}{x + 1}\,dx = \int \frac{2y^4 dy}{y^2 + 3} \text{ if } y^2 = 2x - 1$$
$$= \frac{2y}{3}(y^2 - 9) + 6\sqrt{3}\arctan\left(\frac{y}{\sqrt{3}}\right)$$
$$= \frac{4(x - 5)}{3}(2x - 1)^{\frac{1}{2}} + 6\sqrt{3}\arctan\left(\frac{2x - 1}{3}\right)^{\frac{1}{2}}.$$

(ii) $a \neq 0$, $y^2 = ax^2 + 2bx + c$ $(b^2 \neq ac)$. The substitution of $ax^2 + 2bx + c$ for y^2 in $R(x, y)$ reduces $R(x, y)$ to the form $\frac{A + yB}{C + yD}$ where A, B, C, D are polynomials in x. Assuming that C, D do not vanish for any value of x under consideration we may multiply numerator and denominator by $C - yD$, and obtain $R(x, y) = R_1(x) + \frac{R_2(x)}{y}$ ($ax^2 + 2bx + c$ being substituted for y^2 wherever necessary), $R_1(x)$, $R_2(x)$ being rational functions of x.

The integration of $R_1(x)$ has already been considered; in the other part, let $R_2(x)$ be expressed in partial fractions and it will then be seen that the integration of $R_2(x)/y$ depends on integrals of the type
$$(a)\ I_m = \int \frac{x^m}{y}\,dx, \quad (b)\ J_m = \int \frac{dx}{(x - \alpha)^m y},$$
$$(c)\ K_m = \int \frac{x\,dx}{(x^2 + 2px + q)^m y}, \quad (d)\ Z_m = \int \frac{dx}{(x^2 + 2px + q)^m y},$$
where m is a positive integer (or zero) and α, p, q real with $p^2 < q$.

(a) $I_m = \int \frac{x^m}{y} dx$. Differentiation of yx^{m-1} will show that

$$maI_m + b(2m-1)I_{m-1} + c(m-1)I_{m-2} = yx^{m-1}, \ (m > 1)$$

$aI_1 + bI_0 = y$ so that I_m is expressible in terms of I_0.

Also I_0 is easily reduced to one of the standard forms (if $u = x + b/a$),

(i) $\frac{1}{\sqrt{a}}\int \frac{du}{\sqrt{(u^2 + A)}}$, $a > 0$, (ii) $\frac{1}{\sqrt{(-a)}}\int \frac{du}{\sqrt{(B - u^2)}}$ $(a < 0)$ where

$$A = \frac{ac - b^2}{a^2}, \ B = \frac{b^2 - ac}{a^2}.$$

The integral is not real if $a < 0$, $ac > b^2$.

(b) $J_m = \int \frac{dx}{(x-\alpha)^m y}$. Differentiation of $y(x-\alpha)^{1-m}$ will show that

$$(m-1)(a\alpha^2 + 2b\alpha + c)J_m + (2m-3)(a\alpha + b)J_{m-1} + a(m-2)J_{m-2} = -y(x-\alpha)^{1-m} \ (m > 1),$$

$$(a\alpha^2 + 2b\alpha + c)J_2 + (a\alpha + b)J_1 = -y(x-\alpha)^{-1}$$

and $J_1 = \int \frac{dx}{(x-\alpha)\sqrt{(ax^2 + 2bx + c)}}$ reduces to the type I_0 by the substitution $u(x-\alpha) = 1$.

(c), (d) Differentiation of

$$xy(x^2 + 2px + q)^{1-m} \text{ and } y(x^2 + 2px + q)^{1-m}$$

will give two linear relations connecting $K_m, Z_m, K_{m-1}, Z_{m-1}, Z_{m-2}$; so that K_m, Z_m are theoretically expressible in terms of K_1, Z_1, Z_0 $(= I_0)$. No useful purpose, however, is served by obtaining the actual reduction formulae as, owing to the numerical labour involved, they cease to be of practical value. We shall therefore only consider the integration of K_1, Z_1.

5.13. The Integration of

$$K_1 = \int \frac{x\,dx}{(x^2 + 2px + q)y}, \ Z_1 = \int \frac{dx}{(x^2 + 2px + q)y}.$$

(i) Consider first the simpler cases

$$E = \int \frac{x\,dx}{(x^2 + k^2)\sqrt{(Ax^2 + B)}}, \ F = \int \frac{dx}{(x^2 + k^2)\sqrt{(Ax^2 + B)}}.$$

In E, write $u^2 = Ax^2 + B$ and E becomes $\int \frac{du}{u^2 + Ak^2 - B}$.

In F, write $u^2 = A + \frac{B}{x^2}$ and F becomes $\int \frac{du}{k^2u^2 - k^2A + B}$ and these new expressions for E, F are standard forms.

(ii) We shall now express K_1, Z_1 in terms of integrals of the type E, F. The quadratic $(x^2 + 2px + q) - \lambda(ax^2 + 2bx + c)$ is a perfect square if $\phi(\lambda) \equiv (1 - a\lambda)(q - c\lambda) - (p - b\lambda)^2 = 0$.

The roots of this equation in λ are real and different, since $\phi(1/a) < 0$, $\phi(0) = q - p^2 > 0$. If the roots are λ_1, λ_2 we have

$$x^2 + 2px + q - \lambda_1(ax^2 + 2bx + c) = (1 - a\lambda_1)(x - \alpha)^2$$

$$x^2 + 2px + q - \lambda_2(ax^2 + 2bx + c) = (1 - a\lambda_2)(x - \beta)^2, \ (\lambda_1, \lambda_2 \neq 1/a)$$

Thus
$$ax^2 + 2bx + c = A_1(x - \alpha)^2 + B_1(x - \beta)^2,$$
$$x^2 + 2px + q = A_2(x - \alpha)^2 + B_2(x - \beta)^2.$$

Now take $t = \dfrac{x - \alpha}{x - \beta}$, then

$$\int \frac{(Lx + M)\,dx}{(x^2 + 2px + q)\sqrt{(ax^2 + 2bx + c)}}$$

is of the form $\mu\displaystyle\int \frac{(L_1 t + M_1)\,dt}{(t^2 + k^2)\sqrt{(A_1 t^2 + B_1)}}$, i.e. is $\rho E + \sigma F$.

The case when a root of the λ-equation is $1/a$ is left as an exercise to the reader.

Examples. (i) Find $\displaystyle\int \frac{x^2\,dx}{\sqrt{(x^2 + 2x + 2)}}$.

Since $\dfrac{d}{dx}\{x\sqrt{(x^2 + 2x + 2)}\} = \dfrac{2x^2 + 3x + 2}{\sqrt{(x^2 + 2x + 2)}}$, it follows that

$$2I = x\sqrt{(x^2 + 2x + 2)} - \int \frac{(3x + 2)\,dx}{\sqrt{(x^2 + 2x + 2)}}$$

i.e.

$$I = \tfrac{1}{2}x\sqrt{(x^2 + 2x + 2)} - \tfrac{3}{2}\sqrt{(x^2 + 2x + 2)} + \tfrac{1}{2}\log\{x + 1 + \sqrt{(x^2 + 2x + 2)}\}$$

(ii) $$\int \frac{dx}{(x - 2)\sqrt{(1 + x^2)}} = -\int \frac{du}{\sqrt{(5u^2 + 4u + 1)}} \text{ if } u = \frac{1}{x - 2}$$

$$= -\frac{1}{\sqrt{5}}\log\left\{u + \frac{2}{5} + \sqrt{\left(u^2 + \frac{4u}{5} + \frac{1}{5}\right)}\right\}$$

$$= \frac{1}{\sqrt{5}}\log\left\{\frac{x - 2}{2x + 1 + \sqrt{(5x^2 + 5)}}\right\} + C.$$

(iii) $$\int \frac{(x - 2)\,dx}{(x^2 + 4)\sqrt{(x^2 + 1)}} = \int \frac{du}{u^2 + 3} + \int \frac{dv}{4v^2 - 3}$$

$$\left(\text{if } u^2 = x^2 + 1,\ v^2 = 1 + \frac{1}{x^2}\right)$$

$$= \frac{1}{\sqrt{2}}\arctan\sqrt{\left(\frac{x^2 + 1}{3}\right)} - \frac{1}{2\sqrt{3}}\log\left(\frac{2\sqrt{(1 + x^2)} + x\sqrt{3}}{2\sqrt{(1 + x^2)} - x\sqrt{3}}\right).$$

(iv) $\displaystyle\int \frac{(2x + 1)\,dx}{(3x^2 - 10x + 9)\sqrt{(5x^2 - 16x + 14)}}$. The λ-equation is

$$(8 - 5\lambda)^2 = (5 - 3\lambda)(14 - 9\lambda)$$

from which we find $\lambda = 2$, $\frac{3}{2}$ and

$5x^2 - 16x + 14 = 3(x - 2)^2 + 2(x - 1)^2$, $3x^2 - 10x + 9 = 2(x - 2)^2 + (x - 1)^2$.

Thus the integral is $\displaystyle\int \frac{\{5(x - 1) - 3(x - 2)\}\,dt}{(x - 1)(2t^2 + 1)\sqrt{(3t^2 + 2)}}$ if $t = \dfrac{x - 2}{x - 1}$

i.e. is $$-3\int \frac{t\,dt}{(2t^2 + 1)\sqrt{(3t^2 + 2)}} + 2\int \frac{dt}{(2t^2 + 1)\sqrt{(3t^2 + 2)}}$$

$$= -3\int \frac{du}{2u^2 - 1} - 2\int \frac{dv}{v^2 + 1} \text{ if } u^2 = 3t^2 + 2,\ v^2 = 3 + \frac{2}{t^2}$$

Its value is

$$-\frac{3}{2\sqrt{2}}\log\left\{\frac{\sqrt{(10x^2-32x+28)}+x-1}{\sqrt{(10x^2-32x+28)}-x+1}\right\}-5\ \text{arc tan}\left\{\frac{\sqrt{(5x^2-16x+14)}}{x-2}\right\}$$

(v) Evaluate $\int\frac{dx}{(x^2+b^2)^2\sqrt{(x^2+a^2)}}$ in terms of $I=\int\frac{dx}{(x^2+b^2)\sqrt{(x^2+a^2)}}$.

Differentiating $\frac{x\sqrt{(x^2+a^2)}}{x^2+b^2}$ we find $\frac{d}{dx}\left(\frac{x\sqrt{(x^2+a^2)}}{(x^2+b^2)}\right)=\frac{(2b^2-a^2)x^2+a^2b^2}{(x^2+b^2)^2\sqrt{(x^2+a^2)}}$.

Thus the integral is $\frac{1}{2b^2(a^2-b^2)}\left\{\frac{x\sqrt{(x^2+a^2)}}{(x^2+b^2)}-(2b^2-a^2)I\right\}$.

Notes. (i) An alternative substitution for $\int\frac{dx}{(x^2+b^2)\sqrt{(x^2+a^2)}}$ is $x=a\tan\theta$.

(ii) An alternative method for the original integral $\int\frac{(Lx+M)\,dx}{(x^2+2px+q)y}$ is to use Greenhill's substitution $u^2=y/(x^2+2px+q)$. (*Ref. Hardy, Integration of Functions of a Single Variable*, 24.)

5.2. Integration of Transcendental Functions. When the integrand is a rational function whose arguments are powers of x, exponential, logarithmic and circular functions, then, apart from particular cases, there appear to be only two types of integrand that are easily determined (theoretically) in terms of elementary functions. These are considered in the following paragraphs.

5.21. $\int R(e^{\alpha x}, e^{\beta x}, e^{\gamma x}, \ldots, e^{\kappa x})\,dx$, *where R is rational and the numbers $\alpha, \beta, \gamma, \ldots, \kappa$ are commensurable.* This concise description of a suitable integrand is used here for convenience but we shall regard it as inclusive of the type $R(\sin px, \cos px, \sin qx, \cos qx, \ldots)$ (when p, q, . . . are commensurable) since the latter function may be shown to be of the same type as the former by the use of complex numbers. The integration of the latter function will however be considered separately.

If $\alpha, \beta, \gamma, \ldots$, are commensurable, a number c $(\neq 0)$ exists such that $\alpha=m_1c$, $\beta=m_2c$, . . ., where m_1, m_2, . . . are integers. If $y=e^{cx}$; $e^{\alpha x}=y^{m_1}$, $e^{\beta x}=y^{m_2}$, . . ., $\frac{dy}{dx}=cy$ and the integral

$$\int R(e^{\alpha x}, e^{\beta x}, \ldots)\,dx=\int R_1(y)\,dy$$

where $R_1(y)$ is rational.

Example. $\int\frac{e^x+2}{e^{2x}+1}\,dx=\int\frac{(2+y)\,dy}{y(y^2+1)}$ if $y=e^x$

$$=2\log y-\log(1+y^2)+\text{arc tan}\,y$$
$$=2x-\log(1+e^{2x})+\text{arc tan}\,(e^x).$$

5.22. The Integration of $\int R(\sin x, \cos x)\,dx$. (*a*) *A general* method consists in using the substitution $t=\tan\frac{1}{2}x$, for then

$$\int R(\sin x, \cos x)\,dx=\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}$$

Example. $$\int \frac{dx}{3 + \cos x - 2 \sin x} = \int \frac{dt}{t^2 - 2t + 2} = \operatorname{arc\,tan}(t - 1)$$
$$= \operatorname{arc\,tan}\left(\frac{\sin x - \cos x - 1}{1 + \cos x}\right)$$

(b) *Before* using the general method, we should take the *simpler* substitutions (i) $u = \sin x$, (ii) $u = \cos x$, or (iii) $u = \tan x$ when these are suitable.

Examples. (i) $\int \sin^8 x \cos^3 x \, dx = \int u^8(1 - u^2)du$ if $u = \sin x$
$= \frac{1}{9} \sin^9 x - \frac{1}{11} \sin^{11} x.$

(ii) $$\int \frac{\sin^3 x}{4 + \cos x} dx = -\int \frac{(1 - u^2)\, du}{4 + u} \text{ if } u = \cos x$$
$$= 15 \log (u + 4) - 8(u + 4) + \tfrac{1}{2}(u + 4)^2$$
$$= \tfrac{1}{2} \cos^2 x - 4 \cos x + 15 \log (4 + \cos x) + C$$

(iii) $$\int \frac{dx}{2 \cos^2 x - 5 \sin x \cos x + 2 \sin^2 x} = \int \frac{du}{(2u - 1)(u - 2)} \text{ if } u = \tan x$$
$$= \frac{1}{3} \log \left|\frac{u - 2}{2u - 1}\right| = \frac{1}{3} \log \left|\frac{\sin x - 2 \cos x}{2 \sin x - \cos x}\right|$$

(c) Arithmetical labour may be saved sometimes by the direct integration of part of the integrand.

For example if
$$R(\cos\theta, \sin\theta) = \frac{Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C}{ax + by + c}$$
where $x = \cos\theta$, $y = \sin\theta$, we can determine uniquely constants p, q, r, λ, μ, ν such that
$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C$$
$$\equiv (ax + by + c)(px + qy + r) + \nu(x^2 + y^2 - 1) + \lambda(bx - ay) + \mu.$$
Then $$\int R(\cos\theta, \sin\theta)d\theta = p \sin\theta - q \cos\theta$$
$$+ r\theta + \lambda \log |a \cos\theta + b \sin\theta + c| + \mu \int \frac{d\theta}{a \cos\theta + b \sin\theta + c}$$

Example. $$\int \frac{2 \cos^2\theta + 4 \sin\theta \cos\theta - \sin^2\theta + 2 \cos\theta - 6 \sin\theta + 3}{3 \cos\theta + \sin\theta + 4} d\theta.$$

On determining the constants, we find that the numerator is equal to
$$(3 \cos\theta + \sin\theta + 4)\left(\frac{13}{10} \cos\theta + \frac{9}{10} \sin\theta - \frac{48}{25}\right) + \frac{19}{10}(\cos^2\theta + \sin^2\theta - 1)$$
$$- \frac{64}{25}(\cos\theta - 3 \sin\theta) + \frac{439}{50}$$
and the integral is
$$\frac{13}{10} \sin\theta - \frac{9}{10} \cos\theta - \frac{48}{25}\theta + \frac{64}{25} \log (3 \cos\theta + \sin\theta + 4)$$
$$+ \frac{439\sqrt{6}}{150} \operatorname{arc\,tan}\left(\frac{1 + \tan \frac{1}{2}\theta}{\sqrt{6}}\right)$$

5.23. *The Integration of* $\int \sin^p x \cos^q x \, dx$ *where* p, q *are integers* ($\pm$)

The substitution $u = \tan \frac{1}{2}x$ makes the integrand rational in u, but it

is better in this case to obtain the reduction formulae directly. The obvious principle is to relate the above integral to a similar one in which one at least of the indices is numerically decreased.

For example, if m, n denote *positive* integers,

(i) $$(m+n)\int \sin^m x \cos^n x\,dx = \sin^{m+1} x \cos^{n-1} x + (n-1)\int \sin^m x \cos^{n-2} x\,dx.$$

(ii) $$(n-1)\int \frac{\sin^m x\,dx}{\cos^n x} = \frac{\sin^{m-1} x}{\cos^{n-1} x} - (m-1)\int \frac{\sin^{m-2} x\,dx}{\cos^{n-2} x}.$$

(iii) $$(m-1)\int \frac{dx}{\sin^m x \cos^n x} = -\frac{1}{\sin^{m-1} x \cos^{n-1} x} + (m+n-2)\int \frac{dx}{\sin^{m-2} x \cos^n x}.$$

Other suitable formulae will be found in *Examples V, 94–8*. Important particular cases arise when $p = 0$ or $q = 0$ or $p + q = 0$, for which formulae of the following type are useful:

(i) $$n\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx.$$

(ii) $$(n-1)\int \tan^n x\,dx = \tan^{n-1} x - (n-1)\int \tan^{n-2} x\,dx.$$

(iii) $$(n-1)\int \sec^n x\,dx = \sin x \sec^{n-1} x + (n-2)\int \sec^{n-2} x\,dx.$$

(*See also Examples V, 99–104.*)

Notes. (i) The substitutions $u = \sin x$, $\cos x$ or $\tan x$ may sometimes be immediately effective.

Thus (i) if p, q are both odd ([illegible]), take $u = \cos x$ or $u = \sin x$; (ii) if p is odd, q even, take $u = \cos x$; (iii) if p is even, q odd, take $u = \sin x$; (iv) if p, q are both even, try $u = \tan x$.

Examples. (i) $$\int \frac{\sin^5 x\,dx}{\cos^2 x} = -\int \frac{(1-u^2)^2\,du}{u^2} \text{ (if } u = \cos x)$$
$$= \sec x + 2\cos x - \frac{1}{3}\cos^3 x.$$

(ii) $$\int \frac{\sin^n x\,dx}{\cos^{n+2} x} = \int u^n\,du \text{ (if } u = \tan x) = \frac{1}{n+1}\tan^{n+1} x \ (n \neq -1).$$

(ii) *Multiple* angles may be introduced when p, q are positive integers. Thus $\sin^p x \cos^q x$ may be expressed as a linear combination of sines and cosines of multiples of x. This expression is most conveniently obtained by the use of the complex number, but can also be quickly found by the addition theorem when p, q are not large.

Example. $$\sin^4 x \cos^2 x = \tfrac{1}{4}(\sin^2 2x)\tfrac{1}{2}(1 - \cos 2x)$$
$$= \tfrac{1}{16} - \tfrac{1}{32}\cos 2x - \tfrac{1}{16}\cos 4x + \tfrac{1}{32}\cos 6x$$
and $$\int \sin^4 x \cos^2 x\,dx = \tfrac{1}{16}x - \tfrac{1}{64}\sin 2x - \tfrac{1}{64}\sin 4x + \tfrac{1}{192}\sin 6x.$$

5.24. The Integration of $\int P(x, e^{ax}, e^{bx}, \ldots, \sin(mx+n), \ldots)\,dx$,

Where the integrand consists of a finite number of terms of the form $x^p e^{ax} \sin^q bx \cos^r cx \sin^s kx \cos^t lx \ldots$ and p, q, r, s, t are positive integers whilst $a, b, c, k, l, \ldots$ are any real numbers. The integrand is therefore equivalent to the sum of a finite number of pairs of terms of the type $Ax^p e^{ax} \cos \lambda x + Bx^p e^{ax} \sin \lambda x$, where p is a positive integer or zero.

5.25. The Integration of $C_p = \int x^p e^{ax} \cos \lambda x\,dx$ and

$$S_p = \int x^p e^{ax} \sin \lambda x\,dx.$$

By differentiation of $x^p e^{ax} \cos \lambda x$, $x^p e^{ax} \sin \lambda x$ we obtain

$$aC_p - \lambda S_p + pC_{p-1} = x^p e^{ax} \cos \lambda x$$
$$aS_p + \lambda C_p + pS_{p-1} = x^p e^{ax} \sin \lambda x \ (p > 0).$$

These reduction formulae, together with $aC_0 - \lambda S_0 = e^{ax} \cos \lambda x$; $aS_0 + \lambda C_0 = e^{ax} \sin \lambda x$, determine C_p, S_p.

In particular $C_0 = \int e^{ax} \cos \lambda x\,dx = e^{ax}(a \cos \lambda x + \lambda \sin \lambda x)/(a^2 + \lambda^2)$

and $S_0 = \int e^{ax} \sin \lambda x\,dx = e^{ax}(a \sin \lambda x - \lambda \cos \lambda x)/(a^2 + \lambda^2)$.

Example. Find $\int xe^{2x} \cos x\,dx\ (= C_1)$.

Here $2C_1 - S_1 + C_0 = xe^{2x} \cos x$, $2S_1 + C_1 + S_0 = xe^{2x} \sin x$

$$2C_0 - S_0 = e^{2x} \cos x, \quad 2S_0 + C_0 = e^{2x} \sin x$$

Thus

$$5C_0 = e^{2x}(2 \cos x + \sin x); \quad 5S_0 = e^{2x}(2 \sin x - \cos x);$$
$$5C_1 = xe^{2x}(2 \cos x + \sin x) - 2C_0 - S_0$$

so that finally $\qquad C_1 = \frac{1}{25} e^{2x}\{(10x - 3) \cos x + (5x - 4) \sin x\}$.

Notes. (i) The reader may verify the alternative reduction formulae

$$\left.\begin{aligned} C_p &= \frac{x^p e^{ax}}{r} \cos(\lambda x - \alpha) - \frac{p}{r}\int x^{p-1} e^{ax} \cos(\lambda x - \alpha)\,dx \\ S_p &= \frac{x^p e^{ax}}{r} \sin(\lambda x - \alpha) - \frac{p}{r}\int x^{p-1} e^{ax} \sin(\lambda x - \alpha)\,dx \end{aligned}\right\}$$

where (r, α) are the polar co-ordinates of the point (a, λ).

Example. $\int xe^{2x} \cos x\,dx = \frac{x}{r} e^{2x} \cos(x - \alpha) - \frac{e^{2x}}{r^2} \cos(x - 2\alpha)$,

$$\left(r = \sqrt{5},\ \sin \alpha = \frac{1}{r},\ \cos \alpha = \frac{2}{r}\right)$$

$$= e^{2x}\left\{\frac{x}{5}(2 \cos x + \sin x) - \frac{1}{25}(3 \cos x + 4 \sin x)\right\}$$

giving the same result as before.

(ii) The integrals may also be found by differentiating under the integral sign, since

$$C_p = \frac{d^p}{da^p}\left\{\int e^{ax} \cos \lambda x\,dx\right\}, \quad S_p = \frac{d^p}{da^p}\left\{\int e^{ax} \sin \lambda x\,dx\right\}.$$

Example. Find $\int x^2 \cos x\,dx$.

Since $\int \cos ax\,dx = \frac{1}{a} \sin ax$, we have $\int x \sin ax\,dx = -\frac{x}{a} \cos ax + \frac{1}{a^2} \sin ax$

and $\quad \int x^2 \cos ax\,dx = \left(\frac{x^2}{a} - \frac{2}{a^3}\right) \sin ax + \frac{2x}{a^2} \cos ax$

$$= (x^2 - 2) \sin x + 2x \cos x \ (\text{if } a = 1).$$

(iii) Other effective methods of determining C_p, S_p are

(a) by the use of the complex number whereby C_p, for example, is the real part of the integral $\int x^p e^{(a+ib)x}\,dx$. (*Chap. X.*)

(b) by using the properties of the linear operator $D\left(=\frac{d}{dx}\right)$.

Examples V (a)

Find the indefinite integrals of the functions given in *Examples 1–56*

1. $\frac{2x+3}{3x^2+4}$ 2. $\tanh x$ 3. $\frac{x^2}{x-1}$ 4. $\frac{e^{2x}+1}{e^{2x}-1}$

5. $\frac{x}{(x-1)(x+2)}$ 6. $\frac{e^x}{e^{2x}-1}$ 7. $\frac{x^2+2}{\sqrt{(4-x^2)}}$

8. $\frac{x^4+1}{(x^2-1)(x+2)}$ 9. $\cosh^3 x$ 10. $\sinh^3 x$ 11. $\tanh^3 x$

12. $\frac{(x+1)}{(x-1)(x^2+2x+5)}$ 13. $\frac{1}{(x-1)^3(x-2)^2}$ 14. $\operatorname{cosech}^4 x$

15. $\operatorname{sech}^3 x$ 16. $\frac{1}{\sin x - \cos x}$ 17. $\frac{x^4}{(x-1)^2(x+1)}$ 18. $\frac{x^3+1}{x^3-1}$

19. $\frac{x^3-1}{x^3+1}$ 20. $\frac{x^3}{\sqrt{(1-x^2)}}$ 21. $x^3\sqrt{(1-x)}$ 22. $x^2\sqrt{\left(\frac{1-x}{1+x}\right)}$

23. $\frac{x^4}{(x^2+1)(x^2+9)}$ 24. $\frac{3x-2}{5-4x+x^2}$ 25. $\frac{x^2+1}{3+4x-4x^2}$

26. $\frac{2x+3}{\sqrt{(4x^2-4x+5)}}$ 27. $\frac{x+2}{\sqrt{(9x^2+12x+5)}}$

28. $\frac{1-x}{\sqrt{(5+24x-16x^2)}}$ 29. $\frac{3x+5}{\sqrt{(7-24x-16x^2)}}$

30. $\frac{x^2+2}{\sqrt{(4-x^2)}}$ 31. $\frac{1}{\sqrt{(x+1)}+\sqrt{(x-1)}}$ 32. $\frac{\sqrt{(x+2)}+1}{\sqrt{(x+2)}-\sqrt{(x-2)}}$

33. $\frac{1}{13-12\cos x}$ 34. $\frac{1}{13-5\cos x}$ 35. $\frac{1}{12+13\cos x}$

36. $\frac{2-\sin x}{5\sin x-4}$ 37. $\frac{1+\cos x-\sin x}{5+7\cos x-\sin x}$

38. $\frac{\cos^2 x+2\sin^2 x}{2\cos^2 x+3\sin^2 x}$ 39. $\cos^2 x\sin^7 x$ 40. $\sin^3 x\cos^3 x$

41. $\cos^3 x\sqrt{(\sin x)}$ 42. $\sec^4 x\operatorname{cosec}^2 x$ 43. $\sec^2 x\operatorname{cosec}^2 x$

44. $\sqrt{(\tan x)}$ 45. $\sqrt{\left(\frac{e^x-1}{e^x+1}\right)}$ 46. $\frac{x^2}{x^6-x^3-1}$

47. $\frac{3x^2-2x+1}{(x^2-1)(x^2-2)(x^2+3)}$ 48. $\frac{1}{(x^2-1)^4}$

49. $\frac{3\cos x-1}{\cos^2 x+\cos x-6}$ 50. $\frac{\sin x+2\cos x}{\sin^2 x+3\sin x+2}$

51. $\cos x\cos 2x$ 52. $\sin 2x\cos 3x$ 53. $\cos^2 x\sin 3x$

54. $\frac{2\sin x+3\cos x}{\cos 2x(\cos x+2\sin x)}$ 55. $\frac{x^5}{\sqrt{(x^6-1)}}$

56. $\frac{\sin^2 x+2\sin x\cos x-10\sin x-2\cos x+2}{2-\cos x-\sin x}$

Integrate the functions given in *Examples 57–79* by the method of "Integration by Parts."

57. $x^n\log x$ 58. $(\log x)^2$ 59. $x^2(\log x)^2$ 60. $\arcsin x$

61. $\arctan x$ **62.** $\arctan\left(\frac{1-x}{1+x}\right)$ **63.** $x \arcsin x$ **64.** $x^2 \sin x$

65. $x^2 \arctan x$ **66.** $x^3 e^{2x}$ **67.** $x^4 e^{-x}$ **68.** $x^3 \cos 2x$

69. $x^3 \sin^2 x$ **70.** $x \sin x \cos x$ **71.** $x \sin 3x$ **72.** $x^5 e^{-x^2}$

73. $x^3 e^{-x^2}$ **74.** $\frac{\arctan x}{x^2}$ **75.** $e^{2x} \cos x$ **76.** $e^{-x} \sin 2x$

77. $\sqrt{(4-x^2)}$ **78.** $x^2\sqrt{(1-x)}$ **79.** $x^3(1-x)^{\frac{1}{2}}$

Obtain Reduction Formulæ for the integrals of the functions given in *Examples 80–93* (n and m being positive integers).

80. $x^n\sqrt{(1-x^2)}$ **81.** $x^m(\log x)^n$ **82.** $x^n e^{ax}$ **83.** $x^n \sin x$

84. $x^n(x-1)^m$ **85.** $(ax^2+2bx+c)^{-n}$ **86.** $(ax^2+2bx+c)^{\frac{1}{2}n}$

87. $x^n(1+x^2)^{-\frac{1}{2}}$ **88.** $x^{-n}(1+x^2)^{-\frac{1}{2}}$

89. $\frac{1}{(x^2+a^2)^n\sqrt{(x^2+b^2)}}$ **90.** $\frac{\cos^m x}{\sin^n x}$ **91.** $\frac{1}{(a+b\cos x)^n}$

92. $\cos nx \cos^m x$ **93.** $\cos nx \sin^m x$

Establish the Reduction Formulæ given in *Examples 94–104.*

94. $(m+n)\int \sin^m x \cos^n x\,dx = \sin^{m+1} x \cos^{n-1} x + (n-1)\int \sin^m x \cos^{n-2} x\,dx$
$= -\sin^{m-1} x \cos^{n+1} x + (m-1)\int \sin^{m-2} x \cos^n x\,dx$

95. $I_{m,n} = \int \frac{\cos^n x}{\sin^m x}\,dx = -\frac{\cos^{n-1} x}{(m-1)\sin^{m-1} x} - \frac{n-1}{m-1} I_{m-2,n-2}$

96. $I_{m,n} = \int \frac{\sin^m x\,dx}{\cos^n x} = \frac{\sin^{m-1} x}{(n-1)\cos^{n-1} x} - \frac{m-1}{n-1} I_{m-2,n-2}$

97. $I_{m,n} = \int \frac{dx}{\sin^m x \cos^n x} = \frac{1}{(n-1)\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{n-1} I_{m,n-2}$

98. $I_{m,n} = \int \frac{dx}{\sin^m x \cos^n x} = -\frac{1}{(m-1)\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1} I_{m-2,n}$

99. $I_n = \int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$

100. $I_n = \int \cos^n x\,dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$

101. $I_n = \int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

102. $I_n = \int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$

103. $I_n = \int \sec^n x\,dx = \frac{\sin x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

104. $I_n = \int \mathrm{cosec}^n x\,dx = -\frac{\cos x\,\mathrm{cosec}^{n-1} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

Integrate the functions given in *Examples 105–34.*

105. $x^3\sqrt{(1-x^2)}$ **106.** $\sin^6 x \cos^7 x$ **107.** $\sin^3 x \cos^5 x$

108. $\sin^4 x \cos^6 x$ **109.** $\sin^6 x \sec^3 x$ **110.** $\cos^4 x\,\mathrm{cosec}^6 x$

111. $\sec^6 x\,\mathrm{cosec}^6 x$ **112.** $\sec^5 x\,\mathrm{cosec}^2 x$ **113.** $x^3(x-1)^{10}$

114. $(2x^2+3)^{\frac{1}{2}}$ **115.** $x^4(\log x)^3$ **116.** $x^3 e^{-2x}$ **117.** $e^{-3x}\cos x$

118. $(2x^2-5)^{-\frac{1}{2}}$ **119.** $(4x^2-3)^{-\frac{1}{2}}$ **120.** $x^4(x^2+1)^{-\frac{1}{2}}$

121. $\frac{1}{x^3\sqrt{(1+x^2)}}$ **122.** $\frac{\cos x - \sin x}{(5+4\cos x)^2}$ **123.** $\frac{1}{(x^2-1)\sqrt{(2x^2+3)}}$

124. $\dfrac{x-1}{(x+1)(x+2)\sqrt{(x^2+1)}}$ **125.** $\dfrac{x}{(x-1)\sqrt{(x^2-4)}}$

126. $\dfrac{1}{(x-2)^2\sqrt{(2x^2+1)}}$ **127.** $\dfrac{x^2-2}{(x^2-1)\sqrt{(x^2-3x+2)}}$

128. $\dfrac{x}{(3x^2-10x+8)\sqrt{(x^2-8x+10)}}$

129. $\dfrac{x^2+x+1}{(2x^2+2x+1)\sqrt{(5x^2+6x+3)}}$

130. $\dfrac{x+1}{(3x^2-2x+3)\sqrt{(5x^2+2x-5)}}$

131. $\dfrac{x^4-2x^2+3}{(x-2)(x^2+1)^{3/2}}$ **132.** $\dfrac{x^3-8x^2+7x-2}{(x-1)^2(x^2-1)^{3/2}}$

133. $\cos 3x \cos^5 x$ **134.** $\sin^2 x \cos x \sin 4x$

Prove the results given in *Examples 135–41*.

135. $\displaystyle\int \frac{dx}{\sqrt{\{(a-x)(x-b)\}}} = 2\arctan\sqrt{\left(\frac{x-b}{a-x}\right)},\ (a>x>b)$

136. $\displaystyle\int \frac{dx}{\sqrt{\{(x-a)(x-b)\}}} = 2\log\{\sqrt{(x-a)}+\sqrt{(x-b)}\},\ (x>a>b)$

137. $\displaystyle\int \frac{dx}{(x-a)^{\frac{3}{2}}(x-b)^{\frac{1}{2}}} = \frac{2}{b-a}\sqrt{\left(\frac{x-b}{x-a}\right)},\ (x>a>b)$

138. $\displaystyle\int \frac{(1+x^2)\,dx}{x^4-\sqrt{2}x^2+1} = \frac{1}{2\sin\frac{\pi}{8}}\arctan\left(\frac{2x\sin\frac{\pi}{8}}{1-x^2}\right)$

139. $\displaystyle\int \frac{dx}{(x^4+1)^2} = \frac{x}{4(1+x^4)} + \frac{3}{8\sqrt{2}}\left\{\arctan\left(\frac{\sqrt{2}x}{1-x^2}\right) + \tanh^{-1}\left(\frac{\sqrt{2}x}{1+x^2}\right)\right\}$

140. $\displaystyle\int (ax^2+c)^{\frac{3}{2}}\,dx = \frac{1}{8}x\{\sqrt{(ax^2+c)}\}(2ax^2+5c) + \frac{3}{8}c^2\int \frac{dx}{\sqrt{(ax^2+c)}}$

141. $\displaystyle\int (ax^2+c)^{-\frac{5}{2}}\,dx = \frac{x}{3c^2}(2ax^2+3c)(ax^2+c)^{-\frac{3}{2}}$

142. If $f(x)$ is a rational function of x satisfying identically the relation $f(x)+f(1/x)=0$, show that $f\{(1+y)/(1-y)\}$ is a rational function $F(y)$ with the property $F(y)+F(-y)=0$.

Hence prove that if $u^2 = ax^4+bx^3+cx^2+bx+a$, the integral $\displaystyle\int \frac{f(x)}{u}\,dx$ can be expressed in terms of elementary functions.

By means of *Example 142* integrate the functions given in *Examples 143–5*.

143. $\dfrac{x-1}{(x+1)\sqrt{(x^3+x)}}$ **144.** $\dfrac{x^2-1}{(x^2+1)\sqrt{(x^4+3x^2+1)}}$

145. $\dfrac{x-1}{(x+1)\sqrt{(x^4+x^2+1)}}$ **146.** $\dfrac{x^2-1}{(x^2+1)\sqrt{(x^4+1)}}$

147. Show how to integrate $\int u^p(1+u)^q\,du$ in the three cases (p, q rational) (i) p integral, (ii) q integral, (iii) $(p+q)$ integral.

Integrate the functions given in *Examples 148–53*.

148. $x^5(1+2x^3)^{\frac{1}{2}}$ **149.** $x^{\frac{1}{2}}(2+x^{\frac{1}{2}})^{\frac{1}{3}}$ **150.** $x^{\frac{1}{3}}(2+3x^{\frac{1}{3}})^{4}$

151. $x^{\frac{1}{2}}(3+2x^{\frac{1}{3}})^2$ **152.** $x^{-\frac{11}{3}}(1+x^{\frac{1}{3}})^{-\frac{1}{2}}$ **153.** $x^{-\frac{1}{2}}(1+x^{\frac{1}{3}})^{\frac{1}{2}}$

154. If $x^3 + y^3 = 3axy$, prove that $\int \frac{y\,dx}{x}$ is equal to

$$y - \frac{1}{2}a\log\left|\frac{(x+y)^2}{xy}\right| - \sqrt{3}\,a\arctan\left(\frac{2y-x}{x\sqrt{3}}\right) + C,\ (x,\ y \neq 0).$$

155. If $y^2(2y-3) = x(x^2-x-1)$, express $\int y\,dx$ as the integral of a rational function.

156. If $x^3 - y^4 = x^2y$, show that

$$\int y\,dx = \frac{1}{2}xy - \frac{y^3}{8x} + \frac{1}{32}\log\left|\frac{y-x}{y+x}\right| + \frac{1}{16}\arctan\frac{y}{x} + C,\ (x,\ y\neq 0).$$

Solutions

1. $\frac{1}{3}\log(3x^2+4) + \frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}x}{2}\right)$ 2. $\log\cosh x$

3. $\frac{1}{2}x^2 + x + \log|x-1|$ 4. $\log|e^{2x}-1| - x$

5. $\frac{1}{3}\log|x-1| + \frac{2}{3}\log|x+2|$ 6. $\frac{1}{2}\log\left|\frac{e^x-1}{e^x+1}\right|$

7. $4\arcsin\frac{1}{2}x - \frac{1}{2}x\sqrt{(4-x^2)}$

8. $\frac{1}{2}x^2 - 2x + \frac{1}{6}\log|x-1| - \log|x+1| + \frac{4}{3}\log|x+2|$

9. $\frac{1}{2}(x+\sinh x\cosh x)$ 10. $\frac{1}{2}(\sinh x\cosh x - x)$ 11. $x - \tanh x$

12. $\frac{1}{3}\log|x-1| - \frac{1}{6}\log(x^2+2x+5) + \frac{1}{3}\arctan\frac{1}{2}(x+1)$

13. $3\log\left|\frac{x-1}{x-2}\right| - \frac{6x^2-15x+8}{2(x-1)^2(x-2)}$ 14. $-\coth x$

15. $\tanh x$ 16. $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{1}{2}x+\frac{1}{8}\pi\right)\right|$

17. $x - \frac{1}{4(x-1)^2} - \frac{7}{4(x-1)} + \frac{17}{8}\log|x-1| - \frac{1}{8}\log|x+1|$

18. $x + \log|x-1| - \log|x+1|$ 19. $x - 2\arctan x$

20. $-\frac{1}{35}\sqrt{(1-x)}(32+16x+12x^2+10x^3)$

21. $-\frac{2}{315}(1-x)^{3/2}(464-564x+240x^2-35x^3)$

22. $\frac{1}{2}\arcsin x - \frac{1}{6}(2x^2-3x+4)\sqrt{(1-x^2)}$

23. $x + \frac{1}{3}\arctan x - \frac{8}{3}\arctan\frac{1}{2}x.$

24. $\frac{3}{2}\log(x^2-4x+5) - 4\arctan(x-2)$

25. $\frac{13}{32}\log|2x-3| + \frac{5}{32}\log|2x+1| - \frac{x}{4}$

26. $\frac{1}{4}\sqrt{(4x^2-4x+5)} - \log\{2x-1+\sqrt{(4x^2-4x+5)}\}$

27. $\frac{1}{9}\sqrt{(9x^2+12x-5)} + \frac{1}{3}\log\{(3x+2)+\sqrt{(9x^2+12x-5)}\}$

28. $\frac{1}{16}\sqrt{(-5-24x-16x^2)} + \frac{7}{16}\arcsin(2x+\frac{3}{2})$

29. $-\frac{3}{16}\sqrt{(7-24x-16x^2)} + \frac{11}{16}\arcsin(x+\frac{3}{4})$

30. $-\frac{1}{2}(x^2+8)\sqrt{(4-x^2)} - 2\arcsin\frac{1}{2}x$ 31. $\frac{2}{3}(x+1)^{3/2} - \frac{2}{3}(x-1)^{3/2}$

32. $\frac{1}{4}(2x+3)\sqrt{\{(x+1)(x+2)\}} - \frac{1}{8}\log[x+\frac{3}{2}+\sqrt{\{(x+1)(x+2)\}}]$
$\qquad -\frac{1}{4}(2x-1)\sqrt{\{(x+1)(x-2)\}} + \frac{9}{8}\log[x-\frac{1}{2}+\sqrt{\{(x+1)(x-2)\}}]$

33. $\frac{2}{315}\sqrt{(105)}\arctan\left\{\frac{\sqrt{105}\tan\frac{1}{2}x}{15}\right\}$ 34. $\frac{1}{2}\arctan(\frac{2}{3}\tan\frac{1}{2}x)$

35. $\frac{1}{5}\log\left|\frac{\sin\frac{1}{2}x+5\cos\frac{1}{2}x}{\sin\frac{1}{2}x-5\cos\frac{1}{2}x}\right|$ 36. $\frac{1}{5}x - \frac{14}{15}\log\left|\frac{2\sin\frac{1}{2}x-\cos\frac{1}{2}x}{\sin\frac{1}{2}x-2\cos\frac{1}{2}x}\right|$

37. $\frac{3x}{25} + \frac{4}{25}\log|5+7\cos x+\sin x| - \frac{2}{25}\log\left|\frac{2+\tan\frac{1}{2}x}{\tan\frac{1}{2}x-3}\right|$

38. $x - \frac{\sqrt{6}}{6}\arctan\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right)$

39. $\frac{1}{9}\cos^9x - \frac{3}{7}\cos^7x + \frac{3}{5}\cos^5x - \frac{1}{3}\cos^3x$ 40. $\frac{1}{4}\sin^4x - \frac{1}{6}\sin^6x$

41. $\frac{2}{21}(\sin x)^{3/2}(7-3\sin^2x)$ 42. $2\tan x - \cot x + \frac{1}{3}\tan^3x$

43. $3\log|\tan x| + \sec^2 x - \frac{1}{2}\operatorname{cosec}^2 x + \frac{1}{4}\sec^4 x$

44. $\frac{1}{2\sqrt{2}}\log\left(\frac{1-\sqrt{(2\tan x)}+\tan x}{1+\sqrt{(2\tan x)}+\tan x}\right)+\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{(2\tan x)}}{1-\tan x}\right)$

45. $\log(e^x+\sqrt{(e^{2x}-1)})+\arcsin e^{-x}$ 46. $\frac{2\sqrt{3}}{9}\arctan\left(\frac{2x^3-1}{\sqrt{3}}\right)$

47. $\frac{3\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}}-\frac{4\sqrt{3}}{3}\arctan\frac{x\sqrt{3}}{3}-\arctan x+\frac{1}{2}\log\left(\frac{(x^2-1)(x^2+3)}{(x^2-2)^2}\right)$

48. $\frac{x}{6(1+x^2)^3}+\frac{5x}{24(1+x^2)^2}+\frac{5x}{16(1+x^2)}+\frac{5}{16}\arctan x$

49. $\sqrt{2}\arctan\left(\frac{\tan\frac{1}{2}x}{\sqrt{2}}\right)-\frac{2\sqrt{3}}{3}\arctan\left(\sqrt{2}\tan\frac{1}{2}x\right)$

50. $\frac{4\sqrt{3}}{3}\arctan\left(\frac{1+2\tan\frac{1}{2}x}{\sqrt{3}}\right)+\frac{2}{1+\tan\frac{1}{2}x}+2\log\left(\frac{1+\sin x}{2+\sin x}\right)$

51. $\frac{1}{6}\sin 3x+\frac{1}{2}\sin x$ 52. $\frac{1}{4}\cos x-\frac{1}{20}\cos 5x$

53. $-\frac{1}{2}\cos x-\frac{1}{20}\cos 5x-\frac{1}{6}\cos 3x$

54. $\frac{4}{5}\log|1+2\tan x|-\frac{1}{3}\log|\tan x-1|-\frac{1}{2}\log|1+\tan x|$

55. $\frac{1}{3}(x^2+2)\sqrt{(x^2-1)}$

56. $3x+2\cos x-\sin x-3\log(2+\cos x-\sin x)$

$$+\frac{8}{\sqrt{2}}\arctan\left(\frac{3\tan\frac{1}{2}x-1}{\sqrt{2}}\right)$$

57. $\frac{x^{n+1}}{n+1}\left(\log x-\frac{1}{n+1}\right)$ 58. $x\{(\log x)^2-2\log x+2\}$

59. $\frac{x^3}{3}\left\{(\log x)^2-\frac{2}{3}\log x+\frac{2}{9}\right\}$ 60. $x\arcsin x+\sqrt{(1-x^2)}$

61. $x\arctan x-\frac{1}{2}\log(1+x^2)$ 62. $\frac{\pi x}{4}-x\arctan x+\frac{1}{2}\log(1+x^2)$

63. $\frac{1}{2}x^2\arcsin x+\frac{1}{4}x\sqrt{(1-x^2)}-\frac{1}{4}\arcsin x$

64. $-x^2\cos x+2x\sin x+2\cos x$

65. $\frac{1}{3}x^3\arctan x-\frac{1}{6}x^2+\frac{1}{6}\log(1+x^2)$

66. $\frac{1}{2}e^{2x}(x^3-\frac{3}{2}x^2+\frac{3}{2}x-\frac{3}{4})$ 67. $-e^{-x}(x^4+4x^3+12x^2+24x+24)$

68. $(\frac{1}{2}x^3-\frac{3}{4}x)\sin 2x+(\frac{3}{4}x^2-\frac{3}{8})\cos 2x$

69. $\frac{1}{4}x^2-\frac{1}{4}x^2\sin 2x-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x$

70. $\frac{1}{2}x\sin^2 x-\frac{1}{4}x+\frac{1}{4}\sin x\cos x$ 71. $-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x$

72. $-e^{-x^2}(\frac{1}{2}x^4+x^2+1)$ 73. $-\frac{1}{2}e^{-x^2}(x^4+2x^2+2)$

74. $-\frac{1}{x}\arctan x+\log x-\frac{1}{2}\log(1+x^2)$

75. $\frac{1}{5}e^{2x}(\sin x+2\cos x)$ 76. $-\frac{1}{5}e^{-x}(2\cos 2x+\sin 2x)$

77. $\frac{1}{2}x\sqrt{(4-x^2)}+2\arcsin\frac{1}{2}x$

78. $\frac{2}{315}(35x^3-30x^2+24x-16)(x+1)^{\frac{3}{2}}$

79. $\frac{1}{6}x^6(1-x)^4+\frac{2}{21}x^7(1-x)^3+\frac{1}{28}x^8(1-x)^2+\frac{x^9}{126}(1-x)+\frac{x^{10}}{1260}$

80. $(n+2)I_n=-x^{n-1}(1-x^2)^{\frac{3}{2}}+(n-1)I_{n-2}$

81. $(m+1)I_n=x^{m+1}(\log x)^n-nI_{n-1}$

82. $aI_n=x^ne^{ax}-nI_{n-1}$

83. $I_n=-x^n\cos x+nx^{n-1}\sin x-n(n-1)I_{n-2}$

84. $(m+1)I_{n,m}=x^n(x-1)^{m+1}-nI_{n-1,m+1}$

85. $2(n-1)(ac-b^2)I_n=(ax+b)(ax^2+2bx+c)^{1-n}+a(2n-3)I_{n-1}$

86. $a(n+1)I_n=(ax+b)(ax^2+2bx+c)^{n/2}-n(b^2-ac)I_{n-2}$

87. $nI_n=x^{n-1}\sqrt{(1+x^2)}-(n-1)I_{n-2}$

88. $(n-1)I_n=-x^{1-n}\sqrt{(1+x^2)}-(n-2)I_{n-2}$

89. $2a^2(a^2-b^2)(n-1)I_n = (2a^2-b^2)(2n-3)I_{n-1} - 2(n-2)I_{n-2}$
$$- x(x^2-a^2)^{1-n}\sqrt{(x^2-b^2)}$$

90. $(n-1)I_{m,n} = -\cos^{m-1}x\,\mathrm{cosec}^{n-1}x - (m-1)I_{m-2,n-2}$

91. $(n-1)(a^2-b^2)I_n = -a(2n-3)I_{n-1} + (n-2)I_{n-2}$
$$b\sin x(a+b\cos x)^{1-n}$$

92. $(m+n)I_{m,n} = \sin nx\cos^m x + mI_{m-1,n-1}$

93. $(m+n)(m+n-2)I_{m,n} = (m+n-2)\sin nx\sin^m x$
$$+ m\cos(n-1)x\sin^{m-1}x - m(m-1)I_{m-2,n-2}$$

105. $x(\frac16x^4 - \frac1{24}x^2 - \frac1{16})\sqrt{(1-x^2)} + \frac1{16}\arcsin x$

106. $(\frac19 - \frac3{11}\sin^2 x + \frac3{13}\sin^4 x - \frac1{15}\sin^6 x)\sin^9 x$

107. $\frac15\cos^5 x - \frac17\cos^7 x$

108. $\frac{3x}{256} + \frac{5}{256}\sin x\cos x + \frac{1}{128}\sin x\cos^3 x + \frac{1}{160}\sin x\cos^5 x$
$$- \frac{3}{80}\sin x\cos^7 x - \frac1{10}\sin^3 x\cos^7 x$$

109. $\sin^5 x\sec x + \frac54\sin^3 x\cos x + \frac{15}{8}\sin x\cos x - \frac{15}{8}x$

110. $-\frac72\sin x\cos x - \frac{7x}{2} - \frac73\cos^3 x\,\mathrm{cosec}\,x + \frac{7}{15}\cos^5 x\,\mathrm{cosec}^3 x$
$$- \frac15\cos^7 x\,\mathrm{cosec}^5 x$$

111. $\frac{1}{5\sin^7 x\cos^4 x} + \frac{4}{5\sin^7 x\cos^2 x} + \frac{8}{\sin^7 x\cos x} - \frac{64\cos x}{7\sin^7 x} - \frac{384\cos x}{35\sin^5 x}$
$$- \frac{512\cos x}{35\sin^3 x} - \frac{1024\cos x}{35\sin x}$$

112. $\frac14\tan^4 x + \frac32\tan^2 x - \frac12\cot^2 x + 3\log|\tan x|$

113. $\frac{(x-1)^{21}(231x^2+21x+1)}{5313}$

114. $\frac{x}{6}(2x^2+3)^{\frac52} + \frac{5x}{8}(2x^2+3)^{\frac32} + \frac{45x}{16}(2x^2+3)^{\frac12} + \frac{135}{16\sqrt2}\log\left\{x+\sqrt{\left(x^2+\frac32\right)}\right\}$

115. $\frac{x^6}{216}\{36(\log x)^3 - 18(\log x)^2 + 6\log x - 1\}$

116. $-\frac18 e^{-2x}(4x^6 + 12x^5 + 30x^4 + 60x^3 + 90x^2 + 90x + 45)$

117. $\frac15 e^{-2x}(\sin x - 2\cos x)$ **118.** $\frac{x(32x^4+200x^2+375)}{1875(2x^2+5)^{\frac52}}$

119. $\frac{x(8x^2+9)}{27(4x^2+3)^{\frac32}}$

120. $\frac{x(8x^4-10x^2+15)\sqrt{(x^2+1)}}{48} - \frac{5}{16}\log\{x+\sqrt{(x^2+1)}\}$

121. $-\frac{\sqrt{(1+x^2)}}{4x^4} + \frac{3\sqrt{(1+x^2)}}{8x^2} - \frac38\log\{1+\sqrt{(1+x^2)}\} + \frac38\log|x|$

122. $\frac{9+20\sin x}{36(5+4\cos x)} - \frac{4}{27}\arcsin\left(\frac{3\sin x}{5+4\cos x}\right)$

123. $\arctan\left(\frac{\sqrt{(2x^2+3)}}{x}\right)$

124. $\sqrt2\log\left|\frac{1-x+\sqrt{(2x^2+2)}}{2(x+1)}\right| - \frac{3}{\sqrt5}\log\left|\frac{1-2x+\sqrt{(5x^2+5)}}{5(x+2)}\right|$

125. $\log\{x+\sqrt{(x^2-4)}\} - \frac{1}{\sqrt3}\arcsin\left(\frac{4-x}{2x-2}\right)$

126. $-\frac{\sqrt{(2x^2+1)}}{9(x-2)} + \frac{4}{27}\log\left|\frac{4x+1+3\sqrt{(2x^2+1)}}{9(x-2)}\right|$

127. $\log\left\{x - \frac{3}{2} + \sqrt{(x^2 - 3x + 2)}\right\} - \sqrt{\left(\frac{x-2}{x-1}\right)}$
$$+ \frac{1}{2\sqrt{6}}\log\left|\frac{7 - 5x - 2\sqrt{(6x^2 - 18x + 12)}}{12(x-1)}\right|$$

128. $\frac{1}{\sqrt{7}}\log\left|\frac{(x-2)\sqrt{7} + \sqrt{(x^2 - 8x + 10)}}{(x-2)\sqrt{7} - \sqrt{(x^2 - 8x + 10)}}\right|$
$$- \frac{1}{\sqrt{14}}\arctan\left(\frac{\sqrt{14}}{7}\cdot\frac{\sqrt{(x^2 - 8x + 10)}}{x - 1}\right)$$

129. $\frac{1}{2\sqrt{5}}\log\left\{x + \frac{3}{5} + \sqrt{\left(x^2 + \frac{6}{5}x + \frac{3}{5}\right)}\right\}$
$$- \frac{1}{4}\log\left\{\frac{\sqrt{(5x^2 + 6x + 3)} - x - 1}{\sqrt{(5x^2 + 6x + 3)} + x + 1}\right\} - \frac{1}{2}\arctan\left(\frac{\sqrt{(5x^2 + 6x + 3)}}{x}\right)$$

130. $-\frac{1}{4}\arctan\left(\frac{\sqrt{(5x^2 + 2x + 5)}}{2(x-1)}\right)$

131. $\frac{11}{250}\log\frac{(x-2)^2}{x^2+1} - \frac{120}{250}\arctan x - \frac{17x + 14}{50(x^2+1)} - \frac{3(2x-1)}{10(x^2+1)^2}$

132. $\frac{2 - x^2}{(x-1)(x^2+1)} - \frac{1}{2}\log|x - 1| + \frac{1}{4}\log(x^2+1) - \frac{1}{2}\arctan x$

133. $\frac{3}{8}x + \frac{3}{16}\sin 2x + \frac{3}{32}\sin 4x + \frac{1}{48}\sin 6x$

134. $\frac{1}{4}\cos x - \frac{1}{24}\cos 3x - \frac{1}{40}\cos 5x + \frac{1}{56}\cos 7x$

143. $\frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{(x^2+1)} - \sqrt{(2x)}}{\sqrt{(x^2+1)} + \sqrt{(2x)}}\right)$

144. $\frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{(x^2 + 3x + 1)} - \sqrt{(2x)}}{\sqrt{(x^2 + 3x + 1)} + \sqrt{(2x)}}\right)$

145. $\frac{1}{\sqrt{3}}\log\left\{\left(\frac{x-1}{x+1}\right)^2 + \frac{5}{3} + 4\sqrt{\left(\frac{x^4 - x^2 + 1}{3(x+1)^4}\right)}\right\}$

146. $\frac{1}{3\sqrt{2}}\log\left\{\frac{2(x^2 - x + 1) + \sqrt{(2x^4 + 2)}}{(x-1)^2}\right\} - \frac{2}{3}\arcsin\left(\frac{\sqrt{2}(x-1)^2}{2(x^2 + x + 1)}\right)$

147. (i) If $q = \frac{r}{s}$, take $1 + u = v^s$. (ii) $p = \frac{r}{s}$, take $u = v^s$. (iii) $q = \frac{r}{s}$, take $(1 + u) = uv^s$.

148. $\frac{1}{315}(15x^4 - 6x^2 + 2)(1 + 2x^2)^{\frac{1}{2}}$

149. $(\frac{2}{13}x^{\frac{1}{2}} - [illegible]x + [illegible]x^{\frac{3}{2}} - [illegible]x^{\frac{1}{2}} + [illegible])(2 + x^{\frac{1}{2}})^{\frac{1}{2}}$

150. $x^{\frac{1}{2}}([illegible]x + [illegible]x^{\frac{3}{2}} + [illegible]x^{\frac{1}{2}} + [illegible]x^{\frac{1}{2}} + 6)$

151. $[illegible]x^{\frac{4}{3}}(2x^2 + 27) + [illegible]x^{\frac{1}{3}}(4x^2 + 9)$

152. $\frac{10(x^{\frac{1}{2}} + 1)^{\frac{3}{2}}(5x^{\frac{1}{2}} - 2)}{7x^{\frac{7}{4}}}$ **153.** $-\frac{3}{16}x^{-\frac{16}{3}}(1 + x^{\frac{4}{3}})^{[illegible]}$

155. Take $x = \frac{(t-1)^2(2t+1)}{2t^3 - 1}$, $y = -\frac{(t-1)(t^2 + t - 1)}{2t^3 - 1}$.
$$\int y\,dx = 6\int\frac{(t-1)^2(t^2 + t - 1)^2 t\,dt}{(2t^3 - 1)^3}.$$

5.3. The Definite Integral. *Definition.* Let $f(x)$ be bounded in the interval $a \leqslant x \leqslant b$ and let $x_1, x_2, \ldots, x_{n-1}$ be $(n - 1)$ points of the interval for which
$$a\,(= x_0) < x_1 < x_2 < \ldots < x_{n-1} < b\,(= x_n). \quad (\textit{Fig. 1.})$$
Let M, m be the upper and lower bounds of $f(x)$ in (a, b) and M_r, m_r

the upper and lower bounds of $f(x)$ in the interval (x_{r-1}, x_r). Also let x_r' be any number in the interval (x_{r-1}, x_r).

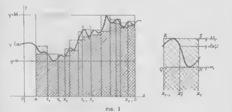

FIG. 1

Form the sums S_n, s_n given by

$$S_n = \sum_1^n M_r(x_r - x_{r-1}); \quad s_n = \sum_1^n m_r(x_r - x_{r-1}).$$

In the figure, S_n is the sum of the areas of shaded rectangles like $UVNQ$, and s_n is the sum of the rectangles like $UVRS$.

Then obviously

$$M(b - a) \geq S_n \geq \sum_1^n f(x_r')(x_r - x_{r-1}) \geq s_n \geq m(b - a).$$

If the sums S_n, s_n tend to the *same* limit S when n tends to infinity in such a way that every sub-interval tends to zero and when this limit S is *independent* of the particular mode of subdivision, S is called the *Definite Integral* of $f(x)$ from $x = a$ to $x = b$ and is written $\int_a^b f(x)dx$. The function is then said to be *integrable* (in the sense of Riemann). Since $\sum_1^n f(x_r')(x_r - x_{r-1})$ lies between S_n and s_n, the limit of

$$\sum_1^n f(x_r')(x_r - x_{r-1})$$

must also be S, when S exists.

5.31. Step-Functions. If $\phi(x) = c_s$ for $a_{s-1} < x < a_s$, $(s = 1 \text{ to } m)$, (where $a_0 = a$, $a_m = b$) then $\phi(x)$ is called a *step-function*. (*Fig. 2.*)

It is easy to see that $\int_a^b \phi(x)dx$ exists and is equal to $\sum_1^m c_s(a_s - a_{s-1})$. For let $x_1, x_2, \ldots, x_{n-1}$ be chosen so that $\max(x_r - x_{r-1}) < \delta$, where δ is less than the smallest of the fixed intervals $a_s - a_{s-1}$. The sum S_n

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differs from $\sum_1^m c_s(a_s - a_{s-1})$ by less than $(m-1)\delta \sum_1^{m-1} |c_s - c_{s+1}|$ which tends to zero when δ tends to zero, i.e. S_n tends to the limit

$$\sum_1^m c_s(a_s - a_{s-1}).$$

Similarly s_n tends to the same limit.

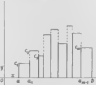


FIG. 2

Corollary. If $\phi_1(x)$, $\phi_2(x)$ are two step-functions in (a, b) and

$$\phi_2(x) \geq \phi_1(x)$$

then $\int_a^b \phi_1(x)dx \leq \int_a^b \phi_2(x)dx$.

5.32. A Continuous Function is Integrable. Let $f(x)$ be continuous

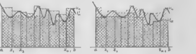


FIG. 3

in (a, b). (*Fig. 3.*) Take $u_n(x) = \max f(x)$ in each sub-interval (x_{r-1}, x_r). Then $U_n = \int_a^b u_n(x)dx$ exists since $u_n(x)$ is a step-function. As n increases, $u_n(x)$ cannot increase so that U_n is a non-increasing monotone and must therefore tend to a limit U.

Similarly if $l_n(x) = \min f(x)$ for each sub-interval then

$$L_n = \int_a^b l_n(x)dx$$

(a non-decreasing monotone) tends to a limit L.

But, given ε, since $f(x)$ is continuous, we can divide the interval into a finite number of parts such that $u_n(x) - l_n(x) < \varepsilon$ for every sub-interval, i.e. $0 < U_n - L_n < \varepsilon(b-a)$ or $U = L$, since both limits exist.

Now take any other mode of subdivision where the corresponding symbols are accented. Then (*Fig.* 3) $l_n'(x) < u_n(x)$ so that $L_n' < U_n$ and $L' < U$; also $u_n'(x) > l_n(x)$ and therefore $U_n' > L_n$ and $U' > L$. But $U' = L'$ and $U = L$; i.e. $U' = L' < U$ and $U = L < U'$ or $L = U = U' = L'$.

5.321. *The General Case.* The set of numbers $S_n = \sum_1^n M_r(x_r - x_{r-1})$ for all possible subdivisions has a finite lower bound S since $f(x)$ is bounded; and therefore there exists one integer m (at least) such that in the corresponding method of subdivision (A) we have $S_m = \sum_1^m M_m(x_r - x_{r-1}) < S + \varepsilon$ where ε is any given number > 0.

Take a sequence of numbers $\delta_p (> 0)$ ($p = 1, 2, 3, \ldots$) that decreases steadily to zero. Divide (a, b) into m_1 sub-intervals each of which is of length $< \delta_1$. Then by introducing $(m_2 - m_1)$ further points of subdivision obtain a second set of intervals (m_2 in number) each of which is of length $< \delta_2$: and let this process of subdivision (B) be continued. To the pth set of intervals ($< \delta_p$) there corresponds a sum $S(m_p)$. In this set there can be only a finite number of sub-intervals containing an end point of the original subdivision (A), and the contribution to $S(m_p)$ due this finite number of intervals is $< M(m+1)\delta_p$. The contribution due to the remaining intervals must obviously be less than or equal to $S_m < S + \varepsilon$ i.e.

$$S(m_p) < S + \varepsilon + M(m+1)\delta_p$$

and therefore since $\delta_p \to 0$ and $M(m+1)$ is finite, $S(m_p)$ tends to the limit S when $\delta_p \to 0$. Thus if the number of subdivisions $(x_1, x_2, \ldots, x_{n-1})$ of (a, b) tends to infinity in such a way that $\max(x_r - x_{r-1})$ tends to zero, then the upper Riemann Integral S exists. Similarly the lower Riemann Integral s exists. The necessary and sufficient condition that $S = s$, is that given ε, n_0 can be found such that $S_n - s_n < \varepsilon$ for all $n > n_0$, the subdivision being of an appropriate character. The function is then *integrable* (R). Thus a continuous function is integrable (R), since the interval (a, b) can be divided into n parts such that in each part

$$|f(x') - f(x'')| < \varepsilon$$

i.e. such that in the interval $(x_r - x_{r-1})$, $M_r - m_r < \varepsilon$, ($r = 1$ to n). Thus

$$S_n - s_n = \Sigma(M_r - m_r)(x_r - x_{r-1}) < \varepsilon(b-a).$$

Again, a bounded function whose only discontinuities are of the first kind is integrable (R), for these discontinuities are isolated (and enumerable, or finite in number): and may be enclosed in a set of intervals of total breadth $< \varepsilon$. The contribution to $S_n - s_n$ due to the intervals containing the discontinuities is therefore $< (M - m)\varepsilon$ and therefore tends to zero when $\varepsilon \to 0$.

5.322. *Monotone Functions.* If $f(x)$ is defined for every point of (a, b) and $f(x_1) \geq f(x_2)$ when $x_1 > x_2$, then $f(x)$ is called a *monotone function* (increasing). Similarly we may define a decreasing monotone function. It follows that $f(x+0)$, $f(x-0)$ exist for all x in (a, b) and therefore since the only discontinuities are of the first kind, the function is integrable (R).

Otherwise,
$$S_n - s_n = \Sigma\{f(x_r) - f(x_{r-1})\}(x_r - x_{r-1})$$
$$< \varepsilon[f(b) - f(a)]$$
i.e.
$$S_n - s_n \to 0 \text{ as } n \to \infty.$$

5.323. *Functions of Bounded Variation.* A function $f(x)$ defined for (a, b) is said to be of *bounded variation* in (a, b) if for all possible subdivisions $x_1, x_2, \ldots, x_{n-1}$,

$$\sum_1^n |f(x_r) - f(x_{r-1})| \; (\equiv \Sigma_n(x_r))$$

has an upper bound $V(a, b)$ (independent of n).

The upper bound $V(a, b)$ is called the *total fluctuation* (or variation) for (a, b). Thus if $f(x)$ is a monotone in (a, b), $V(a, b) = |f(b) - f(a)|$.

Now
$$\Sigma_n(x_r) = \sum_1^n |f(x_r) - f(x_{r-1})|$$
and
$$f(b) - f(a) = \sum_1^n (f(x_r) - f(x_{r-1})).$$

Denote the sum of the *positive* differences $(f(x_r) - f(x_{r-1}))$ by Σ_1, and the sum of the *negative* differences by $-\Sigma_2$.

$$\text{Then } f(b) - f(a) = \Sigma_1 - \Sigma_2 \text{ and } \Sigma_n(x_r) = \Sigma_1 + \Sigma_2$$

It follows easily from these equations that Σ_1, Σ_2 have upper bounds $P(a, b)$, $N(a, b)$ given by $f(b) - f(a) = P(a, b) - N(a, b)$, and $V(a, b) = P(a, b) + N(a, b)$. Now a function of bounded variation in (a, b) is also of bounded variation in (a, x) where $a < x < b$; and therefore
$$V(a, x) = P(a, x) + N(a, x);\quad f(x) - f(a) = P(a, x) - N(a, x).$$
Also $P(a, x_1) > P(a, x_2)$ if $x_1 > x_2$ so that $P(a, x)$ is a monotone increasing function; and similarly $N(a, x)$ is a monotone increasing function.

Thus $f(x) = f(a) + P(a, x) - N(a, x)$ and is therefore expressible as the difference of two monotone (increasing) functions. From the previous paragraph it follows that a function of bounded variation is integrable (R).

Note. More generally, the necessary and sufficient condition that a bounded function should be integrable (R) is that the function should be continuous *almost everywhere* (i.e. at a set of points of zero measure). (*See Lebesgue, Annali di Mat.* (3), *VII*, 234.)

For example, if $f(x) = 0$ when x is irrational and $f(x) = 1/q$, when x is a rational number expressed in the form p/q, (p, q being integers prime to each other), the integral $\int_a^b f(x)dx$ exists and has the value zero.

5.33. The Mean Value of $f(x)$ in (a, b). When $f(x)$ is bounded and integrable
$$M(b - a) > \int_a^b f(x)dx > m(b - a).$$

Therefore $\int_a^b f(x)dx = k(b - a)$ where $M > k > m$. This number k is called the *mean value* of $f(x)$ for the interval (a, b).

If $f(x)$ is continuous in (a, b), there must be at least one number ξ in $a < x < b$ for which $f(\xi) = k$,

i.e. $$\int_a^b f(x)dx = (b - a)f(\xi) \text{ for some number } \xi \text{ in } (a, b).$$

5.34. Relation between the Definite Integral and the Indefinite Integral. Since the definite integral is the limit of a sum
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
when $f(x)$ is integrable in (a, c) and (c, b).

In particular
$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Let $G(x) = \int_a^x f(t)dt$, then $G(x + h) - G(x) = \int_x^{x+h} f(t)dt$. If $f(t)$ is

continuous near $t = x$, $\int_x^{x+h} f(t)dt = hf(x + \theta h)$ where $0 < \theta < 1$. Thus $G'(x) = \lim\limits_{h \to 0} f(x + \theta h) = f(x)$.

If $F(x) + C$ is the indefinite integral of $f(x)$, $G(x) = F(x) + C$. But in this case $G(a) = 0$ and therefore $\int_a^x f(t)dt = G(x) = F(x) - F(a)$, where $F(x)$ is any function whose derivative is $f(x)$.

In particular $\int_a^b f(t)dt = F(b) - F(a)$.

Example. $\int_0^{\frac{\pi}{2}} \sin^n x\, dx = I_n$

Here $I_n = \left(-\frac{1}{n}\sin^{n-1} x \cos x\right)_0^{\frac{\pi}{2}} + \frac{n-1}{n} I_{n-2}$ (§ 5.23) and therefore

$$I_n = \frac{n-1}{n} I_{n-2} = \frac{(n-1)(n-3)\ldots 4.2}{n(n-2)\ldots 5.3}, \ (n \text{ odd})$$

and

$$= \frac{(n-1)(n-3)\ldots 3.1}{n(n-2)\ldots 4.2} \cdot \frac{\pi}{2}, \ (n \text{ even})$$

5.35. Change of Variable. Let $x = \phi(u)$ be a continuous function of u, varying from $a\ (= \phi(u_1))$ to $b\ (= \phi(u_2))$ as u varies from u_1 to u_2. Also let $\phi'(u)$ be continuous in (u_1, u_2) and $f(x)$ continuous in (a, b).

Then $\int_a^{\phi(u)} f(x)dx = \int_{u_1}^{u} f\{\phi(u)\}\phi'(u)du$, for the derivatives of both integrals with regard to u are equal and the integrals vanish when $u = u_1$.

In particular $\int_a^b f(x)dx = \int_{u_1}^{u_2} f\{\phi(u)\}\phi'(u)du$.

Note. This result may be proved more generally from the sum-definition of the definite integral.

Examples. (i) $\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$ since

$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = -\int_{\frac{\pi}{2}}^{0} \sin^n y\, dy$ (where $y = \frac{1}{2}\pi - x$) $= \int_0^{\frac{\pi}{2}} \sin^n x\, dx$. (*See Example*, § 5.34.)

(ii) $\int_0^{\frac{1}{2}} \frac{x^2\, dx}{\sqrt{(1 - x^2)}} = \int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta$ (where $\theta = \text{arc} \sin x$) $= \frac{1}{2}(\theta - \sin\theta\cos\theta)\Big|_0^{\frac{\pi}{6}}$

$$= \frac{2\pi - 3\sqrt{3}}{24}.$$

Note. Care must be exercised in the use of the formula when functions that are not single-valued are introduced. Thus $I = \int_0^{\pi} d\theta = \pi$, but it is not correct to take $\theta = \text{arc} \sin x$ in this integral to obtain $I = \int_0^0 \frac{dx}{\sqrt{(1 - x^2)}} = 0$.

Strictly we should write $I = \lim\limits_{\epsilon_1,\epsilon_2\to\frac{1}{2}\pi}\left[\int_0^{\epsilon_1} d\theta + \int_{\epsilon_2}^{\pi} d\theta\right]$
where $0 < \epsilon_1 < \frac{1}{2}\pi$, $\frac{1}{2}\pi < \epsilon_2 < \pi$,
i.e. $I = \lim\limits_{\epsilon_1\to1}\int_0^{\epsilon_1}\frac{dx}{\sqrt{(1-x^2)}} + \lim\limits_{\epsilon_2\to1}\int_{\epsilon_2}^{0}\left\{-\frac{dx}{\sqrt{(1-x^2)}}\right\} = \pi$, and this introduces the idea of the *infinite* integral. (§ 5.43.)

5.4. Discontinuous Integrands. If $f(x)$ is discontinuous at a number of points $c_1, c_2, \ldots, c_m$ where $a < c_1 < c_2 \ldots < c_m < b$, then we may *define* $\int_a^b f(x)dx$ as the limit, if it exists, of

$$I = \int_a^{c_1-\epsilon_1} f(x)dx + \sum_{r=1}^{r=m-1}\int_{c_r+\epsilon_r'}^{c_{r+1}-\epsilon_{r+1}} f(x)dx + \int_{c_m+\epsilon_m'}^{b} f(x)dx$$

when $\epsilon_r, \epsilon_r'(>0)$ tend independently to zero ($r = 1$ to m). For if this limit exists, the above definition is consistent with the general definition. If a is a discontinuity of $f(x)$, $\int_a^b f(x)$ is defined to be $\lim\limits_{\epsilon\to+0}\int_{a+\epsilon}^b f(x)dx$, if this exists; and if b is a discontinuity the integral is defined to be

$$\lim_{\epsilon\to+0}\int_a^{b-\epsilon} f(x)dx.$$

Similarly, a simplified definition may be taken when the points of discontinuity form an enumerable set. Thus, to take an easy example, if there are discontinuities at $x = c_1, c_2, \ldots, c_m, \ldots$ when

$$a < c_1 < c_2 < \ldots < c_m \ldots < b$$

then the integral may be defined as $\lim\limits_{m\to\infty}\int_a^{c_m} f(x)dx + \int_c^b f(x)dx$ where $c = \lim c_m$, the first integral of this expression being defined as above.

5.41. A Finite Number of Finite Discontinuities (of the First Kind). Suppose that $f(x)$ has a single finite discontinuity of the first kind at $x = c$ $(a < c < b)$, so that $f(c-0)$, $f(c+0)$ both exist and need not be equal.

Then $\lim\limits_{\epsilon_1\to0}\int_a^{c-\epsilon_1} f(x)dx = \int_a^c f(x)dx$, if $f(c)$ is assumed to be $f(c-0)$

and $\lim\limits_{\epsilon_2\to0}\int_{c+\epsilon_2}^b f(x)dx = \int_c^b f(x)dx$, if $f(c)$ is assumed to be $f(c+0)$.

Thus $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $f(x)$ is now completely defined as a continuous function in each interval.

Similarly, if there are m discontinuities of this type at $c_1, c_2, \ldots, c_m$, we may write $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \sum_1^{m-1}\int_{c_r}^{c_{r+1}} f(x)dx + \int_{c_m}^b f(x)dx$. It should be noted that if $F(x) = \int_a^x f(x)dx$, then

$$F'(c_r - 0) = f(c_r - 0) \text{ and } F'(c_r + 0) = f(c_r + 0).$$

Example. Let $f(x)$ be given as follows:

$10 - 2x$, $(0 < x < 1)$; $5 - 2x$, $(1 < x < 2)$; $5 - 2x$, $(2 < x < 3)$; $3 - 2x$, $(3 < x < 4)$, $-1 - 2x$, $(4 < x < 5)$.

The graph of this function is typical of the *shear diagram* of a loaded beam; and $\int_0^x f(x)dx$ is the *bending-moment* at x (*Fig. 4.*)

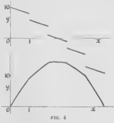


FIG. 4

Here $F(x) \equiv \int_0^x f(x)dx = 10x - x^2$, $(0 < x < 1)$.

If $1 < x < 2$, $F(x) = F(1) + \int_1^x (5 - 2x)dx = 5 + 5x - x^2$.

Similarly, $F(x) = 5 + 5x - x^2$, $(2 < x < 3)$; $14 + 3x - x^2$, $(3 < x < 4)$; $30 - x - x^2$, $(4 < x < 5)$. $F(x)$ is of course continuous in the interval $0 < x < 5$.

Note. The integrand may have a discontinuity of the second kind at points within the interval. For example, let

$$f(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \text{ for } 0 < x \leq \frac{2}{\pi} \text{ and } f(0) = 0$$

Then $f(x)$ is continuous in $0 < x \leq \frac{2}{\pi}$, but $\overline{f(+0)} = 1$; $\underline{f(+0)} = -1$. But if $\int_0^{\frac{2}{\pi}} f(x)dx$ is defined to be $\lim\limits_{\epsilon \to 0} \int_\epsilon^{\frac{2}{\pi}} f(x)dx$, we find that the integral is

$$\lim_{\epsilon \to 0} \left\{x^2 \sin \frac{1}{x}\right\}_\epsilon^{\frac{2}{\pi}} = \frac{4}{\pi^2}.$$

5.42. *An Infinite Number of Finite Discontinuities.* A function possessing an infinite number of discontinuities may or may not possess a Riemann integral; and it has been indicated above that the integral does exist when the points of discontinuity can be enclosed in a set of intervals whose total length can be made arbitrarily small.

Examples. (i) Let $f(x) = 2x$, $(0 \leq x \leq \frac{1}{2})$, $4x - 2$, $(\frac{1}{2} < x < \frac{3}{4})$, . . . , $2^n(x - 1) + 2$, $(1 - 2^{1-n} < x < 1 - 2^{-n})$, ... Here there is an enumerable set of finite discontinuities at $x = 1 - 2^{-n}$. We may define the integral from 0 to 1 by the equation $\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_1^n \int_{c_r}^{c_{r+1}} (2^r(x - 1) + 2)dx$ where $c_r = 1 - 2^{1-r}$. This is easily verified to be $\lim_{n\to\infty} (2^{-2} + 2^{-3} + \ldots + 2^{-n-1})$, i.e. $\frac{1}{2}$, as is otherwise obvious from the figure. (*Fig.* 5.)

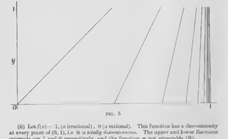


FIG. 5

(ii) Let $f(x) = 1$, (x irrational), 0 (x rational). This function has a discontinuity at every point of $(0, 1)$, i.e. it is *totally discontinuous*. The upper and lower Riemann integrals are 1 and 0 respectively, and the function is not integrable (R).

5.43. *The Lebesgue Integral and Measurable Functions.* More general definitions of the definite integral have been given that apply to cases where, as in Example (ii) above, the Riemann Integral does not exist. The most important of these from a theoretical point of view is that of the Lebesgue Integral, for a description of which it is necessary to introduce the notion of *measurable function*.

Suppose that $f(x)$ is bounded in (a, b), then $f(x)$ is said to be a *measurable* function (bounded) in (a, b) if the set of points of (a, b) for which $f(x) > c$ is measurable for all c. Denoting this set of points by $E(f > c)$ we can deduce that the sets given by (i) $E(f \leq c)$, (ii) $E(f \geq c)$, (iii) $E(f < c)$ are also measurable.

For (i) $E(f \leq c)$ being complementary to $E(f > c)$ is measurable.

(ii) $E\left(f > c - \frac{1}{n}\right) - E\left(f > c - \frac{1}{n+1}\right) = E\left(c - \frac{1}{n} < f \leq c - \frac{1}{n+1}\right)$,

$(n = 1, 2, 3, \ldots)$ and therefore the set on the right is measurable.

But $E(c - 1 < f < c) = E(c - 1 < f \leq c - \frac{1}{2}) + E(c - \frac{1}{2} < f \leq c - \frac{1}{3}) + \ldots + E\left(c - \frac{1}{n} < f \leq c - \frac{1}{n+1}\right) + \ldots$ and therefore $E(c - 1 < f < c)$ is measurable.

Now $E(f > c - 1) = E(c - 1 < f < c) + E(f = c) + E(f > c)$; and therefore $E(f = c)$ is measurable, so that $E(f \geq c)$ is measurable.

(iii) $E(f < c)$, being complementary to $E(f \geq c)$ is also measurable.

In the above we have used the fundamental theorem on sets that the sum of a finite number (or an enumerable infinity) of measurable sets is measurable.

To define the Lebesgue Integral of $f(x)$, assume that $f(x)$ is a bounded measurable function whose upper and lower bounds in (a, b) are M, m respectively. Divide the interval (m, M) of y into n parts by means of the lines parallel to the x-axis given by

$$y = y_0 (= m);\ y = y_1;\ y = y_2;\ \ldots\ y = y_n (= M).$$

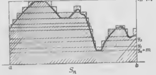


FIG. 6

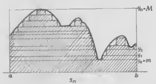


FIG. 7

Form the sums S_n, s_n where

$$S_n = y_0(b - a) + (y_1 - y_0)m(E_0) + (y_2 - y_1)m(E_1) + \ldots + (y_n - y_{n-1})m(E_{n-1}),$$
$$s_n = y_0(b - a) + (y_1 - y_0)m(E_1) + (y_2 - y_1)m(E_2) + \ldots + (y_n - y_{n-1})m(E_n)$$

where $E_r = E(f > y_r)$, $(r = 0, 1, \ldots n)$, so that $E_n = E(f = y_n)$.

The two sums are the sums of the rectangles shown in *Figs.* 6 and 7, where, in order to make the definition clear, $f(x)$ is taken to be a continuous function of a simple type. The illustration is therefore inadequate for the general type of function which possesses a Lebesgue Integral.

If further points of subdivision are introduced, the corresponding sum $S_{n'}$ is obviously $< S_n$ and the corresponding sum $s_{n'}$ is $> s_n$; so that if the number of subdivisions tends steadily to infinity in some specified way, the sequences S_n, s_n being monotonic must tend to limits. In particular, if n tends to infinity in such a way that $\max(y_{r+1} - y_r)$ tends to zero, S_n, s_n tend to limits S, s respectively. We can now prove that $S = s$, but for this purpose we shall obtain expressions for S_n, s_n in a more convenient (and more usual) form.

Let $e_r = E(y_r < f < y_{r+1})$, $(r = 0, 1, \ldots, (n - 1)$ and $e_n = E(f = y_n = M)$.

Then $E_r - E_{r+1} = E(f > y_r) - E(f > y_{r+1}) = e_r$, $(r = 0, 1, \ldots (n - 1))$ and $E_n = e_n$

Then
$$\begin{aligned} S_n &= y_0\{b - a - m(E_0)\} + \dots + y_r\{m(E_{r-1}) - m(E_r)\} + \dots \\ &\qquad + y_{n-1}\{m(E_{n-2}) - m(E_{n-1})\} + y_n m(E_{n-1}) \\ &= y_1 m(e_0) + y_2 m(e_1) + \dots + y_{n-1} m(e_{n-2}) + y_n\{m(e_{n-1}) + m(e_n)\} \\ &= \sum_0^n y_{r+1} m(e_r) \text{ if } y_{n+1} \text{ is taken to be } y_n (= M), \end{aligned}$$

and
$$\begin{aligned} s_n &= y_1\{b - a - m(E_1)\} + \dots + y_r\{m(E_r) - m(E_{r+1})\} + \dots \\ &\qquad + y_{n-1}\{m(E_{n-1}) - m(E_n)\} + y_n m(E_n) \\ &= \sum_0^n y_r m(e_r). \end{aligned}$$

Thus $S_n - s_n = \sum_0^n (y_{r+1} - y_r) m(e_r) < \epsilon \sum_0^n m(e_r)$ where $\epsilon = \max(y_{r+1} - y_r)$, i.e. $= \epsilon(b - a)$, and therefore the limits of S_n, s_n are equal.

If the common limit be denoted by I, then the *Lebesgue Integral* of $f(x)$ over (a, b) is defined to be I.

Also, it may be proved by a method analogous to that used for the Riemann Integral, that the value of I is the same for all choices of the points of subdivision y_r, provided $\max(y_{r+1} - y_r)$ tends to zero.

It may be of some interest to mention here without proof the main properties of the Lebesgue Integral.

The Lebesgue Integral is more general than the Riemann Integral and may exist when the Riemann Integral does not exist. Thus if $f(x) = 1$ when x is irrational and $f(x) = 0$ when x is rational, the Riemann Integral $\int_a^b f(x)dx$ does not exist, but the Lebesgue Integral is obviously equal to $(b - a)$. It may be proved that when the Riemann Integral exists so also does the Lebesgue Integral and their values are equal.

The measure of a set of points may be expressed as a Lebesgue Integral. Thus if $f(x) = 1$ over a set E and $f(x) = 0$ elsewhere, then $\int_a^b f(x)dx$ where E is within the interval (a, b) is obviously equal to $m(E)$. The function $f(x)$ is called the *characteristic-function* of the set E.

Measurable functions are therefore more general than the class of functions that are integrable (R), and include all such functions. If f, ϕ are measurable so also are $f \pm \phi$ and $f\phi$. Also if f_n is a sequence of measurable functions that tends to a limit f, the limit is also measurable; and if the limit f does not exist, the upper and lower limits of f_n are measurable. Thus not only are continuous functions measurable but also all those functions of analysis that are defined as the limits (where they exist) of continuous functions.

Two functions that are equal almost everywhere (i.e. except at a set of measure zero) have the same Lebesgue Integral, and the derivative of the Lebesgue indefinite integral of $f(x)$ is almost everywhere equal to $f(x)$.

If $f_n(x)$ is a bounded sequence of measurable functions (i.e. such that $|f_n(x)| < M$, all n, where x is any point of an interval), then

$$\lim \int_a^b f_n(x)dx = \int_a^b \lim f_n(x)dx.$$

Thus if the series $\sum_1^\infty u_n(x)$ is boundedly convergent (i.e. such that $\left|\sum_1^n u_n(x)\right| < M$ for all n) then the series may be integrated term-by-term.

The meaning of the Lebesgue Integral may be extended to cases where $f(x)$ is an unbounded measurable function; and the integrals of such functions, if integrable, have similar properties to those of bounded functions. It may be shown, for example, that if $|f_n(x)| < F(x)$, all n, and all x in the interval (a, b) (the functions

concerned being integrable and the conditions being satisfied almost everywhere), then $\lim \int_a^b f_n(x)dx = \int_a^b \lim f_n(x)dx$.

Such properties give some indication of the importance of Lebesgue integration in general theory; and whilst it is true that functions that are integrable (R) will usually be found adequate for the applications of mathematics, the conditions under which many limiting processes are valid (for example, the conditions under which

$$\Sigma\int f_n(x)dx = \int\Sigma f_n(x)dx \text{ or } \int\{\int f(x, \alpha)dx\}d\alpha = \int\{\int f(x, \alpha)d\alpha\}dx)$$

are obtained more satisfactorily by the use of Lebesgue integration. (*Ref. Lebesgue, Leçons sur l'intégration et la recherche des fonctions primitives, Paris 1928.*)

5.44. Infinite Discontinuities. Infinite Integrals. An integral is called *Infinite* (or *Improper*) if either (i) there is at least one infinite discontinuity in the range or (ii) the range is infinite.

Although it is more useful in practice to distinguish between these two types of infinite integrals, they are not theoretically distinct, for by such a substitution as $u = \dfrac{bx}{x - a + b}$, we can convert the infinite range (a, ∞) for x into the finite range (a, b) for u.

Note. The rule for change of variable (or integration by parts) may easily be extended to apply to infinite integrals (*Ref. Hardy, Pure Mathematics, 162*), but it should be observed that when a change of variable is made, the new integral may be finite.

Examples. (i) $\displaystyle\int_0^\infty \frac{dx}{(1 + x^2)} = \int_0^{\frac{\pi}{2}} du = \tfrac{1}{2}\pi$, where $u = \arctan x$, or

$$\int_0^\infty \frac{dx}{1 + x^2} = \int_0^1 \frac{du}{1 - 2u + 2u^2} = \left\{\arctan(2u - 1)\right\}_0^1 = \tfrac{1}{2}\pi, \text{ where } u = x/(x + 1).$$

(ii) $\displaystyle\int_0^2 \frac{dx}{(x + 1)\sqrt{(4 - x^2)}}$ has an infinity in the integrand at $x = 2$. Taking $x = 2\sin\theta$, we find the integral to be $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{d\theta}{1 + 2\sin\theta} = \frac{1}{\sqrt{3}}\log(2 + \sqrt{3})$.

When the limit indicated by the integral exists, the latter is said to be *convergent*; and when the indefinite integral is known, the convergence may be established directly.

Examples. (i) $\displaystyle\int_0^\infty xe^{-ax}dx$.

Here $\displaystyle\int_0^x xe^{-ax}dx = \frac{1}{a^2}\{1 - e^{-ax}(1 + ax)\}$ which converges as $x \to \infty$ only when $a > 0$.

Thus $\displaystyle\int_0^\infty xe^{-ax}dx = 1/a^2 \; (a > 0)$.

(ii) $$\int_0^2 \frac{dx}{(x - 1)^{\frac{2}{3}}} = \lim_{\epsilon_1 \to 0}\int_0^{1-\epsilon_1} \frac{dx}{(x - 1)^{\frac{2}{3}}} + \lim_{\epsilon_2 \to 0}\int_{1+\epsilon_2}^2 \frac{dx}{(x - 1)^{\frac{2}{3}}} \; (\epsilon_1, \epsilon_2 > 0).$$

Thus the integral is $\lim 3(-\epsilon_1)^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}} + 3(1 - \epsilon_2^{\frac{1}{3}}) = 6$.

(iii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0}(2 - 2\sqrt{\varepsilon}) = 2 \ (\varepsilon > 0).$

(iv) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = \lim_{\varepsilon\to 0} \text{arc sin}\,(1-\varepsilon) = \tfrac{1}{2}\pi.$

(v) $\displaystyle\int_0^1 x\log x\,dx = \lim_{\varepsilon\to 0}\left(-\tfrac{1}{4} - \tfrac{1}{2}\varepsilon^2\log\varepsilon + \tfrac{1}{4}\varepsilon^2\right) = -\tfrac{1}{4}.$

(vi) $\displaystyle\int_0^{\frac{1}{2}\pi} \sin x\log\sin x\,dx$

$$\int^x \sin x\log\sin x\,dx = -\cos x\log\sin x + \int\frac{\cos^2 x}{\sin x}\,dx$$
$$= -\cos x\log\sin x + \log\tan\tfrac{1}{2}x + \cos x,$$

i.e.
$$\int_\varepsilon^{\frac{1}{2}\pi} \sin x\log\sin x\,dx = (1+\cos\varepsilon)\log\cos\tfrac{1}{2}\varepsilon - (1-\cos\varepsilon)\log\sin\tfrac{1}{2}\varepsilon$$
$$+ (\log 2 - 1)\cos\varepsilon$$

Now $(1 - \cos\varepsilon) = \frac{1}{2}\varepsilon^2 + O(\varepsilon^4)$; $\log\sin\frac{1}{2}\varepsilon = \log(\frac{1}{2}\varepsilon + O(\varepsilon^3))$ and therefore $(1-\cos\varepsilon)\log\sin\frac{1}{2}\varepsilon \to 0$ as $\varepsilon \to 0$ since $\varepsilon^2\log\varepsilon \to 0$. The value of the integral is therefore $(\log 2 - 1)$.

5.45. *Principal Value of an Infinite Integral.* If $x = c$ is an infinity of $f(x)$ in the interval (a, b), the limit $\int_a^{c-\varepsilon_1} f(x)dx + \int_{c+\varepsilon_2}^b f(x)dx$ may not exist when $\varepsilon_1, \varepsilon_2\ (> 0)$ tend independently to zero; whilst the limit may exist when ε_2 is some function of ε_1. Thus if $f(x) = \dfrac{1}{x-c}$, its value would be $\lim\left(\log\dfrac{b-c}{c-a} + \log\dfrac{\varepsilon_1}{\varepsilon_2}\right)$ which does not exist when $\varepsilon_1, \varepsilon_2$ tend independently to zero. If however $\varepsilon_1 = k\varepsilon_2$ (k fixed), its value is $\log\left(\dfrac{b-c}{c-a}\right) + \log k$. In particular if $\varepsilon_1 = \varepsilon_2$, the limit is $\log\left(\dfrac{b-c}{c-a}\right)$ and is called the *Principal Value* and written $P\int_a^b f(x)dx$. Generally then if $x = c_1, c_2, \ldots c_m$ are infinities within (a, b),

$$P\int_a^b f(x)dx = \lim\left\{\int_a^{c_1-\varepsilon_1} f(x)dx + \sum_1^{m-1}\int_{c_r+\varepsilon_r}^{c_{r+1}-\varepsilon_{r+1}} f(x)dx + \int_{c_m+\varepsilon_m}^b f(x)dx\right\}$$

when $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \to 0$ and $a < c_1 < c_2 \ldots < c_m < b$.

Example. $\displaystyle\int_a^b \frac{x^3 - 2x^2 + x - 1}{x^2(x-1)}\,dx.$

The integrand is $\dfrac{1}{x^2} + \dfrac{2}{x} - \dfrac{1}{x-1}$; $\displaystyle\int_{-\varepsilon}^{+\varepsilon}\frac{dx}{x^2}, \int_{-\varepsilon}^{+\varepsilon}\frac{dx}{x}$, and $\displaystyle\int_{1-\varepsilon}^{1+\varepsilon}\frac{dx}{x-1}$ are all zero

Thus $P\int_a^b f(x)dx = \frac{1}{2a^2} - \frac{1}{2b^2} + \log\left\{\frac{(1-a)b^2}{(1-b)a^2}\right\}$ when $a < 0 < b < 1$

and $P\int_a^b f(x)dx = \frac{1}{2a^2} - \frac{1}{2b^2} + \log\left\{\frac{(1-a)b^2}{(b-1)a^2}\right\}$ when $a < 1 < b$.

For other values of a, b the integral is not infinite.

5.5. Convergence of Infinite Integrals. When the indefinite integral is not known or is not easily determined, the convergence of an infinite integral may sometimes be established by direct comparison with a known infinite integral (of positive integrand).

5.51. Comparison Theorem for Convergence of Integrals. (Positive Integrands.) If (i) $g(x) > 0$, (ii) $0 < f(x) < g(x)$, (iii) $\int_a^b g(x)dx$ is convergent, then $\int_a^b f(x)$ is convergent. This follows immediately from the sum-definition of an integral.

Here we suppose that a may be $-\infty$ and b may be $+\infty$.

Notes. (i) It is sufficient for direct comparison that $f(x)$ should be of *constant sign* in the neighbourhood of the significant values of x in the interval.

Thus (*a*) if $x = c$ is a discontinuity at an internal point, we need only consider the neighbourhood of $x = c$ on *both sides*.

(*b*) If $x = a$ is the discontinuity, we should consider the neighbourhood of $x = a$ *on the right*.

(*c*) If $x = b$ is the discontinuity, we should take the neighbourhood of $x = b$ *on the left*.

(*d*) If b is $+\infty$, we should consider only *large positive* values of x, and

(*e*) If a is $-\infty$, we should take only *large negative* values of x.

Again, since the change of variable $u = x - c$ transfers $x = c$ to $u = 0$, comparison *tests* for convergence need only refer to the neighbourhood of *small* x and of *large* x.

(ii) If $f(x) < 0$ throughout a neighbourhood the comparison tests may be applied to $\int\{-f(x)\}dx$.

(iii) The necessary and sufficient condition that $\int^\infty f(x)dx$ should be convergent is that, given ϵ, any positive number however small, we can find a value x_0 such that $\left|\int_{x_1}^{x_2} f(x)dx\right| < \epsilon$ *for all* $x_1, x_2 > x_0$. For if a be taken sufficiently large to ensure that $f(x)$ has no discontinuities for finite $x > a$, the above inequality becomes $|F(x_2) - F(x_1)| < \epsilon$ for all $x_1, x_2 > x_0$; (where $F(x) = \int_a^x f(x)dx$). This is precisely the necessary and sufficient condition that $F(x)$ should tend to a limit as x tends to $+\infty$. A similar necessary and sufficient condition may be framed for the other type of infinite integral.

5.52. Absolute Convergence of an Infinite Integral. Since

$$\left|\int_{x_1}^{x_2} f(x)dx\right| < \int_{x_1}^{x_2} |f(x)|dx$$

then the convergence of $\int^\infty |f(x)|dx$ implies that of $\int^\infty f(x)dx$; the latter integral is then said to be *absolutely convergent*. A similar result may

be obtained for $\int_{-\infty} f(x)dx$ and for $\int_a^b f(x)dx$ when there is a discontinuity within the interval.

Example $\displaystyle\int_0^1 \frac{\cos x\, dx}{x^\alpha}$.

$$\int_0^1 \frac{dx}{x^\alpha} = \lim_{\epsilon \to 0}\left(\frac{1 - \epsilon^{1-\alpha}}{1-\alpha}\right), \ (\alpha \neq 1) \text{ and } = -\lim_{\epsilon \to 0} (\log \epsilon), \ (\alpha = 1).$$

Thus $\displaystyle\int_0^1 \frac{dx}{x^\alpha}$ is convergent if $\alpha < 1$; therefore since $\left|\dfrac{\cos x}{x^\alpha}\right| \leq \dfrac{1}{x^\alpha}$ $(x > 0)$, the integral $\displaystyle\int_0^1 \frac{\cos x}{x^\alpha}dx$ is absolutely convergent when $\alpha < 1$.

5.53. Practical Forms of the General Comparison Theorem. (Positive Integrands.) Let $f(x)$, $g(x)$ be non-negative functions.

(*a*) *Convergence* :

If $\dfrac{f(x)}{g(x)} < K$, (x large, K finite constant > 0) and $\displaystyle\int^\infty g(x)dx$ is convergent then $\displaystyle\int^\infty f(x)$ is convergent.

Thus, if $\dfrac{f(x)}{g(x)} \to k \ (> 0)$ when $x \to +\infty$ and $\displaystyle\int^\infty g(x)dx$ is convergent so also is $\displaystyle\int^\infty f(x)dx$.

For if k_1 is any fixed number $> k$, $f(x) < k_1 g(x)$ for x large.

(*b*) *Divergence* :

If $\dfrac{f(x)}{g(x)} > K$ (x large, K finite constant > 0) and $\displaystyle\int^\infty g(x)dx$ is divergent then $\displaystyle\int^\infty f(x)dx$ is divergent.

Thus, if $\dfrac{f(x)}{g(x)} \to k \ (> 0)$ when $x \to +\infty$ and $\displaystyle\int^\infty g(x)dx$ is divergent, so also is $\displaystyle\int^\infty f(x)dx$

For if k_1 is any fixed number such that $0 < k_1 < k$, then $f(x) > k_1 g(x)$ when x is large.

It follows from the above that if $f(x) = g(x) + o(g(x))$ when x is large, $\displaystyle\int^\infty f(x)dx$ converges or diverges with $\displaystyle\int^\infty g(x)dx$.

Similar results may be obtained with suitable modifications for other significant neighbourhoods.

5.54. The Comparison Integrals. (*a*) Consider $\displaystyle\int^\infty x^m e^{-ax}dx$.

If $a > 0$, $x^m < e^{\frac{1}{2}ax}$, when x is large (all m).

Thus $\int_c^\infty x^m e^{-ax}\,dx < \int_c^\infty e^{-\frac{1}{2}ax}\,dx$, ($c$ large).

But $\int_c^\infty e^{-\frac{1}{2}ax}\,dx = \frac{2}{a}e^{-\frac{1}{2}ac}$ is convergent and therefore $\int^\infty x^m e^{-ax}\,dx$ converges.

If $a = 0$, $\int^\infty x^m\,dx$ converges when $m < -1$ and diverges when $m \geq -1$.

If $a < 0$, $x^m > e^{\frac{1}{2}ax}$ (x large, all m) and since $\int^\infty e^{-\frac{1}{2}ax}\,dx$ is divergent it follows that $\int^\infty x^m e^{-ax}\,dx$ diverges for $a < 0$.

Thus $\int^\infty x^m e^{-ax}\,dx$ converges for $a > 0$ all m, and for $a = 0$, $m < -1$ but otherwise diverges.

(b) Consider $\int^\infty x^m (\log x)^n\,dx$.

Let $x = e^\xi$ and the integral becomes $\int^\infty e^{(m+1)\xi}\xi^n\,dx$.

Thus $\int^\infty x^m (\log x)^n\,dx$ converges for $m < -1$, all n, and for $m = -1$, $n < -1$, but otherwise diverges.

(c) Consider $\int_0 x^m \left(\log\frac{1}{x}\right)^n dx$, ($x \to 0$).

Let $x = \frac{1}{\xi}$ and the integral becomes $\int^\infty \xi^{-m-2}(\log \xi)^n d\xi$.

Thus $\int_0 x^m \left(\log\frac{1}{x}\right)^n dx$ converges for $m > -1$, all n, and for $m = -1$, $n < -1$ but otherwise diverges.

5.55. Comparison Tests for Infinite Integrals (*Positive Integrands.*) If $f(x) < Ax^m e^{-ax}$ (x large), then $\int^\infty f(x)dx$ converges with $\int^\infty x^m e^{-ax}\,dx$ and if $f(x) > Ax^m e^{-ax}$ (x large), then $\int^\infty f(x)dx$ diverges with $\int^\infty x^m e^{-ax}\,dx$ where A is a positive number independent of x.

Thus if $f(x)/(x^m e^{-ax})$ tends to a *positive* or *zero* limit when x tends to ∞, $\int^\infty f(x)dx$ converges with $\int^\infty x^m e^{-ax}\,dx$; whilst if $f(x)/(x^m e^{-ax})$ tends to a *positive* limit, $\int^\infty f(x)dx$ diverges with $\int^\infty x^m e^{-ax}\,dx$.

Similarly we may compare $f(x)$ with $x^m(\log x)^n$ when x is large or with $x^m\left(\log \dfrac{1}{x}\right)^n$ when x is small.

Note. If $f(x) = O(x^m e^{-ax})$, the convergence of $\int^\infty x^m e^{-ax}\,dx$ implies that of $\int^\infty f(x)dx$: but it is not true in general that the divergence of $\int^\infty x^m e^{-ax}\,dx$ implies that of $\int^\infty f(x)dx$. If, however, $f(x) = kx^m e^{-ax} + o(x^m e^{-ax})$, the $\int^\infty f(x)dx$ diverges (or converges) with $\int^\infty x^m e^{-ax}\,dx$ when $k \neq 0$.

Examples. (i) $\displaystyle\int_0^\infty \frac{x^{a-1}}{x^m+1}\,dx \;(m > 0).$

Near $x = 0$, $f(x) = x^{a-1} + o(x^{a-1})$, integral converges there if $a > 0$ and is otherwise divergent.

Near $x = \infty$, $f(x) = x^{a-1-m} + o(x^{a-1-m})$, integral converges if $a < m$ and is otherwise divergent.

Thus the integral converges only if $0 < a < m$.

(ii) $\displaystyle\int_0^\infty \log(1 + 2\operatorname{sech} x)dx.$

Near $x = \infty$, $\log(1 + 2\operatorname{sech} x) < 2\operatorname{sech} x < 4e^{-x}$, therefore the integral converges since $\int^\infty e^{-x}\,dx$ converges.

(iii) $\displaystyle\int_1^\infty \frac{|\sin x|dx}{x^a}.$

$\left|\dfrac{\sin x}{x^a}\right| < \dfrac{1}{x^a}$; but $\displaystyle\int^\infty \frac{dx}{x^a}$ converges when $a > 1$, and therefore $\displaystyle\int_1^\infty \frac{\sin x}{x^a}dx$ converges absolutely when $a > 1$.

5.6. Differentiation of Finite Definite Integrals. Let $F(x, \alpha)$ be a continuous function of both variables x, α possessing the continuous derivatives $F_{x\alpha}$, $F_{\alpha x}$, which are therefore equal.

Then $\displaystyle I = \int_a^b \frac{\partial}{\partial x}(F(x, \alpha))dx = F(b, \alpha) - F(a, \alpha)$

and
$$\int_a^b F_{x\alpha}\,dx = F_\alpha(b, \alpha) - F_\alpha(a, \alpha)$$

If a, b are functions of α, possessing derivatives a_α, b_α,
$$\frac{dI}{d\alpha} = F_\alpha(b, \alpha) - F_\alpha(a, \alpha) + F_b\frac{db}{d\alpha} - F_a\frac{da}{d\alpha}$$

i.e.
$$\frac{d}{d\alpha}\int_a^b F_x(x, \alpha)dx = \int_a^b F_{x\alpha}\,dx + F_b\frac{db}{d\alpha} - F_a\frac{da}{d\alpha}$$

or writing $f(x, \alpha)$ for F_x we obtain
$$\frac{d}{d\alpha}\int_a^b f(x, \alpha)dx = \int_a^b f_\alpha(x, \alpha)dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}.$$

Examples. (i) $\displaystyle\int_0^b \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\left(\frac{b}{a}\right), \; (a > 0).$

Therefore $\displaystyle 2a\int_0^b \frac{dx}{(x^2 + a^2)^2} = \frac{1}{a^2}\arctan\left(\frac{b}{a}\right) + \frac{b}{a(b^2 + a^2)}$ (if b is independent of a),

i.e.
$$\int_0^b \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2a^3}\arctan\frac{b}{a} + \frac{b}{2a^2(b^2 + a^2)}$$

Thus $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$.

(ii) $\int_0^a \frac{dx}{a^2+x^2} = \frac{\pi}{4a}$, $(a > 0)$.

On differentiating we find $\int_0^a \frac{-2a\,dx}{(x^2+a^2)^2} + \frac{1}{2a^2} = -\frac{\pi}{4a^2}$

so that $\int_0^a \frac{dx}{(a^2+x^2)^2} = \frac{\pi+2}{8a^3}$.

(iii) $\int_0^\pi \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{(a^2-b^2)}}$, $(a > 0,\ |b| < a)$.

Differentiation with regard to a gives $\int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}$ and differentiation with regard to b gives $\int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^2} = -\frac{\pi b}{(a^2-b^2)^{\frac{3}{2}}}$.

Notes. (i) The above process cannot be applied to *infinite integrals* without further justification. The discussion of this case is given in *Chapter XI*. If, of course, the indefinite integral is known as in *Example* (i) above, the value of the infinite integral may be obtained by a limiting process.

(ii) We may similarly obtain a formula for *integration* with respect to a parameter, but no useful purpose appears to be served by considering such a formula here because (a) the integrals occurring in practice are usually *infinite* and (b) a repeated integral is often best expressed as a *double integral* (*Chapter IX*).

5.7. Integration of Power Series. Let

$$F(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

and let the radius of convergence of the series be R.

Let x_1, x_2 belong to the interval, i.e. be such that

$$-R < x_1 < x_2 < R$$

Let $x_1 < x < x_2$ and let c lie between the greater of $|x_1|$, $|x_2|$ and R. Then

$$|a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \ldots| < |a_{n+1}|\,|x^{n+1}| + |a_{n+2}|\,|x^{n+2}| + \ldots$$
$$< |a_{n+1}|c^{n+1} + |a_{n+2}|c^{n+2} + \ldots$$

But since $F(c)$ is absolutely convergent, we can find n_0 such that

$$|a_{n+1}|c^{n+1} + |a_{n+2}|c^{n+2} + \ldots < \varepsilon \text{ for all } n > n_0,$$

i.e. $|F(x) - \sum\limits_0^n a_nx^n| < \varepsilon$ for all $n > n_0$ and for *all* x in the interval $x_1 < x < x_2$ (n_0 being independent of x), i.e. $F(x) = \sum\limits_0^n a_nx^n + \lambda$ where $|\lambda| < \varepsilon$.

Integrating, we find $\int_{x_1}^{x_2} F(x)dx = \sum\limits_0^n a_n\frac{x_2^{n+1} - x_1^{n+1}}{n+1} + \mu$, where $|\mu| = \left|\int_{x_1}^{x_2}\lambda\,dx\right| < \varepsilon(x_2 - x_1)$ so that $\mu \to 0$ when $n \to \infty$, i.e.

$$\int_{x_1}^{x_2} F(x)dx = \sum_0^\infty \frac{a_n}{n+1}(x_2^{n+1} - x_1^{n+1}).$$

In particular $\int_0^x F(x)dx = a_0 x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \dots$ $(|x| < R)$, the radius of convergence of this last series being also R.

Notes. (i) That the radius of convergence of the integrated series is also R is verified by the fact that $\lim\left(\frac{a_n}{n+1}\right)^{\frac{1}{n}} = \lim (a_n)^{\frac{1}{n}} = R$.

(ii) If the integrated series converges at $x = R$ (even when the original series does not), Abel's Theorem shows that its value for $x = R$ is $\int_0^R F(x)dx$. A similar result holds for $x = -R$.

5.71. The Expansion of $\log(1 + x)$.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots, \text{ when } |x| < 1.$$

Therefore $\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$ when $|x| < 1$, by integrating from 0 to x.

But the series on the right converges for $x = 1$ (but not for $x = -1$), i.e. $\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ for $-1 < x \leq 1$.

5.72. The Binomial Series. Let

$$F(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1.2}x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + \dots,$$

(α *real*). This, the *Binomial Series*, is considered here for convenience although we do not actually use the method of integration of series to find its value.

Denoting the coefficient of x^n by a_n, we have

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{n+1}{\alpha - n}\right| \text{ which} \to 1 \text{ when } n \to \infty \text{ (all } \alpha).$$

Thus *unity* is the radius of convergence.

(*a*) When $x = -1$, the terms are *ultimately* of the *same sign* and since $\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n-\alpha} = 1 + \frac{1+\alpha}{n} + O\left(\frac{1}{n^2}\right)$, ($n$ large), the series is convergent for $x = -1$ if $\alpha > 0$ and otherwise is divergent.

(*b*) When $x = 1$, the terms are ultimately of alternate signs and the series (by Leibniz's Rule) converges if the nth term tends to zero.

Now (i) if $\alpha + 1 > 0$, $\frac{n+1}{n-\alpha} > 1 + \frac{1+\alpha}{2n}$ since $\lim\left(\frac{n+1}{n-\alpha} - 1\right)n = 1 + \alpha\ (> 0)$,

i.e.

$$\left|\frac{a_m}{a_n}\right| > \left(1 + \frac{k}{m}\right)\left(1 + \frac{k}{m+1}\right)\dots\left(1 + \frac{k}{n-1}\right) \text{ where } k = \frac{1+\alpha}{2} > 0$$

$$> 1 + k\left(\frac{1}{m} + \frac{1}{m+1} + \dots + \frac{1}{n-1}\right), \text{ where } m\ (< n) \text{ is fixed,}$$

so that $a_n \to 0$ as $n \to \infty$ since $\sum^{\infty}\frac{1}{n}$ is divergent.

(ii) If $\alpha = -1$, $\dfrac{a_n}{a_{n+1}} = -1$, $|a_n| \to 1$ and the series oscillates finitely.

(iii) If $\alpha + 1 < 0$, $\left|\dfrac{a_{n+1}}{a_n}\right| = 1 + \dfrac{\beta}{n+1}$ where

$$\beta = -1 - \alpha > 0 \; (n \geqslant m)$$

so that

$$\left|\frac{a_n}{a_m}\right| = \left(1 + \frac{\beta}{m+1}\right)\left(1 + \frac{\beta}{m+2}\right) \cdots \left(1 + \frac{\beta}{n}\right)$$
$$> 1 + \beta\left(\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}\right)$$

i.e. $|a_n| \to \infty$ when $n \to \infty$, the series oscillating infinitely.

Thus the series converges for $x = 1$ if $\alpha + 1 > 0$ but otherwise is not convergent.

Now $F'(x) = \alpha + \alpha(\alpha - 1)x + \dfrac{\alpha(\alpha - 1)(\alpha - 2)}{2!}x^2 + \ldots$, when x belongs to the interval,

i.e.
$$(1 + x)F'(x) = \alpha F(x)$$

since
$$\frac{\alpha(\alpha - 1) \ldots (\alpha - n)}{n!} + \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{(n - 1)!}$$
$$= \alpha\frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!},$$

so that $\dfrac{d}{dx}\{\log F(x)\} = \dfrac{\alpha}{1 + x}$ or $F(x) = k(1 + x)^\alpha$ where k is independent of x. But $F(0) = 1$, so that

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{1.2}x^2 + \ldots = (1 + x)^\alpha$$

the function on the right being defined as $e^{\alpha \log(1 + x)}$ and the result being true for $|x| < 1$, all α; for $x = 1$, $\alpha > -1$; and for $x = -1$, $\alpha > 0$. In particular therefore

$$1 + \alpha + \frac{\alpha(\alpha - 1)}{2!} + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} + \ldots = 2^\alpha \; (\alpha > -1),$$

$$1 - \alpha + \frac{\alpha(\alpha - 1)}{2!} - \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} + \ldots = 0 \; (\alpha > 0).$$

Note. The expansion is of course true for *all* x when α is a positive integer.

5.73. The Expansions for arc tan x and arc sin x.

(a) $\dfrac{1}{1 + x^2} = 1 - x^2 + x^4 - \ldots$ if $|x| < 1$.

Integration gives $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots$ for $|x| < 1$. But the series on the right converges for $|x| = 1$ (by Leibniz's Rule). The expansion is therefore valid for $|x| \leqslant 1$,

i.e.
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

(b) $(1 - x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \dots$ when $|x| < 1$. (§ 5.72.)

Integration gives the result

$$\text{arc} \sin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \dots \text{ for } |x| \leq 1.$$

Thus $$\frac{\pi}{2} = 1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \dots$$

5.8. The Area bounded by a Curve. Let $f(x) > 0$ in the interval $a < x < b$. Since the definite integral $\int_a^b f(x)dx$ is a number lying between the sums $\sum_1^n M_r(x_r - x_{r-1})$ and $\sum_1^n m_r(x_r - x_{r-1})$ and is the common limit of these sums, when this exists (§ 4.3), it gives a natural definition of the *measure* of the area bounded by $y = 0$, $y = f(x)$, $x = a$, $x = b$.

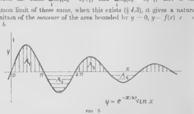


FIG. 8

In general, $f(x)$ may have both signs in the interval and then the definite integral gives the sum of the magnitudes of the areas above the x-axis less the sum of those below the axis.

Note. The word 'area' is often used for the measure of an area when there is no likelihood of confusion.

Example. Consider $I(x) = \int_0^x e^{-\frac{x}{10}} \sin x \, dx$.

$$I(x) = -\tfrac{10}{101}(\sin x + 10 \cos x)e^{-\frac{x}{10}} + \tfrac{100}{101}.$$

Thus $I(\pi) = \frac{100}{101}(1 + e^{-\frac{\pi}{10}}) = 1{\cdot}71 \dots (= A_1)$, $I(2\pi) = \frac{100}{101}(1 - e^{-\frac{\pi}{5}}) = 0{\cdot}46 \dots (= A_1 - A_2)$. $I(\infty) = \frac{100}{101} = A_1 - A_2 + A_3 - A_4 + \dots$ (*Fig. 8*)

5.81. Approximate Integration. It is important in certain cases of practical interest to determine a definite integral (i.e. an *area* or *quadrature*) approximately. Approximate methods are required when $f(x)$ is given empirically by a finite number of values in the interval or by an approximate graph. They may be used when the indefinite integral cannot be expressed in terms of known functions or when it is

too complicated for direct calculation. They may be used also for the approximate determination of the constants of analysis such as π or Euler's constant γ. We shall suppose therefore in what follows that $n+1$ values of $f(x)$ are given in an interval (a, b) and that it is required to find an approximate value for $\int_a^b f(x)dx$. These values need not be equally spaced, but the assumption of equal intervals simplifies some of the results.

5.82. Finite Differences. If $y = f(x)$ and Δx is an increment of x, then Δy, the corresponding increment of y is $f(x+\Delta x) - f(x)$ and is called the *first finite difference* of $f(x)$ with respect to the increment Δx.

The *second difference* is

$$\{f(x+2\Delta x) - f(x+\Delta x)\} - \{f(x+\Delta x) - f(x)\}$$

i.e. $f(x+2\Delta x) - 2f(x+\Delta x) + f(x)$, and is written $\Delta^2 y$. Similarly we may obtain the third and higher differences.

The operator E is defined by the equation $Ef(x) = f(x+\Delta x)$ and therefore $Ef(x) = (1+\Delta)f(x)$ or E is equivalent to $1+\Delta$. Thus if we assume Δx to be a constant increment, any expression of the form $\phi(E, \Delta)f(x)$ has an obvious interpretation when ϕ is a polynomial with constant coefficients.

Thus $\Delta^2 y = (E-1)^2 y = f(x+2\Delta x) - 2f(x+\Delta x) + f(x)$ and generally

$$\Delta^n y = (E-1)^n y = f(x+n\Delta x) - nf(x+(n-1)\Delta x)$$
$$+ \frac{n(n-1)}{1.2}f(x+(n-2)\Delta x) - \ldots + (-1)^n f(x).$$

Now let the interval (a, b) be divided into n equal parts by the points $x_0, x_1, \ldots, x_n$ where $x_0 = a$, $x_n = b$, so that $x_r - x_{r-1} = h$ ($r = 1$ to n), where $nh = b - a$; also $x_r - x_0 = rh$. For simplicity of statement take the origin at $x = a$ and let h be the *unit* of the x axis. Let the ordinates corresponding to $x_0, x_1, \ldots, x_n$ be $y_0, y_1, \ldots, y_n$ respectively.

When the values of y are given as in the following example, the successive differences are easily tabulated.

	x	0	1	2	3	4	5	6	7	8	9	10
y	$f(x)$	999	1984	2919	3744	4375	4704	4599	3904	2439	0	3641
	Δy	985	935	825	631	329	−105	−695	−1465	−2439	3641	
	$\Delta^2 y$	−50	−110	−194	−302	−434	−590	−770	−974	1292		
	$\Delta^3 y$	−60	−84	−108	−132	−156	−180	−204	−228			
	$\Delta^4 y$	−24	−24	−24	−24	−24	−24	−24				
	$\Delta^5 y$	0	0	0	0	0	0					

In the above, any place in the table of differences is filled up by writing in it the result of subtracting the number k above it from the number on the right of k.

If $f(x)$ is a polynomial of degree m, then $\Delta f(x)$ is a polynomial of degree $m-1$, $\Delta^m f(x)$ is constant and $\Delta^{m+1}f(x)$ is zero. Conversely, if $(n+1)$ ordinates are given at equal distances apart and if the $(m+1)$th differences vanish $(< n)$, a polynomial of degree m can be found to

pass through the given points. Thus in the above example, since $\Delta^5 y = 0$, we may assume the polynomial to be,

$$y = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

so that

$$\Delta y = a_0(4x^3 + 6x^2 + 4x + 1) + a_1(3x^2 + 3x + 1) + a_2(2x + 1) + a_3,$$
$$\Delta^2 y = a_0(12x^2 + 24x + 14) + a_1(6x + 6) + 2a_2,$$
$$\Delta^3 y = a_0(24x + 36) + 6a_1,$$
$$\Delta^4 y = 24a_0.$$

Putting $x = 0$ we find

$$24a_0 = -24\,;\quad 36a_0 + 6a_1 = -60\,,\quad 14a_0 + 6a_1 + 2a_2 = -50\,,$$
$$a_0 + a_1 + a_2 + a_3 = 985\,,\quad a_4 = 999$$

from which we obtain $y = -x^4 - 4x^3 - 6x^2 + 996x + 999$

A more rapid method of obtaining the polynomial is given in the next paragraph.

(For a comprehensive treatment of Finite Differences, the reader should consult *Milne-Thomson, Calculus of Finite Differences*.)

5.83. *The Approximate Polynomial.* The polynomial curve of lowest degree that passes through the $(n + 1)$ points (r, y_r) $(r = 0$ to $n)$ may be regarded as an approximation to the curve given by $y = f(x)$. Its degree is therefore n.

$$f(x + n) = (1 + \Delta)^n f(x)$$
$$= f(x) + n\Delta f(x) + \frac{n(n-1)}{1.2}\Delta^2 f(x) + \ldots + \Delta^n f(x).$$

Putting $x = 0$ in this result we obtain

$$y_n = y_0 + n\Delta y_0 + \frac{n(n-1)}{1.2}\Delta^2 y_0 + \ldots + \Delta^n y_0.$$

Consider the following polynomial in x

$$F(x) = y_0 + x\Delta y_0 + \frac{x(x-1)}{1.2}\Delta^2 y_0$$
$$+ \ldots + \frac{x(x-1)\ldots(x-n+1)}{n!}\Delta^n y_0.$$

When $x = r$, its value is y_r $(r = 0$ to $n)$. It is therefore the required polynomial, since it is of degree n. Thus in the example § 5.82 above

$$F(x) = 999 + x985 + x(x-1)(-25) + x(x-1)(x-2)(-10)$$
$$+ x(x-1)(x-2)(x-3)(-1)$$
$$= 999 + 996x - 6x^2 - 4x^3 - x^4 \text{ as before.}$$

Example. Find the quintic polynomial through the points

$(0, 3)$, $(1, -2)$, $(2, -61)$, $(3, -240)$, $(4, -509)$, $(5, -622)$.

Here $y =$	3	-2	-61	-240	-509	-622
$\Delta y =$	-5	-59	-179	-269	-113	
$\Delta^2 y =$	-54	-120	-90	156		
$\Delta^3 y =$	-66	30	246			
$\Delta^4 y =$	96	216				
$\Delta^5 y =$	120					

$$F(x) = 3 - 5x - 27x(x-1) - 11x(x-1)(x-2) + 4x(x-1)(x-2)(x-3)$$
$$+ x(x-1)(x-2)(x-3)(x-4)$$
$$= x^5 - 6x^4 + 3.$$

The formula for $F(x)$ may be used for interpolating other values of $f(x)$ and is called *Newton's Interpolation Formula.*

5.84. The Error in the Interpolation Formula. Suppose that $f(x)$ is continuous and possesses a derivative of order $(n+1)$ in the interval.

Let $f(x) = y_0 + x\Delta y_0 + \frac{x(x-1)}{1.2}\Delta^2 y_0 + \ldots$

$$+ \frac{x(x-1)\ldots(x-n+1)}{n!}\Delta^n y_0 + \frac{x(x-1)\ldots(x-n)}{(n+1)!}G(x).$$

Then $f(r) = y_r$, $(r = 0$ to $n)$. If x be some other value in the interval, the above equation determines $G(x)$ in terms of $f(x)$, $y_0, \ldots, y_n$. Consider the following function of ξ

$$H(\xi) = f(\xi) - y_0 - \xi\Delta y_0 - \frac{\xi(\xi-1)}{1.2}\Delta^2 y_0 - \ldots$$

$$- \frac{\xi(\xi-1)\ldots(\xi-n+1)}{n!}\Delta^n y_0 - \frac{\xi(\xi-1)\ldots(\xi-n)}{(n+1)!}G(x)$$

where x is not one of the values $0, 1, 2, \ldots, n$.

$$H(0) = H(1) = H(2) = \ldots = H(n) = H(x) = 0,$$

and therefore by continued application of Rolle's Theorem, $H^{(n+1)}(\xi)$ must vanish for some number θ in the interval. But

$$H^{(n+1)}(\xi) = f^{(n+1)}(\xi) - G(x)$$

Therefore $\qquad G(x) = f^{n+1}(\theta)$

or $f(x) = y_0 + x\Delta y_0 + \frac{x(x-1)}{1.2}\Delta^2 y_0$

$$\ldots + \frac{x(x-1)\ldots(x-n+1)}{n!}\Delta^n y_0 + R_n$$

where $R_n = \frac{x(x-1)\ldots(x-n)}{(n+1)!}f^{n+1}(\theta)$, when x is not one of the numbers $0, 1, \ldots, n$, and the result is obviously true when x is equal to any of these numbers. Thus the error in taking the polynomial as the value of $f(x)$ is R_n.

If $f^{n+1}(\theta)$ is bounded in the interval, it follows that the error is less than $M\frac{x(x-1)\ldots(x-n)}{(n+1)!}$ where $M = \max|f^{(n+1)}(x)|$.

Example. The following table gives the value of $\log_{10} N$ from $N = 40$ to $N = 45$. Find log 40·1, 40·2, . . ., 40·9.

N	40	41	42	43	44	45
$\log_{10} N$	1·60206	·61278	·62325	·63347	·64345	·65321

The differences are

Δy	1072	1047	1022	998	976
$\Delta^2 y$	−25	−25	−24	−22	

The third differences may be ignored.

Using the formula $f(x) = y_0 + x\Delta y_0 + \frac{x(x-1)}{1.2}\Delta^2 y_0$, we find for $x = 0{\cdot}1$,

$\log_{10} 4{\cdot}01 = {\cdot}60206 + {\cdot}00107\,(2) + {\cdot}00001\,(1) = {\cdot}60314$

Similarly by taking $x = 0{\cdot}2, 0{\cdot}3, \ldots, 0{\cdot}9$, we obtain

N	40·1	40·2	40·3	40·4	40·5
$\log_{10} N$	1·60314	60422	60530	60638	60745
N	40·6	40·7	40·8	40·9	
$\log_{10} N$	1·60852	60959	61066	61172	

The error due to taking the polynomial for the function is $\frac{1}{6}x(x-1)(x-2)f'''(x)$ where $f(x) = \log_{10}(40+x)$ so that $|f'''(x)| = \dfrac{2 \times {\cdot}4343}{(40+x)^3} < \dfrac{1{\cdot}3}{10^5}$.

The greatest value of $x(x-1)(x-2)$ that occurs is $0{\cdot}384$ (when $x = 0{\cdot}4$) and therefore the error throughout is $< 9 \times 10^{-7}$ and does not affect the result.

The logarithm found above are all correct to five figures except those for 40·2, 40·6 which are wrong by one unit in the last place. This is due to the fact that the given logarithms are correct only to 5 places and that there are consequent errors in the differences.

Note. The interpolation formula may obviously be used in the reverse direction. Thus in the example above

$$\log 44{\cdot}5 = 1{\cdot}65321 + (0{\cdot}5)(-976) + \tfrac{1}{2}(0{\cdot}5)(-0{\cdot}5)(-22) = 1{\cdot}64836$$

5.85. *The Approximate Area.* If we integrate the interpolation formula

$$f(x) = y_0 + x\Delta y_0 + \frac{x(x-1)}{2}\Delta^2 y_0 + \ldots$$
$$+ \frac{x(x-1)\ldots(x-n+1)}{n!}\Delta^n y_0 + \frac{x(x-1)\ldots(x-n)}{(n+1)!}f^{(n+1)}(\theta)$$

from 0 to n we obtain the corresponding area

$$A = ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{6} - \frac{n^2}{4}\right)\Delta^2 y_0 + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6}$$
$$+ \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right)\frac{\Delta^4 y_0}{24}$$
$$+ \ldots + \frac{f^{(n+1)}(\theta)}{(n+1)!}\int_0^n x(x-1)\ldots(x-n)dx.$$

(i) Let $n = 1$, and we find $A = \frac{1}{2}(y_0 + y_1) - \frac{1}{12}f''(\theta)$.

If the interval is of length $h\ (= b - a)$, an inspection of the dimensions shows that

$$\int_a^b f(x)dx = \frac{1}{2}h(y_0 + y_1) - \frac{h^3}{12}f''(\theta)$$

or

$$\int_a^b f(x)dx = \frac{1}{2}(b-a)(f(a) + f(b)) + R$$

where $R = -\frac{1}{12}(b-a)^3 f''(\theta)$

This is the *Trapezoidal Rule* and may be used when the curve is nearly straight.

If the interval is divided into n equal parts and the rule applied to each part, we obtain

$$\int_a^b f(x)dx = \frac{b-a}{n}\left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right) - \frac{(b-a)^3}{12\, n^2}f''(\theta).$$

(ii) Let $n = 3$ and we find

$$\int_0^{3h} f(x)dx = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3) - \frac{3h^5}{80}f^{iv}(\theta).$$

Thus $\displaystyle\int_a^b f(x)dx = \frac{(b-a)}{8}(y_0 + 3y_1 + 3y_2 + y_3) - \frac{(b-a)^5}{6480}f^{iv}(\theta).$

This is known as the '*Three-Eighths' Rule*'.

If we write $(n+1)$ for n in the interpolation formula and integrate from 0 to n we obtain

$$A = ny_0 + \frac{1}{2}n^2\Delta y_0 + \left(\frac{1}{6}n^3 - \frac{1}{4}n^2\right)\Delta^2 y_0 + \left(\frac{1}{4}n^4 - n^3 + n^2\right)\frac{\Delta^3 y_0}{6}$$

$$+ \left(\frac{1}{5}n^5 - \frac{3}{2}n^4 + \frac{11}{3}n^3 - 3n^2\right)\frac{\Delta^4 y_0}{24}$$

$$+ \left(\frac{1}{6}n^6 - 2n^5 + \frac{35}{4}n^4 - \frac{50}{3}n^3 + 12n^2\right)\frac{\Delta^5 y_0}{120} + \dots + R_n'$$

where $\displaystyle R_n' = \frac{f^{(n+2)}(\theta)}{(n+2)!}\int_0^n x(x-1)(x-2)\dots(x-n-1)dx$

The reason for taking the $(n+1)$-formula is that, owing to symmetry, the coefficient of $\Delta^{n+1}y_0$ is zero when n is *even*. For this coefficient is

$$\frac{1}{(n+1)!}\int_0^n x(x-1)\dots(x-n)dx$$

$$= \frac{1}{(n+1)!}\int_{-m}^{+m}\xi(\xi^2-1^2)(\xi^2-2^2)\dots(\xi^2-m^2)d\xi$$

where $x = m + \xi$ and $n = 2m$.

Also in this case

$$R_n' = \frac{f^{(n+2)}(\theta)}{(n+2)!}\int_{-m}^{m}\xi(\xi-m-1)(\xi^2-1^2)\dots(\xi^2-m^2)d\xi$$

and the integral on the right is $2\displaystyle\int_0^m \xi^2(\xi^2-1^2)\dots(\xi^2-m^2)d\xi$.

(iii) Let $n = 2$, then

$$\int_0^2 f(x)dx = 2y_0 + 2\Delta y_0 + \tfrac{1}{3}\Delta^2 y_0 - \tfrac{1}{90}f^{(iv)}(\theta).$$

For an interval h, $\displaystyle\int_0^{2h} f(x)dx = \frac{1}{3}h(y_0 + 4y_1 + y_2) - \frac{h^5}{90}f^{(iv)}(\theta),$

or $$\int_a^b f(x)dx = \frac{b-a}{6}(y_0 + 4y_1 + y_2) - \frac{(b-a)^5}{2880}f^{(iv)}(\theta).$$

This is *Simpson's Rule*; and if we divide the whole interval into $2n$

parts and apply Simpson's Rule to each successive pair, we obtain

$$A = \frac{b-a}{6n}\{(y_0 + y_{2n}) + 4(y_1 + y_3 + \ldots + y_{2n-1})$$
$$+ 2(y_2 + y_4 + \ldots + y_{2n-2})\} + R$$

where $$R = -\frac{(b-a)^5}{2880n^4}f^{(iv)}(\theta).$$

(iv) Let $n = 6$ and then it will be found that

$$A = 6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0 + R$$

where $$R = -\frac{9}{1400}f^{(viii)}(\theta).$$

If we write $\frac{41}{140}\Delta^6 y_0$ as

$$\frac{3}{10}(y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0) - \frac{\Delta^6 y_0}{140}$$

and substitute also for $\Delta y_0, \ldots \Delta^5 y_0$, it may be verified that

$$A = \frac{3}{10}\{y_0 + y_2 + y_4 + y_6 + 5(y_1 + y_5) + 6y_3\} - \frac{1}{140}\Delta^6 y_0 - \frac{9}{1400}f^{(viii)}(\theta).$$

Now $\Delta^6 y_0 = f^{(vi)}(\theta_1)$

(since $f(x) - y_0 - x\Delta y_0 - \ldots - \frac{x(x-1)\ldots(x-n+1)}{n!}\Delta^n y_0$ vanishes at $0, 1, \ldots, n$, and therefore its nth derivative, viz. $f^{(n)}(x) - \Delta^n y_0$ must vanish at some point in the interval).

i.e. $$A = \frac{3h}{10}\{y_0 + y_2 + y_4 + y_6 + 5(y_1 + y_5) + 6y_3\} - \frac{h^7}{140}f^{(vi)}(\theta_1) - \frac{9h^9}{1400}f^{(viii)}(\theta)$$

or $$\int_a^b f(x)dx = \frac{b-a}{20}\{y_0 + y_2 + y_4 + y_6 + 5(y_1 + y_5) + 6y_3\} + R$$

where

$$R = -\frac{(b-a)^7}{140 \times 6^7}f^{(vi)}(\theta_1) - \frac{9(b-a)^9}{1400 \times 6^9}f^{(viii)}(\theta)$$
$$= -(2\cdot55 \times 10^{-8})(b-a)^7 f^{(vi)}(\theta_1) - (6\cdot37 \times 10^{-10})(b-a)^9 f^{(viii)}(\theta)$$

the coefficients being given to 3 significant figures.

This is known as *Weddle's Rule.*

Example. Find the area determined by $\int_{-1}^{+1}\sqrt{(4-x^2)}dx$. (The correct result being $\frac{2}{3}\pi + \sqrt{3} = 3\cdot82644591 \ldots$)

Rule	Number of Ordinates	Area	Percentage Error
Trapezoidal	2	3·464 . .	9·4
	3	3·732 . .	2·4
	5	3·8025	0·6
	7	3·8158 . .	0·3
Simpson's	3	3·8214 .	0·13
	5	3·8260 .	0·01
	7	3·8263 .	0·003
Three-Eighths	4	3·8241 . . .	0·06
	7	3·8262	0·005
Weddle's	7	3·826426 .	0·0005

5.86. *The Use of Legendre Polynomials.* A question of theoretical interest arises when we consider the case of non-equidistant ordinates.

Let the interval be $-1 \leq x \leq 1$ and let the n ordinates be $y_1, y_2, \ldots y_n$ corresponding to the n abscissae $a_1, a_2, \ldots, a_n$. Then a polynomial curve $y = E_{n-1}(x)$ of degree $(n-1)$ may be drawn through the n points (a_r, y_r). Let us therefore consider the problem of choosing a_r so that the most general polynomial of degree m $(> n-1)$ gives the same area as that given by $y = E_{n-1}$. If this polynomial be denoted by F_m, there are $(m-n+1)$ undetermined coefficients since the curve $y = F_m$ must pass through (a_r, y_r) $(r = 1$ to $n)$. Since there are n numbers a_r we can therefore assume that $m - n + 1 = n$, i.e. $m = 2n - 1$. We may therefore write

$$F_{2n-1} = E_{n-1} + (x-a_1)(x-a_2)\ldots(x-a_n)G_{n-1}$$

The area determined by $y = F_{2n-1}$ is the same as that determined by $y = E_{n-1}$ for an arbitrary G_{n-1} if

$$\int_{-1}^{+1}(x-a_1)(x-a_2)\ldots(x-a_n)G_{n-1}\,dx$$

for all polynomials G_{n-1}.

If we denote $(x-a_1)\ldots(x-a_n)$ by $\dfrac{d^nH_{2n}}{dx^n}$ (so that there are n arbitrary constants in H_{2n}), we have

$$\int_{-1}^{+1}G_{n-1}(x-a_1)\ldots(x-a_n)dx$$
$$= G_{n-1}H_{2n}^{(n-1)} - G'_{n-1}H_{2n}^{(n-2)} + \ldots + (-1)^{n-1}G_{n-1}^{(n-1)}H_{2n}$$

and this integral vanishes for all polynomials G_{n-1} if

$$H_{2n}, H_{2n}', \ldots, H_{2n}^{(n-1)} \text{ vanish at } x = \pm 1$$

i.e. H_{2n} must be a multiple of $(x^2-1)^n$ and therefore $a_1, a_2, \ldots, a_n$, are the roots of the equation

$$\frac{d^n}{dx^n}(x^2-1)^n = 0$$

which are obviously all real and distinct.

The function given by $\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ is called the *Legendre Polynomial* of degree n and is usually denoted by $P_n(x)$.

Example. Consider $\int_{-1}^{+1} \sqrt{(4-x^2)}dx$.

(i) $n=1$; $P_1(x)=x$; one ordinate at $x=0$, $y=2$.
Polynomial is $y=2$; $A=4$; Error 4·5 per cent.
This has the same accuracy as any straight line through $(0, 2)$.

(ii) $n=2$, $P_2(x)=\frac{3}{2}x^2-\frac{1}{2}x$, two ordinates; $x_1=\frac{1}{3}\sqrt{3}$, $y_1=\sqrt{\frac{11}{3}}$; $x_2=-\frac{1}{3}\sqrt{3}$, $y_2=\sqrt{\frac{11}{3}}$. The polynomial is $y=\sqrt{\frac{11}{3}}$; $A=2\sqrt{\frac{11}{3}}=3·8297$, error 0·09 per cent. This has the same value as the area determined by any *cubic* polynomial through $\left(\pm\frac{\sqrt{3}}{3}, \sqrt{\frac{11}{3}}\right)$

(iii) $n=3$; $P_3(x)=0$ when $x=0, \pm\sqrt{\frac{3}{5}}$. The three ordinates are $\sqrt{\frac{17}{5}}$, 2, $\sqrt{\frac{17}{5}}$. The polynomial is $2-\left(\frac{10}{3}-\frac{5\sqrt{17}}{3\sqrt{5}}\right)x^2$

$$A=4-\left(\frac{20}{9}-\frac{10\sqrt{17}}{9\sqrt{5}}\right)=3·82857$$

error is 0·003 per cent.

There is the same accuracy as that of a quintic through the points.

(iv) $n=4$; $P_4(x)=0$ when $x=\pm 0·861136$, $\pm 0·339981$, and the area will be found to be 3·826450 (error 0·0001 per cent.)

Owing to the fact that the roots of $P_n(x)$ are not simple numbers, the use of the polynomial is not of practical value.

5.9. Definite Integrals of Frequent Occurrence.

$$(1) \qquad \int_0^\pi \sin mx \sin nx\,dx=\int_0^\pi \cos mx \cos nx\,dx=0$$

where m, n are unequal positive integers.

$$\int_0^\pi \sin^2 mx\,dx=\int_0^\pi \cos^2 mx\,dx=\tfrac{1}{2}\pi \text{ (m being a positive integer)}$$

$$(2) \qquad \int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx=\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx=\int_0^{\frac{\pi}{2}} \cos^{2n+1} x\,dx=\frac{2.4.6\ldots 2n}{3.5.7\ldots(2n+1)}$$

where n is a positive integer

(*See* § 5.34, *Example*, § 5.35, *Example* (i).)

$$(3) \quad \int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n} x\,dx=\frac{(2m)!(2n)!}{2^{2m+2n}m!n!(m+n)!}\frac{\pi}{2},$$

$$\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \cos^n x\,dx=\int_0^{\frac{\pi}{2}} \cos^{2m+1} x \sin^n x\,dx$$
$$=\frac{2^m m!}{(n+1)(n+3)\ldots(n+2m+1)},$$

m, n being positive integers.

These may be established by Reduction Formulæ

Let $J(m, n)=\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx$ and suppose $m \geqslant n$.

From the result $(2m+2n)J(m, n) = -\left[\sin^{2m-1}x\cos^{2n+1}x\right]_0^{\frac{\pi}{2}} + (2m-1)J(m-1, n)$

we deduce that $J(m, n) = \dfrac{2m-1}{2m+2n}J(m-1, n)$ and therefore by repeated applications

$$J(m, n) = \frac{(2m-1)(2m-3)\ldots 3.1}{(2m+2n)(2m+2n-2)\ldots(2n+2)}J(0, n);$$

$$\text{and } J(0, n) = \frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\,\frac{\pi}{2}.$$

From these the required result follows since $J(m, n) = J(n, m)$. Similarly the other integrals may be determined.

Note. The integral $\int_0^{\frac{\pi}{2}} \sin^p x\cos^q x\,dx$ can be more concisely expressed in terms of *Gamma Functions* (*see Chapter XII*), and the expression so obtained is applicable to *all* real values of $p, q > -1$.

(4) $$\int_0^\infty e^{-ax}\cos bx\,dx = \int_0^\infty e^{-bx}\sin ax\,dx = \frac{a}{a^2+b^2},$$

$$\int_0^\infty x^n e^{-cx}\,dx = \frac{n!}{c^{n+1}}$$

where a is positive in the first integral, b is positive in the second and c in the third.

Examples V(b)

Evaluate the integrals given in *Examples 1–27*.

1. $\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^4\theta\,d\theta$ 2. $\int_0^{\frac{\pi}{2}} \sin^3 2\theta\cos\theta\,d\theta$ 3. $\int_{\frac{1}{2}}^1 \frac{dx}{x^2\sqrt{(1-x^2)}}$

4. $\int_0^\infty \frac{\sqrt{x}\,dx}{(4x+1)(2x+1)(4x+3)}$ 5. $\int_0^1 x^4(1-x)\,dx$ 6. $\int_0^\infty \frac{dx}{(x^2+2)^7}$

7. $\int_0^\infty \frac{dx}{(1+x^2+x^4)}$ 8. $\int_0^\infty \frac{x^2\,dx}{1+x^2+x^4}$ 9. $\int_0^1 \frac{(1+x^4)dx}{(1+x^6)}$

10. $\int_0^\infty \frac{dx}{25x^4-14x^2+25}$ 11. $\int_0^1 \frac{(1+x^2)dx}{(1+x^4)}$ 12. $\int_1^\infty \frac{dx}{(3x+1)\sqrt{(x^2-1)}}$

13. $\int_{-1}^{+1} \frac{dx}{(1-2x\cos\alpha+x^2)}$ 14. $\int_0^1 x^4(1-x^2)^{\frac{3}{2}}\,dx$

15. $\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$ 16. $\int_0^\pi \frac{p\,dx}{p^2\cos^2 x+\sin^2 x}$. $(p>0)$

17. $\int_0^\pi \frac{(a-b\cos x)dx}{a^2-2ab\cos x+b^2}$ 18. $\int_{x_0}^{x_0+h} \frac{dx}{x^2-2px+p^2+b^2}$, $(b\neq 0)$

19. $\int_0^\pi e^{-x}\sin^3 x\,dx$ 20. $\int_0^\pi \frac{2-\cos x}{5-4\cos x}dx$ 21. $\int_0^\pi \frac{1-2\cos x}{5-4\cos x}dx$

22. $\int_0^{\frac{\pi}{2}} \frac{3\sin x+4\cos x+5}{2\sin x+\cos x+2}dx$ 23. $\int_0^\infty \operatorname{sech} ax\,dx$

24. $\displaystyle\int_0^{\pi} \frac{\sin^2 x \cos x\, dx}{(3+\cos x)^2}$ **25.** $\displaystyle\int_0^{\infty} \frac{x^2\, dx}{1+x^2+x^4}$

26. $\displaystyle\int_0^{\infty} \frac{x^2\, dx}{1+x^4}$ **27.** $\displaystyle\int_0^{1} \frac{x\, dx}{1-x^4}$

28. Prove that $\displaystyle\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\frac{\pi}{2}} f(\sin x)dx.$

29. Deduce from *Example 28* that $\displaystyle\int_0^{\pi} \frac{x \sin x}{2-\sin^2 x}\,dx = \frac{1}{4}\pi^2.$

30. Prove that $\displaystyle\int_0^{\frac{1}{2}\pi} \log \sin x\, dx = \int_0^{\frac{1}{2}\pi} \log \cos x\, dx = -\tfrac{1}{2}\pi \log 2.$

31. Find $\displaystyle\int_0^{3} f(x)dx$ if $f(x) = x^2$, $(0 < x < 1)$, 1, $(1 < x < 2)$, $(x-3)^2$, $(2<x<3)$.

Prove the results given in *Examples 32–41.*

32. $\displaystyle\int_0^{2} |x-1|dx = 1.$

33. $\displaystyle\int_0^{x} f(x)dx = (x-\tfrac{1}{2})f(x) - \tfrac{1}{2}\{f(x)\}^2$ where $f(x)$ is the greatest integer $< x$.

34. $\displaystyle\int_0^{\pi} \frac{x\, dx}{4-\cos^2 x} = \frac{\pi^2}{4\sqrt{3}}$

35. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-e^2\cos^2\theta)}} = \frac{\pi}{2}\sum_0^{\infty}\frac{(2n!)^2}{2^{4n}(n!)^4}e^{2n}$ where $|e| < 1$

36. $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(a^2x^2+b^2)^n} = \frac{(2n-2)!\pi}{2^{2n-2}\{(n-1)!\}^2 ab^{2n-1}}$, $(a, b > 0)$

37. $\displaystyle\int_0^{\pi} \frac{dx}{(2+\cos x)^n} = \frac{\sqrt{3}}{3^n}\int_0^{\pi}(2-\cos x)^{n-1}\,dx$

38. $\displaystyle\int_0^{\pi} \frac{dx}{(2-\cos x)^2} = \frac{\pi}{2\sqrt{3}}$

39. $\displaystyle\int_0^{1} x^n(\log x)^m\, dx = (-1)^m\frac{m!}{(n+1)^{m+1}}$ (m being a positive integer and $n > -1$)

40. $\displaystyle\int_0^{\pi} \frac{dx}{(1+\cos\alpha\cos x)^n} = \operatorname{cosec}^{2n-1}\alpha\int_0^{\pi}(1-\cos\alpha\cos x)^{n-1}\,dx$

41. $\displaystyle\int_0^{\pi} \log(1+\alpha\cos x)dx = \pi\log\{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-\alpha^2)}\}$, $(|\alpha| < 1)$

42. If $\displaystyle u = \int_{-1}^{+1} \frac{dx}{\sqrt{((1-2ax+a^2)(1-2bx+b^2))}}$ $(0 < a < 1,\ 0 < b < 1)$, show that $\tanh \tfrac{1}{2}u\sqrt{ab} = \sqrt{ab}$

43. Prove that $\int_0^1 \frac{dt}{1+t^3} = \frac{\pi}{3\sqrt{3}} + \frac{1}{3}\log 2$ and deduce that

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \ldots = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \log 2\right)$$

44. Show that $\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \ldots = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} - \log 2\right)$.

45. Prove that $\int_0^t \frac{t^2\,dt}{1+t^4} = \frac{1}{4\sqrt{2}}\log\left(\frac{t^2 - t\sqrt{2} + 1}{t^2 + t\sqrt{2} + 1}\right) + \frac{\sqrt{2}}{4}\operatorname{arc\,tan}\left(\frac{t\sqrt{2}}{1-t^2}\right)$

and deduce that $\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \ldots = \frac{1}{2\sqrt{2}}\left\{\frac{\pi}{2} + \log(\sqrt{2} - 1)\right\}$.

46. By integrating the power series for $\frac{\operatorname{arc\,tan} x}{1+x^2}$ show that when $|x| < 1$

$(\operatorname{arc\,tan} x)^2 = \sum_1^\infty (-1)^{n-1}\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right)\frac{x^{2n}}{2n}$.

Prove the results given in *Examples 47–54.*

47. $1 - \frac{1}{7} + \frac{1}{13} - \frac{1}{19} + \ldots = \frac{\pi}{6} + \frac{\sqrt{3}}{6}\log(2 + \sqrt{3})$

48. $\frac{1}{1.2.3} - \frac{1}{2.3.4} + \frac{1}{3.4.5} - \ldots = \log 4 - \frac{5}{4}$

49. $\frac{1}{1.3.5} + \frac{x}{3.5.7} + \frac{x^2}{5.7.9} + \ldots = \frac{5}{24x} - \frac{1}{8x^2} + \frac{(1-x)^2}{16x^{5/2}}\log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$, $(0 < x < 1)$

50. $\frac{1}{1.3.5} - \frac{x}{3.5.7} + \frac{x^2}{5.7.9} - \ldots = \frac{5}{24x} - \frac{1}{8x^2} - \frac{(1+x)^2}{8x^{5/2}}\operatorname{arc\,tan}\sqrt{x}$, $(0 < x < 1)$

51. $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \ldots = \frac{1}{12}$

52. $\frac{1}{1.3.5} - \frac{1}{3.5.7} + \frac{1}{5.7.9} - \ldots = \frac{\pi}{8} - \frac{1}{3}$

53. $\frac{1}{3!} - \frac{3!}{6!} + \frac{6!}{9!} - \ldots = \frac{2}{3}\log 2 - \frac{\pi}{6\sqrt{3}}$

54. $\frac{1}{2^0.2!} - \frac{4!}{2^4.6!} + \frac{8!}{2^8.10!} - \ldots = \frac{1}{16}\log 5 + \frac{1}{8}\operatorname{arc\,tan}\left(\frac{2}{11}\right)$

Discuss the convergence of the integrals given in *Examples 55–63.*

55. $\int_0^\infty e^{-x^a}\,dx$ **56.** $\int_0^1 x^\alpha(1-x)^\beta\,dx$ **57.** $\int_0^\infty e^{-t}t^{a-1}\,dt$

58. $\int_0^\infty e^{-x}x^m\,dx$ **59.** $\int_0^1 \frac{x^p \log x\,dx}{(1-x)^q}$ **60.** $\int_0^\infty \frac{\cosh bt}{\cosh at}\,dt$

61. $\int_0^\infty \frac{\sinh bt}{\sinh at}\,dt$ **62.** $\int_0^\infty \frac{\sin bx\,dx}{e^{ax}-1}$

63. $\int_0^\infty \frac{x^\alpha(1+x^\beta)}{1+x^\gamma}\,dx$, $(\beta, \gamma > 0)$

Prove the results given in *Examples 64–7.*

64. $P\int_{-2}^{2} \frac{x^4 + 3x^3 + x^2 - 1}{x^2(x^2-1)}\,dx = \ldots 3\log 3$

65. $\lim\limits_{\epsilon \to 0}\left\{\int_0^{1-\epsilon} + \int_{1+\epsilon}^{2-\epsilon} + \int_{2+\epsilon}^{\infty}\right\}\left(\dfrac{2x^2 - 4x + 1}{(x-1)^2(x-2)^2}\right)dx = \dfrac{1}{2} - 2\log 2$

66. $\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x\,dx$ lies between 1·178 and 1·321

67. $\int_0^1 \dfrac{dx}{\sqrt{(1-x^4)}}$ lies between 1 and $\dfrac{\pi}{2}$

68. Obtain Lagrange's polynomial for the $(n+1)$ points (a_r, y_r), ($r = 0$ to n)

$$f(x) = \sum_0^n \frac{(x-a_0)(x-a_1)\dots(x-a_{r-1})(x-a_{r+1})\dots(x-a_n)}{(a_r-a_0)(a_r-a_1)\dots(a_r-a_{r-1})(a_r-a_{r+1})\dots(a_r-a_n)}\,y_r.$$

69. Prove that the cubic through (r, y_r), ($r = 0$ to 3), is given by

$$6y = -(x-1)(x-2)(x-3)y_0 + 3x(x-2)(x-3)y_1 - 3x(x-1)(x-3)y_2 + x(x-1)(x-2)y_3$$

Find the polynomial of lowest degree for the points given in *Examples 70–1*.

70. $(-5, 1000)$, $(-3, 228)$, $(-1, 0)$, $(1, 4)$, $(3, 96)$, $(5, 900)$

71. $(-5, 1150)$, $(-3, 120)$, $(-1, 6)$, $(1, 4)$, $(3, 150)$, $(5, 1200)$

72. The *divided difference* $[x_1 x_2 \dots x_n]$ is defined by means of the relations:

$$[x_1 x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2},\quad [x_1 x_2 \dots x_n] = \frac{[x_1 x_2 \dots x_{n-1}] - [x_2 x_3 \dots x_n]}{x_1 - x_n}.$$

Prove that

(i) $[x_1 x_2 \dots x_n]$

$$= \sum_1^n \frac{f(x_r)}{(x_r - x_1)(x_r - x_2)\dots(x_r - x_{r-1})(x_r - x_{r+1})\dots(x_r - x_n)}$$

(ii) The value of $[x_1 x_2 \dots x_n]$ is independent of the order of its arguments.

(iii) $f(x) = f(x_1) + \sum\limits_1^{n-1}(x - x_1)(x - x_2)\dots(x - x_s)[x_1 x_2 \dots x_{s+1}] + R_n(x)$,

where

$$R_n(x) = (x - x_1)(x - x_2)\dots(x - x_n)[x\,x_1 x_2 \dots x_n]$$

(iv) $[x_1 x_2 \dots x_n] = \dfrac{f^{(n-1)}(\theta)}{(n-1)!}$ where $\min x_r < \theta < \max x_r$.

(v) $R_n(x) = (x - x_1)(x - x_2)\dots(x - x_n)\dfrac{f^{(n)}(\theta)}{n!}$.

73. Corresponding values of x, y are given in the following table

x	0	1	2	3	4	5
y	30·00	121·67	249·43	419·25	637·11	909·00

Deduce the value of y for $x = 1·3$ and $x = 2·5$.

74. From the following values of $\sin x$, calculate $\sin 30^\circ\,10'$, $\sin 39^\circ\,40'$

x°	30	31	32	33	34	35	36	37	38	39	40
$\sin x$	·50000	·51504	·52992	·54464	·55919	·57358	·58779	·60182	·61566	·62932	·64279

Also find an upper limit to the error for the first interval due to the polynomial approximation.

75. Prove that $\int_a^b f(x)dx = \dfrac{b-a}{90}(7(y_0 + y_4) + 32(y_1 + y_3) + 12\,y_2) + \dfrac{(b-a)^7}{1935360} f^{(vi)}(\theta)$ for 5 equidistant ordinates, where $y_0 = f(a)$, $y_4 = f(b)$.

76. Show that

$$\int_a^b f(x)dx = \tfrac{1}{3}(b-a)\{(0{\cdot}14)(y_0+y_6)+(0{\cdot}81)(y_1+y_5)+(1{\cdot}10)y_3\}+R$$

where $R = (4.6)\times 10^{-6}\,(b-a)^7 f^{(vi)}(\theta_1) - (6.4)\,(b-a)^9 f^{(viii)}(\theta)\times 10^{-10}$, the numerical coefficients in R being approximate, and where $y_r = f(a+rh)$, $h = (b-a)/6$. (This is *G. F. Hardy's Rule*.)

77. Show that

$$\int_0^{3mh} f(x)dx = \frac{3h}{8}\{(y_0+y_{3m})+3(y_1+y_2+y_4+y_5+\ldots y_{3m-1}) + 2(y_3+y_6+\ldots+y_{3m-3})\}$$

approximately, the ordinates being equidistant.

78. Show that Simpson's rule may be obtained by eliminating the error term between the trapezoidal formulæ corresponding to the sub-intervals h and $2h$.

79. Calculate $\displaystyle\int_0^{10}\frac{10dx}{x^2+100}$ by (i) the trapezoidal rule (ii) Simpson's rule, using in each case 11 ordinates at intervals of 1 unit.

80. Calculate $\displaystyle\int_0^{6}\frac{24dx}{x^2+36}$ by (i) the trapezoidal rule (ii) Simpson's rule, (iii) the three-eighths rule, (iv) Weddle's rule, (v) G. F. Hardy's rule (*Example 76*), using in each case 7 ordinates at intervals of 1 unit.

81. (i) Prove that $\displaystyle\int_0^{\infty}\frac{dx}{1+x^4}$ differs from $\displaystyle\int_0^{10}\frac{dx}{1+x^4}$ by less than $\frac{1}{3}\times 10^{-3}$.

(ii) Calculate $\displaystyle\int_0^{2}\frac{dx}{1+x^4}$ by Simpson's rule, taking 11 ordinates.

(iii) Calculate $\displaystyle\int_2^{10}\frac{dx}{1+x^4}$ by Simpson's rule, taking 9 ordinates.

(iv) Find the approximate value of $\displaystyle\int_0^{\infty}\frac{dx}{1+x^4}$.

82. (i) Calculate $\displaystyle\int_0^{2} e^{-x^2}dx$ with 11 ordinates.

(ii) Calculate $\displaystyle\int_2^{3} e^{-x^2}dx$ with 3 ordinates,

using Simpson's rule in each case.

(iii) Deduce the approximate value of $\displaystyle\int_0^{\infty} e^{-x^2}dx$.

83. The semi-ordinates of the deck plan of a ship in feet are respectively 1·25, 6·75, 10·75, 14·00, 15·50, 15·00, 13·25, 10·00, 3·50, and they are 16 feet apart. Find the approximate area of the plan in square yards.

84. The areas of the horizontal sections of a vessel floating in salt water at intervals of $1\frac{1}{2}$ ft. are 2,100, 2,060, 1,630, 1,260 and 320 sq. ft., the first referring to the water line section and the others being lower. Taking the weight of salt water as 64 lb. per cubic ft., find the displacement of the vessel in tons.

85. A curve is given between $x = -1$ and $x = +1$ and it is required to choose n intermediate ordinates at $x_1, x_2, \ldots, x_n$ such that the area determined by the polynomial curve through (x_r, y_r) is equal to $\dfrac{2}{n}(y_1+y_2+\ldots+y_n)$. Show that

the n values $x_1, x_2, \ldots x_n$ are the zeros of the polynomial $T_n(x)$ which is asymptotic to the function

$$\left(\frac{(x+1)^{x+1}}{(x-1)^{x-1}}\cdot\frac{1}{e^2}\right)^{\frac{1}{2}n}.$$

In particular, show that (i) $T_2 = x^2 - \frac{1}{3}$, with zeros $\pm 0{\cdot}5773$.

(ii) $T_3 = x(x^2 - \frac{1}{2})$, with zeros 0, $\pm 0{\cdot}7071$. (iii) $T_4 = x^4 - \frac{2}{3}x^2 + \frac{1}{45}$, with zeros $\pm 0{\cdot}1876$, $\pm 0{\cdot}7947$. (iv) $T_5 = x(x^4 - \frac{5}{6}x^2 + \frac{7}{72})$, with zeros 0, $\pm 0{\cdot}3745$, $\pm 0{\cdot}8325$. (*Tschebyscheff's method.*)

Solutions.

1. $\frac{\pi}{32}$ **2.** $\frac{8}{15}$ **3.** $\sqrt{3}$ **4.** $\frac{\pi}{8}(2\sqrt{2} - \sqrt{3} - 1)$ **5.** $\frac{32}{3003}$

6. $\frac{(12!)\pi\sqrt{2}}{2^{20}(6!)^2}$ **7.** $\frac{\pi}{2}$ **8.** $\frac{\pi}{2\sqrt{3}}$ **9.** $\frac{\pi\sqrt{2}}{4}$ **10.** $\frac{\pi}{60}$

11. $\frac{1}{3}\left(2\log 2 + \frac{\pi}{\sqrt{3}}\right)$ **12.** $\frac{\sqrt{2}}{4}\operatorname{arc\,cos}\left(\frac{1}{3}\right)$ **13.** $\frac{\pi}{2\,|\sin\alpha|}$ **14.** $\frac{8}{315}$

15. $\frac{2\pi}{\sqrt{3}}$ **16.** $\frac{\pi}{2}$ **17.** $\frac{\pi}{a}$, $(|a| > |b|)$; 0, $(|a| < |b|)$

18. $\frac{1}{b}\operatorname{arc\,tan}\left\{\frac{2bh}{(x_0+p)^2 + b^2 - h^2}\right\}$ **19.** $\frac{3}{10}(1 - e^{-\pi})$ **20.** $\frac{\pi}{2}$

21. 0 **22.** $\pi + \log 2$ **23.** $\frac{\pi}{2a}$, $(a > 0)$ **24.** $\frac{\pi}{4}(24 - 17\sqrt{2})$

25. $\frac{\pi}{2}$ **26.** $\frac{\pi\sqrt{2}}{4}$ **27.** $\frac{2\pi}{3\sqrt{3}}$

30. $I = \frac{1}{2}\int_0^{\frac{\pi}{2}}(\log\sin x + \log\cos x)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}\{\log\sin 2x + \log\frac{1}{2}\}dx$

$$= \frac{\pi}{4}\log\frac{1}{2} + \frac{1}{2}I$$

31. $\frac{1}{2}$ **34.** *See Example 28.* **36.** Let $ax = b\tan\theta$.

37. Take $(2 - \cos x)(2 - \cos y) = 3$, so that $(2 + \cos x)\sin y = \sqrt{3}\sin x$ and $\frac{dy}{dx} = \frac{(2 - \cos y)\sin x}{(2 + \cos x)\sin y}$.

38. Use *Example 37.*

40. Take $(1 + \cos\alpha\cos x)(1 - \cos\alpha\cos y) = \sin^2\alpha$.

41. Integrate $\int_0^{\pi}\frac{\cos x\,dx}{1 + a\cos x} = \frac{\pi}{a}\left(1 - \frac{1}{\sqrt{(1 - a^2)}}\right)$

44. Find $\int_0^1\frac{t\,dt}{1 + t^6}$. **47.** Find $\int_0^1\frac{dt}{1 + t^6}$.

48. Integrate the integral of $\log(1 + t)$.

49, 50. Integrate the series $\sum_0^{\infty}(1 - t)^2x^nt^{2n} = \frac{(1-t)^2}{1 - xt^2}$ with respect to t, the justification of this process being a simple extension of that given for a power series.

53. Integrate $\frac{(1 - t)^2}{1 - t^3}$. **54** Integrate $\frac{(1 - t)}{4 + t^4}$.

55. Convergent; take $x = \sqrt{u}$ and *see Example 57.*

56. Convergent when $p, q > -1$. **57.** Convergent when $a > 0$.

58. Take $x^m = u$; convergent if $m > 0$. **59.** Convergent for $p > -1$.

60, 61. Convergent for $|b| < |a|$. **62.** Convergent for $a > 0$.

63. Convergent for $\gamma - \beta > \alpha + 1 > 0$ **66.** $\frac{1}{2}\pi < l < \frac{1}{2}\pi - \frac{1}{2}$
70. $2x^4 - 3x^3 + 5x$ **71.** $2x^4 - 3x^3 + 5x$
72. *See Milne-Thomson, Finite Differences, I.*
73. 155·94, 326·71 **74.** 0·50252, 0·63632, error $< 4 \times 10^{-7}$.
76. Eliminate y_3, y_4 from the Area-formula for $n = 6$ by means of the relation $\Delta^6 y_0 = y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0$.
79. (i) 0·784983, (ii) 0·785398
80. (i) 3·1369631, (ii) 3·1415918, (iii) 3·1415834, (iv) 3·1415984, (v) 3·1415212
81. (i) The difference is $\int_{10}^{\infty} \frac{dx}{x^4} = \frac{1}{3} \cdot 10^{-3}$, (ii) 1·07, (iii) 0·04, (iv) 1·11 (the correct value being $\frac{1}{4}\pi\sqrt{2} = 1·1107$ approx.).
82. (i) 0·882 (ii) 0·004 (iii) 0·89 (the correct value being $\frac{1}{2}\sqrt{\pi} = 0·886$ approx.).
83. 311·4 **84.** 272
85. The equations to be satisfied by x_m are $s_n = \frac{1}{2}\{1 + (-1)^n\}/(n + 1)$, where $s_r = \sum_{m=1}^{n} x_m^r$. The required result may be proved by using the fact that $f'(x)/f(x) = n/x + \Sigma s_m/x^{m+1}$, when x is large and $f(x) = \Pi(x - x_m)$.

CHAPTER VI

JACOBIANS. IMPLICIT FUNCTION THEOREM. TRANSFORMATIONS.

6. Jacobians. If $y_1, y_2, \ldots, y_n$ are functions of n variables $x_1, x_2, \ldots, x_n$, possessing partial derivatives, the determinant

$$\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

is called a *Jacobian* and is often written $\dfrac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}$.

Note. The functions y_r may of course be functions of other variables in addition to $x_1, x_2, \ldots, x_n$.

6.01. A Characteristic Property of a Jacobian. Let the variables x_r be expressed as functions of n other variables z_r so that

$$x_r = x_r\,(z_1, z_2, \ldots, z_n)$$

where x_r on the right is a functional symbol.

Then
$$\frac{\partial y_r}{\partial z_s} = \sum_{t=1}^{t=n} \frac{\partial y_r}{\partial x_t} \cdot \frac{\partial x_t}{\partial z_s} \quad (r = 1 \text{ to } n,\ s = 1 \text{ to } n)$$

where on the left, y_r is expressed as a function of $z_1, z_2, \ldots, z_n$. Therefore
$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(z_1, z_2, \ldots, z_n)} = \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} \cdot \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(z_1, z_2, \ldots, z_n)}$$
by the rule for the multiplication of determinants.

Note. This relation may be regarded as an analogue for 'functions of functions' of the simple result $\dfrac{dy}{dz} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dz}$ for 'function of a function'.

Example. If F, G are functions of x, y and x, y are changed to polar coordinates by the transformation $x = r\cos\theta$, $y = r\sin\theta$, then $\dfrac{\partial(x, y)}{\partial(r, \theta)} = r$ and therefore $\dfrac{\partial(F, G)}{\partial(x, y)} = \dfrac{1}{r}\dfrac{\partial(F, G)}{\partial(r, \theta)}$.

6.1. The General Implicit Function Theorem. In *Chapter II*, we have shown that under certain conditions the relation $f(x, y) = 0$

will determine a unique function y of x taking the value b when $x = a$ (if $f(a, b) = 0$). In particular, if $f_y \neq 0$ at (a, b), where $f(x, y)$ is continuous and possesses first partial derivatives, the function y exists and possesses a derivative given by $f_x + f_y \dfrac{dy}{dx} = 0$. By a similar proof it is easily shown that under analogous conditions, the relation

$$f(y, x_1, x_2, \ldots, x_n) = 0 \quad (\text{when } f(b, a_1, a_2, \ldots a_n) = 0)$$

determines a function $y(x_1, x_2, \ldots x_n)$ if $f_y \neq 0$ at $(b, a_1, a_2, \ldots a_n)$. Also the derivatives of y are given by the n equations,

$$f_y \frac{\partial y}{\partial x_r} + f_{x_r} = 0 \quad (r = 1 \text{ to } n).$$

By the method of induction we can generalize this result to obtain the *General Implicit Function Theorem.*

If $f_r(y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_n)$ $(r = 1$ to $m)$ are continuous functions possessing partial derivatives and if

$$J = \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(y_1, y_2, \ldots, y_m)} \neq 0,$$

the equations $f_r = 0$ determine in the neighbourhood of $(b_1, b_2, \ldots, b_m, a_1, a_2, \ldots, a_n)$ (where $f_r(b_1, \ldots, b_m, a_1, \ldots, a_n) = 0$) a unique set of functions y_r.

Assume the theorem to be true for $(m - 1)$ equations connecting $(m - 1)$ functions y, the theorem having been proved for $m = 1$. The expansion of J in terms of its first row gives

$$J = \frac{\partial f_1}{\partial y_1} J_1 + \frac{\partial f_1}{\partial y_2} J_2 + \ldots + \frac{\partial f_1}{\partial y_m} J_m$$

where $\quad J_r = (-1)^r \dfrac{\partial(f_2, f_3, \ldots, f_m)}{\partial(y_1, \ldots, y_{r-1}, y_{r+1}, \ldots, y_m)}.$

Since $J \neq 0$, one at least of the terms $\dfrac{\partial f_1}{\partial y_r} J_r$ does not vanish and the order of the equations can be taken in such a way that $\dfrac{\partial f_1}{\partial y_1} J_1$ is a term that does not vanish. It follows therefore that $\dfrac{\partial f_1}{\partial y_1} \neq 0$ and $J_1 \neq 0$.

Since $\dfrac{\partial f_1}{\partial y_1} \neq 0$, we can from the relation

$$f_1(y_1, \ldots, y_m, x_1 \ldots x_n) = f_1$$

determine y_1 as a function of $f_1, y_2, y_3, \ldots, y_m, x_1, \ldots, x_n$ which reduces to b_1 when $f_1 = 0$, $y_2 = b_2, \ldots, y_m = b_m$, $x_r = a_r$.

When this function is substituted in $f_r(y_1, \ldots, y_m, x_1, \ldots, x_n)$ $(r = 2$ to $m)$, the latter become functions of $f_1, y_2, \ldots, y_m, x_1, \ldots, x_n$ which we may denote by $F_2, F_3, \ldots, F_m$. We have thus changed the variables $y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_n$ to $f_1, y_2, \ldots, y_m, x_1, \ldots, x_n$ and the m functions $f_1, f_2, \ldots, f_m$ are changed to $f_1, F_2, \ldots, F_m$.

Now

$$J = \frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(y_1, y_2, \ldots, y_m)} = \frac{\partial(f_1, F_2, F_3, \ldots, F_m)}{\partial(f_1, y_2, y_3, \ldots, y_m)} \frac{\partial(f_1, y_2, y_3, \ldots, y_m)}{\partial(y_1, y_2, y_3, \ldots, y_m)}$$

$$= \frac{\partial(F_2, F_3, \ldots, F_m)}{\partial(y_2, y_3, \ldots, y_m)} \cdot \frac{\partial f_1}{\partial y_1}.$$

But $J \neq 0$ and $\dfrac{\partial f_1}{\partial y_1} \neq 0$; therefore $\dfrac{\partial(F_2, F_3, \ldots, F_m)}{\partial(y_2, y_3, \ldots, y_m)} \neq 0.$

The theorem being assumed true for $m - 1$ variables y, we can determine $y_2, y_3, \ldots, y_m$ from $F_2 = F_3 = \ldots = F_m = 0$ as functions of $x_1, x_2, \ldots, x_n$ (f_1 being zero): and the substitution in the expression for y_1 as a function of $(f_1, y_2, \ldots, y_m, x_1, \ldots, x_n)$ determines y_1 as a function of $(x_1, x_2, \ldots, x_n)$. Since the theorem is true for $m = 1$, it is generally true.

Notes. (i) The condition $J \neq 0$ is not a necessary condition.

Example.
$$f_1(u, v, x, y) = xu^3 - v^5 - x^2 - 4x = 0,$$
$$f_2(u, v, x, y) = yu^3 - v^5 - y^2 - 4y = 0.$$

When $x \neq y$, $u^3 = x + y + 4$, $v^5 = xy$, and near $x = -1$, $y = -3$ (for example) these equations determine the functions $u = (x + y + 4)^{1/3}$, $v = (xy)^{1/5}$ which are continuous and tend to the values 0, $3^{1/5}$ respectively when $x \to -1$, $y \to -3$. But $J = -15(x - y)u^2v^4$ which is zero when $x = -1$, $y = -3$.

6.11. The Derivatives of Implicit Functions. If

$$f_r(y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_n) = 0 \ (r = 1 \text{ to } m),$$

the derivatives, when they exist, are obtained by solving the equations

$$\frac{\partial f_r}{\partial y_1}\frac{\partial y_1}{\partial x_s} + \frac{\partial f_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + \ldots + \frac{\partial f_r}{\partial y_m}\frac{\partial y_m}{\partial x_s} + \frac{\partial f_r}{\partial x_s} = 0, \quad \begin{matrix} r = 1 \text{ to } m \\ s = 1 \text{ to } n \end{matrix}$$

Example. If u, v, w are given by the equations

$$f(u, v, w, x, y) = 0, \quad \phi(u, v, w, x, y) = 0; \quad \psi(u, v, w, x, y) = 0$$

find expressions for their first derivatives.

Taking the differentials of the functions, we have

$$f_u du + f_v dv + f_w dw + f_x dx + f_y dy = 0, \quad \phi_u du + \phi_v dv + \phi_w dw + \phi_x dx + \phi_y dy = 0;$$
$$\psi_u du + \psi_v dv + \psi_w dw + \psi_x dx + \psi_y dy = 0.$$

If then $\dfrac{\partial(f, \phi, \psi)}{\partial(u, v, w)} \neq 0$, we deduce that

$$\frac{\partial(f, \phi, \psi)}{\partial(u, v, w)} du + \frac{\partial(f, \phi, \psi)}{\partial(x, v, w)} dx + \frac{\partial(f, \phi, \psi)}{\partial(y, v, w)} dy = 0$$

with similar results for dv, dw.

Thus
$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(f, \phi, \psi)}{\partial(x, v, w)}}{\dfrac{\partial(f, \phi, \psi)}{\partial(u, v, w)}}, \quad \frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(f, \phi, \psi)}{\partial(y, v, w)}}{\dfrac{\partial(f, \phi, \psi)}{\partial(u, v, w)}}$$

with similar results for v_x, v_y, w_x, w_y.

6.2. The Vanishing Jacobian. In this paragraph we shall consider the case when $J = 0$ *identically*. Since for the moment we are not concerned with the occurrence of the variables $x_1, x_2, \ldots x_n$ we shall omit them from the functional expressions.

For simplicity of exposition let us take the case of four functions $f(x, y, z, u)$, $\phi(x, y, z, u)$, $\psi(x, y, z, u)$, $\chi(x, y, z, u)$ where

$$J = \frac{\partial(f, \phi, \psi, \chi)}{\partial(x, y, z, u)} = 0 \text{ identically.}$$

(i) Suppose that the first minors of J do not all vanish identically; without loss of generality we may then assume that $\dfrac{\partial(\phi, \psi, \chi)}{\partial(y, z, u)} \neq 0$.

Consider the relations

$$\phi = \phi(x, y, z, u), \; \psi = \psi(x, y, z, u), \; \chi = \chi(x, y, z, u)$$

where ϕ, ψ, χ are functional symbols on the right and dependent variables on the left. wow! Since $\dfrac{\partial(\phi, \psi, \chi)}{\partial(y, z, u)} \neq 0$, we can, by §*6.10*, express y, z, u as functions of x, ϕ, ψ, χ. When these are substituted in $f(x, y, z, u)$ the latter becomes a function of x, ϕ, ψ, χ, which for clearness we may denote by $F(x, \phi, \psi, \chi)$. We have thus changed the independent variables from x, y, z, u to x, ϕ, ψ, χ obtaining the functions $F(x, \phi, \psi, \chi)$, ϕ, ψ, χ. Then

$$0 = \frac{\partial(f, \phi, \psi, \chi)}{\partial(x, y, z, u)} = \frac{\partial(F, \phi, \psi, \chi)}{\partial(x, \phi, \psi, \chi)} \frac{\partial(x, \phi, \psi, \chi)}{\partial(x, y, z, u)} = F_x \frac{\partial(\phi, \psi, \chi)}{\partial(y, z, u)}.$$

Therefore $F_x = 0$ since $\dfrac{\partial(\phi, \psi, \chi)}{\partial(y, z, u)} \neq 0$, i.e. F does not contain x or $f = F(\phi, \psi, \chi)$.

Thus if $J = 0$ but not all its first minors, there is a functional relation connecting $f(x, y, z, u)$, $\phi(x, y, z, u)$, $\psi(x, y, z, u)$, $\chi(x, y, z, u)$.

(ii) Suppose that J and *all* its first minors vanish but *not* all its second minors. Without loss of generality we may assume then that

$$\frac{\partial(\psi, \chi)}{\partial(z, u)} \neq 0.$$

From the relations $\psi = \psi(x, y, z, u)$, $\chi = \chi(x, y, z, u)$ we can then determine z, u as functions of x, y, ψ, χ and when these functions are substituted in $f(x, y, z, u)$, $\phi(x, y, z, u)$ the latter become functions of x, y, ψ, χ which may be denoted by $F(x, y, \psi, \chi)$, $G(x, y, \psi, \chi)$. The independent variables have thus been changed from x, y, z, u to x, y, ψ, χ.

$$\text{Then} \quad 0 = \frac{\partial(f, \psi, \chi)}{\partial(x, z, u)} = \frac{\partial(F, \psi, \chi)}{\partial(x, \psi, \chi)} \cdot \frac{\partial(x, \psi, \chi)}{\partial(x, z, u)} = F_x \cdot \frac{\partial(\psi, \chi)}{\partial(z, u)},$$

i.e. $F_x = 0$ since $\dfrac{\partial(\psi, \chi)}{\partial(z, u)} \neq 0$. Similarly by considering other vanishing first minors, we may prove that $F_y = 0$, $G_x = 0$, $G_y = 0$. Thus $f = F(\psi, \chi)$, $\phi = G(\psi, \chi)$ or *two* functional relations exist among the four functions.

(iii) If finally J and all its first and second minors vanish but not all its third minors (in this case the sixteen first derivatives), we may assume that $\chi_u \neq 0$. Then from the relation $\chi = \chi(x, y, z, u)$ we can determine u as a function of x, y, z, χ and when this is substituted

in $f(x, y, z, u)$, $\phi(x, y, z, u)$, $\chi(x, y, z, u)$ the latter become functions $F(x, y, z, \chi)$, $G(x, y, z, \chi)$, $H(x, y, z, \chi)$. The dependent variables have thus been changed to x, y, z, χ.

$$\text{But } 0 = \frac{\partial(f, \chi)}{\partial(x, u)} = \frac{\partial(F, \chi)}{\partial(x, \chi)} \cdot \frac{\partial(x, \chi)}{\partial(x, u)} = F_x \chi_u \text{ so that } F_x = 0.$$

Similarly $F_y = 0$, $F_z = 0$, $G_x = 0$, $G_y = 0$, $G_z = 0$, $H_x = 0$, $H_y = 0$, $H_z = 0$,

or
$$f = F(\chi), \ \phi = G(\chi), \ \psi = H(\chi)$$

i.e. *three* functional relations exist among the four functions.

Notes (i) By generalizing the above proof, we deduce that if the Jacobian of m functions of m variables vanishes identically and also all its minors up to and including those of order s, there are $(s + 1)$ relations connecting the functions.

(ii) When $J = 0$ identically, the functional relation that exists need not contain all the functions f_r, even when all the first minors do not vanish. For example, if all the first minors of the elements $\frac{\partial f_1}{\partial y_1}, \frac{\partial f_1}{\partial y_2}, \ldots \frac{\partial f_1}{\partial y_m}$ vanish, $J = 0$ and there is a relation connecting $f_2, f_3, \ldots, f_m$.

(iii) If $f_r(y_1, y_2, \ldots, y_n)$ $(r = 1$ to $m)$ are m functions of n variables, then for a functional relation to exist among them it is necessary that all the Jacobians of the mth order obtained by taking the m functions f_r and the n variables y_s, m at a time should vanish.

(iv) The condition $J = 0$ has been proved *sufficient* for the existence of a functional relationship. It is also a *necessary* condition. For if $F(f_1, f_2, \ldots, f_m) = 0$, we can form the m equations

$$\frac{\partial F}{\partial f_1}\frac{\partial f_1}{\partial x_r} + \frac{\partial F}{\partial f_2}\frac{\partial f_2}{\partial x_r} + \ldots + \frac{\partial F}{\partial f_m}\frac{\partial f_m}{\partial x_r} \quad (r = 1 \text{ to } m)$$

in which the derivatives $\frac{\partial F}{\partial f_s}$ are not all zero. Therefore the determinant of the coefficients of $\frac{\partial F}{\partial f_s}$ must vanish, i.e. $J = 0$.

(v) In the general implicit function theorem for $f_r(y_1, \ldots, y_m, x_1, \ldots, x_n) = 0$ $(r = 1$ to $m)$, when $\frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(y_1, y_2, \ldots, y_m)}$ vanishes identically, the equations $f_r = 0$ are inconsistent (except possibly for particular values of $x_1, x_2, \ldots x_n$) or are redundant (see Note (iii) above). In any case they cannot determine *all* the functions y_r in terms of $x_1, x_2, \ldots, x_n$.

Example. Let $f = x^3 + y^3 + z^3 + u^3$, $\phi = x^2 + y^2 + z^2 + u^2$,
$\psi = x + y + z + u$, $\chi = yzu + zux + uxy + xyz$.

$$\text{Then } \frac{\partial(f, \phi, \psi, \chi)}{\partial(x, y, z, u)} = \begin{vmatrix} 3x^2 & 3y^2 & 3z^2 & 3u^2 \\ 2x & 2y & 2z & 2u \\ 1 & 1 & 1 & 1 \\ zu + uy + yz & ux + xz + zu & xy + yu + ux & yz + zx + xy \end{vmatrix}$$

Since this vanishes for $x = y$, $x = z$, $x = u$, $y = z$, $y = u$, $z = u$ and is only of the *fifth* degree, it must vanish identically.

Then
$$x^3+y^3+z^3+u^3 = F(x^2+y^2+z^2+u^2,\ x+y+z+u,\ yzu+zux+uxy+xyz).$$

Let $u = 0$, then $x^3 + y^3 + z^3 = F(x^2 + y^2 + z^2,\ x + y + z,\ xyz)$.

But $\quad x^3 + y^3 + z^3 - 3xyz = (x + y + z)\{\frac{3}{2}(x^2 + y^2 + z^2) - \frac{1}{2}(x + y + z)^2\}$

i.e.
$$f = 3\chi + \psi(\tfrac{3}{2}\phi - \tfrac{1}{2}\psi^2)$$

6.3. Identical Relations. Suppose that $(m + n)$ variables $x_1, x_2, \ldots, x_{m+n}$ are connected by m relations

$$\phi_r(x_1, x_2, \ldots, x_{m+n}) = 0 \ (r = 1 \text{ to } m).$$

These equations, in general, will determine m of the variables as functions of the remaining n; and if none of the Jacobians determined by the bordered determinant

$$\begin{Vmatrix} \dfrac{\partial \phi_1}{\partial x_1} & \dfrac{\partial \phi_1}{\partial x_2} & \cdots & \dfrac{\partial \phi_1}{\partial x_n} \\ \dfrac{\partial \phi_2}{\partial x_1} & \dfrac{\partial \phi_2}{\partial x_2} & \cdots & \dfrac{\partial \phi_2}{\partial x_n} \\ \cdot & \cdot\ \cdot & \cdot & \cdot \\ \dfrac{\partial \phi_m}{\partial x_1} & \dfrac{\partial \phi_m}{\partial x_2} & \cdots & \dfrac{\partial \phi_m}{\partial x_n} \end{Vmatrix}$$

vanishes, the choice of the m dependent variables may be made in $\dfrac{(m+n)!}{m!n!}$ ways; and any one choice may be regarded as a transformation of any other choice. In each selection there are mn first derivatives and therefore there are $\dfrac{(m+n)!}{(m-1)!(n-1)!}$ first derivatives in all. The derivatives for any one selection may in general be expressed in terms of the derivatives of any other selection. There are therefore

$$\frac{(m+n)! - m!n!}{(m-1)!(n-1)!}$$

relations connecting the first derivatives. They are called *Identical Relations*, since they are independent of the given functions ϕ_r.

Example. Let there be 5 relations connecting 8 variables. Then there are 56 ways of choosing the dependent variables. The total number of first derivatives is 840 and the number of identical relations connecting these first derivatives is 825.

6.31. *Method of determining Identical Relations.* Suppose that there are p relations connecting n variables $x_1, x_2, \ldots, x_n$ and let $x_r, x_s, x_t, \ldots$ be a particular choice of p dependent variables $(n > p)$. Form the differentials

$$dx_r = A_1\,dx_1 + A_2\,dx_2 + \ldots; \quad dx_s = B_1\,dx_1 + B_2\,dx_2 + \ldots;$$
$$dx_t = C_1\,dx_1 + \ldots; \quad \ldots$$

where dx_r, dx_s, dx_t are omitted from the right-hand sides.

Then $$A_1 = \frac{\partial x_r}{\partial x_1},\ A_2 = \frac{\partial x_r}{\partial x_2}, \ldots,\ B_1 = \frac{\partial x_s}{\partial x_1}, \ldots$$

where $x_r, x_s, x_t \ldots$ are expressed in terms of the others.

To obtain the corresponding equations for any other selection solve the above system for the appropriate set of differentials. Thus if a new set were $x_p, x_\sigma, x_\tau, \ldots$ we should obtain, on solving

$$dx_\rho = \alpha_1\,dx_1 + \alpha_2\,dx_2 + \ldots; \quad dx_\sigma = \beta_1\,dx_1 + \beta_2\,dx_2 + \ldots,$$
$$dx_\tau = \gamma_1\,dx_1 + \ldots$$

where now dx_ρ, dx_σ, dx_τ, . . . are omitted on the right and where α_1, α_2, . . ., β_1, β_2, . . . are functions of A_1, A_2, . . ., B_1, B_2, . . . Also

$$\alpha_1 = \frac{\partial x_\rho}{\partial x_1},\ \alpha_2 = \frac{\partial x_\rho}{\partial x_2},\ \ldots\ \beta_1 = \frac{\partial x_\sigma}{\partial x_1},\ \ldots \text{ where } x_\rho,\ x_\sigma,\ x_\tau,\ \ldots$$

are expressed in terms of the others

If then, for example, $\alpha_1 = F(A_1, A_2, \ldots B_1, B_2, \ldots)$, an identical relation would be

$$\frac{\partial x_\rho}{\partial x_1} = F\left(\frac{\partial x_\tau}{\partial x_1}, \frac{\partial x_\tau}{\partial x_2}, \ldots, \frac{\partial x_\sigma}{\partial x_1}, \frac{\partial x_\sigma}{\partial x_2}, \ldots\right).$$

It should be noted, however, that the symbol $\dfrac{\partial x_m}{\partial x_n}$ is, in general, ambiguous and that it may be necessary to indicate the particular selection to which it belongs

The above method will determine the derivatives for one selection in terms of any other selection; but we can use the properties of Jacobians to determine symmetrical identities, although it does not appear obvious how to obtain the correct number of independent symmetrical identities.

For example, suppose that there are 6 variables x_1, x_2, x_3, x_4, x_5, x_6 connected by three relations.

There are ${}^6C_3 = 20$ selections, 180 derivatives and 171 identities. The following examples illustrate how these identities may be obtained.

(i) From $\dfrac{\partial(x_1, x_2, x_3)}{\partial(x_1, x_2, x_4)}\dfrac{\partial(x_1, x_2, x_4)}{\partial(x_1, x_2, x_3)} = 1$ where the denominator in a Jacobian indicates the *independent* variables, we find

$$\left(\frac{\partial x_3}{\partial x_4}\right)_{x_1 x_2}\left(\frac{\partial x_4}{\partial x_3}\right)_{x_1 x_2} = 1$$

where the suffixes indicate the variables kept constant.

(ii) From $\dfrac{\partial(x_1, x_2, x_3)}{\partial(x_1, x_4, x_5)}\cdot\dfrac{\partial(x_1, x_4, x_5)}{\partial(x_1, x_2, x_3)} = 1$ we find

$$\left\{\frac{\partial(x_2, x_3)}{\partial(x_4, x_5)}\right\}_{x_1}\left\{\frac{\partial(x_4, x_5)}{\partial(x_2, x_3)}\right\}_{x_1} = 1.$$

(iii) From $\dfrac{\partial(x_1, x_2, x_3)}{\partial(x_4, x_2, x_3)}\cdot\dfrac{\partial(x_3, x_5, x_4)}{\partial(x_1, x_5, x_4)}\cdot\dfrac{\partial(x_2, x_4, x_1)}{\partial(x_3, x_4, x_1)}\cdot\dfrac{\partial(x_6, x_1, x_5)}{\partial(x_2, x_1, x_5)} = 1$

we obtain

$$\left(\frac{\partial x_1}{\partial x_4}\right)_{x_2 x_3}\left(\frac{\partial x_3}{\partial x_1}\right)_{x_4 x_5}\left(\frac{\partial x_2}{\partial x_3}\right)_{x_4 x_1}\left(\frac{\partial x_6}{\partial x_2}\right)_{x_1 x_5} = 1.$$

(iv) From $\dfrac{\partial(x_1, x_2, x_3)}{\partial(x_4, x_2, x_3)}\cdot\dfrac{\partial(x_2, x_6, x_5)}{\partial(x_4, x_1, x_5)}\cdot\dfrac{\partial(x_5, x_6, x_2)}{\partial(x_1, x_3, x_2)} = 1$, we obtain

$$\left\{\frac{\partial(x_1, x_3)}{\partial(x_4, x_3)}\right\}_{x_2}\left\{\frac{\partial(x_2, x_6)}{\partial(x_4, x_1)}\right\}_{x_5}\left\{\frac{\partial(x_5, x_6)}{\partial(x_1, x_3)}\right\}_{x_2} = 1.$$

6.32. *The Inverse Relations.* Suppose that there are m functions $u_1, u_2, \ldots, u_m$ of the m variables $x_1, x_2, \ldots, x_m$. There are $(2m!)/(m!)^2$ ways in which m of the variables (u_r, x_s) can be expressed in terms of the remaining m. One of special importance consists in expressing $x_1, x_2, \ldots, x_m$ as functions of $u_1, u_2, \ldots, u_m$; and the functions obtained thereby may be called the inverse of the given functions.

Denote $\dfrac{\partial(u_1, u_2, \ldots, u_m)}{\partial(x_1, x_2, \ldots, x_m)}$ by $J(\neq 0)$; then

$$\frac{\partial(u_1, u_2, \ldots, u_m)}{\partial(x_1, x_2, \ldots, x_m)} \cdot \frac{\partial(x_1, x_2, \ldots, x_m)}{\partial(u_1, u_2, \ldots, u_m)} = \frac{\partial(u_1, u_2, \ldots, u_m)}{\partial(u_1, u_2, \ldots, u_m)} = 1.$$

Note. This is the analogue of the result $\dfrac{dy}{dx}\dfrac{dx}{dy} = 1$ for a function of one variable and its inverse.

Again $\dfrac{\partial(u_1, u_2, \ldots, u_{r-1}, x_s, u_{r+1}, \ldots, u_m)}{\partial(u_1, u_2, \ldots \ldots \ldots \ldots \ldots, u_m)} \cdot \dfrac{\partial(u_1, u_2, \ldots, u_m)}{\partial(x_1, x_2, \ldots, x_m)}$

$$= \frac{\partial(u_1, \ldots, u_{r-1}, x_s, u_{r+1}, \ldots, u_m)}{\partial(x_1, x_2, \ldots \ldots \ldots \ldots \ldots, x_m)}$$

i.e. $J.\dfrac{\partial x_s}{\partial u_r} = A_{rs}$ where A_{rs} is the co-factor of $\dfrac{\partial u_r}{\partial x_s}$ in J.

Examples (i) Let u, v, w be functions of x, y, z, so that x, y, z may be expressed in terms of u, v, w.

If $$J = \frac{\partial(u, v, w)}{\partial(x, y, z)},\ J_1 = \frac{\partial(x, y, z)}{\partial(u, v, w)},\ \text{then } JJ_1 = 1$$

The nine relations connecting the 18 first derivatives are given by $Jx_u = \dfrac{\partial(v, w)}{\partial(y, z)}$ and 8 similar relations giving y_u, z_u, x_v, etc. These are, of course, equivalent to $J_1u_x = \dfrac{\partial(y, z)}{\partial(v, w)}$ and 8 similar relations. It should be noted also that the 9 relations are given by equations of the type

$$x_uu_x + x_vv_x + x_ww_x = 1;\ x_uu_y + x_vv_y + x_ww_y = 0, \text{ etc.}$$

(ii) If $u_1, u_2, \ldots, u_m$ are given by the set of equations

$$\phi_r(u_1, u_2, \ldots, u_m, x_1, x_2, \ldots, x_m) = 0\ (r = 1 \text{ to } m)$$

find an expression for $\dfrac{\partial(u_1, u_2, \ldots, u_m)}{\partial(x_1, x_2, \ldots, x_m)}$.

The equation giving the m^2 derivatives $\dfrac{\partial u_r}{\partial x_s}$ are

$$\left\{\sum_{t=1}^{t=m} \frac{\partial \phi_r}{\partial u_t}\frac{\partial u_t}{\partial x_s}\right\} + \frac{\partial \phi_r}{\partial x_s} = 0\ (r = 1 \text{ to } m,\ s = 1 \text{ to } m)$$

and therefore

$$\frac{\partial(\phi_1, \phi_2, \ldots, \phi_m)}{\partial(u_1, u_2, \ldots, u_m)} \cdot \frac{\partial(u_1, u_2, \ldots, u_m)}{\partial(x, x_2, \ldots, x_m)} = (-1)^m \frac{\partial(\phi_1, \phi_2, \ldots, \phi_m)}{\partial(x_1, x_2, \ldots, x_m)}.$$

Note. This is the analogue of the result $f_y\dfrac{dy}{dx} + f_x = 0$ for a function y given by $f(x, y) = 0$.

(iii) If $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1u_2 = x_2 + x_3 + x_4$, $u_1u_2u_3 = x_3 + x_4$, $u_1u_2u_3u_4 = x_4$, find $J \equiv \dfrac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)}$.

Here
$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ u_2 & u_1 & 0 & 0 \\ u_2u_3 & u_1u_3 & u_1u_2 & 0 \\ u_2u_3u_4 & u_1u_3u_4 & u_1u_2u_4 & u_1u_2u_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} J$$
i.e. $J = u_1^3u_2^2u_3$.

(iv) If
$$u^2x^3 + v^2y^3 + 2a^2(u^3 + v^3) = c_1^3$$
and
$$u^3x^2 + v^3y^2 + 2a^2(x^3 + y^3) = c_2^3$$
find $J = \dfrac{\partial(u, v)}{\partial(x, y)}$.

Here
$$\begin{vmatrix} 2ux^3 + 6a^2u^2 & 2vy^3 + 6a^2v^2 \\ 3u^2x^2 & 3v^2y^2 \end{vmatrix} J = -\begin{vmatrix} 3u^2x^2 & 3v^2y^2 \\ 2u^3x + 6a^2x^2 & 2v^3y + 6a^2y^2 \end{vmatrix}$$
i.e.
$$J = \frac{u^2v^2xy(ux - vy) + 3a^2x^2y^2(u^2 - v^2)}{uvx^2y^2(vx - uy) + 3a^2u^2v^2(y^2 - x^2)}.$$

6.33. A Functional Relation connecting n Variables $x_1, x_2, \ldots, x_n$. The dependent variable may be chosen in n ways, the number of derivatives is $n^2 - n$ and the number of relations is $(n - 1)^2$. The symbol $\dfrac{\partial x_r}{\partial x_s}$ is not ambiguous. The functional relation leads to
$$A_1\,dx_1 + A_2\,dx_2 + \ldots + A_n\,dx_n = 0$$
so that $\dfrac{\partial x_r}{\partial x_s} = -\dfrac{A_s}{A_r}$.

Suppose for example that $n = 4$, then
$$dx_1 = A\,dx_2 + B\,dx_3 + C\,dx_4 \text{ gives } dx_2 = \frac{1}{A}\,dx_1 - \frac{B}{A}\,dx_3 - \frac{C}{A}\,dx_4$$
i.e.
$$\frac{\partial x_2}{\partial x_1}\frac{\partial x_1}{\partial x_2} = 1, \quad \frac{\partial x_2}{\partial x_3}\frac{\partial x_1}{\partial x_2} + \frac{\partial x_1}{\partial x_3} = 0, \quad \frac{\partial x_2}{\partial x_4}\frac{\partial x_1}{\partial x_2} + \frac{\partial x_1}{\partial x_4} = 0,$$
equations that connect the 3 derivatives of one selection with the 3 derivatives of another. The 9 identities that may be obtained in this way may be written symmetrically:
$$\binom{1}{2}\binom{2}{1} = 1; \quad \binom{1}{3}\binom{3}{1} = 1, \quad \binom{1}{4}\binom{4}{1} = 1; \quad \binom{2}{3}\binom{3}{2} = 1,$$
$$\binom{2}{4}\binom{4}{2} = 1, \quad \binom{3}{4}\binom{4}{3} = 1,$$
$$\binom{1}{2}\binom{2}{3}\binom{3}{1} = -1, \quad \binom{2}{3}\binom{3}{4}\binom{4}{2} = -1$$
$$\binom{1}{2}\binom{2}{3}\binom{3}{4}\binom{4}{1} = 1,$$
where $\left(\dfrac{r}{s}\right)$ denotes $\dfrac{\partial x_r}{\partial x_s}$.

These symmetrical results may also be obtained by using the appropriate Jacobians.

Thus if we denote $\dfrac{\partial(x_m, x_n, x_p)}{\partial(x_r, x_s, x_t)}$ by $\begin{pmatrix} m & n & p \\ r & s & t \end{pmatrix}$ we obviously have

$$\begin{pmatrix} m & n & p \\ m & n & q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \begin{pmatrix} m & n & p \\ n & q & p \end{pmatrix} = -\begin{pmatrix} m \\ q \end{pmatrix}, \quad \begin{pmatrix} m & n & p \\ n & p & q \end{pmatrix} = \begin{pmatrix} m \\ q \end{pmatrix},$$

and other similar results, so that

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix} = 1$ leads to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 1$ &c.

(ii) $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 & 4 \\ 1 & 2 & 4 \end{pmatrix} = 1$ gives $\begin{pmatrix} 1 \\ 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = -1$ &c.

and (iii) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 & 1 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = -1$ gives $\begin{pmatrix} 1 \\ 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 1.$

6.34. $(n-1)$ Functional Relations connecting n Variables $x_1, x_2, \ldots, x_n$. The independent variables may be chosen in n ways; there are $n^2 - n$ first derivatives and $(n-1)^2$ identities; and the symbol $\dfrac{\partial x_r}{\partial x_s} = \left(\dfrac{r}{s}\right)$ may be used without ambiguity.

If one selection is indicated by

$$x_2 = x_2(x_1); \quad x_3 = x_3(x_1); \quad \ldots; \quad x_n = x_n(x_1), \text{ we have}$$

$$dx_2 = A_{21}\,dx_1; \quad dx_3 = A_{31}\,dx_1; \quad \ldots; \quad dx_n = A_{n1}\,dx_1$$ so that for another

$$dx_1 = \frac{1}{A_{21}}\,dx_2; \quad dx_3 = \frac{A_{31}}{A_{21}}\,dx_2, \quad \ldots; \quad dx_n = \frac{A_{n1}}{A_{21}}\,dx_2$$

or $$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1; \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \ldots; \quad \begin{pmatrix} n \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ 1 \end{pmatrix}.$$

Thus if there were 4 variables, the 9 results could be written symmetrically

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1; \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 1; \quad \begin{pmatrix} 1 \\ 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1; \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1;$$

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 1; \quad \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 1;$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 1; \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 1.$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1.$$

These of course follow immediately from the formulæ for functions of *one* variable such as $\dfrac{dx_r}{dx_s}\cdot\dfrac{dx_s}{dx_r} = 1$; $\dfrac{dx_r}{dx_s}\cdot\dfrac{dx_s}{dx_t}\cdot\dfrac{dx_t}{dx_r} = 1$, &c.

6.35. Four Variables x_1, x_2, x_3, x_4 connected by Two Relations. The independent variables may be chosen in 6 ways giving 24 derivatives and 20 identities. The symbol $\begin{pmatrix} r \\ s \end{pmatrix}$ is now ambiguous since for example

$\binom{1}{2}$ may mean either $\dfrac{\partial x_1}{\partial x_2}(x_2, x_3)$ or $\dfrac{\partial x_1}{\partial x_2}(x_2, x_4)$. We shall therefore denote $\dfrac{\partial x_r}{\partial x_s}(x_s, x_t)$ by the symbol $\left(\dfrac{\partial x_r}{\partial x_s}\right)_{x_t}$ or $\binom{r}{s}_t = \binom{r\,t}{s\,t}$.

If, for example, x_2, x_4 are the independent variables, we may write

$$dx_1 = A\,dx_2 + B\,dx_4; \quad dx_3 = C\,dx_2 + D\,dx_4$$

and the derivatives for any of the other 5 selections may be expressed in terms of A, B, C, D.

Thus $dx_1 = \dfrac{A}{C}dx_3 - \dfrac{AD - BC}{C}dx_4$, $dx_2 = \dfrac{1}{C}dx_3 - \dfrac{D}{C}dx_4$ so that 4 of the 20 identities are

$$\binom{1}{2}_4\binom{2}{3}_4 = \binom{1}{3}_4; \quad \binom{1}{4}_2\binom{2}{3}_4 = -\binom{1\,2}{3\,4}; \quad \binom{3}{2}_4\binom{2}{3}_4 = 1;$$
$$\binom{3}{4}_2\binom{2}{3}_4 = -\binom{2}{4}_3.$$

From these and the remaining 16 relations obtained similarly it is easy to establish the following symmetrical results.

(a) $\binom{1}{2}_3\binom{2}{1}_3 = 1$ and 11 similar results.

(b) $\binom{1}{2}_3\binom{2}{3}_1\binom{3}{1}_2 = -1$ and 3 similar results.

(c) $\binom{1}{2}_4\binom{2}{3}_4\binom{3}{1}_4 = 1$ and 3 similar results.

(d) $\binom{1\,2}{3\,4}\binom{3\,4}{1\,2} = 1$ and 2 similar results.

These may also be proved by the use of appropriate Jacobians.

Thus (a) $\quad \binom{1\,3}{2\,3}\binom{2\,3}{1\,3} = 1$ gives $\binom{1}{2}_3\binom{2}{1}_3 = 1$

(b) $\binom{1\,2}{2\,3}\binom{2\,3}{3\,1}\binom{3\,1}{1\,2} = -1$ gives $\binom{1}{3}_2\binom{2}{1}_3\binom{3}{2}_1 = -1$

(c) $\binom{1\,4}{2\,4}\binom{2\,4}{3\,4}\binom{3\,4}{1\,4} = 1$ gives $\binom{1}{2}_4\binom{2}{3}_4\binom{3}{1}_4 = 1$.

These 23 symmetrical results are not independent. Thus one in (c) can be deduced from (a), (b) and the remainder in (c). If (a), (b), (c) are satisfied, two of the results in (d) are not independent of the third.

6.36 *Application to a Function of Two Variables.* Let z be a function of (x, y) possessing first and second derivatives $p\,(= z_x)$, $q\,(= z_y)$, $r\,(= z_{xx})$, $s\,(= z_{xy})$, $t\,(= z_{yy})$; then p, q, x, y are 4 variables connected by *two* relations.

Also $dp = r\,dx + s\,dy$, $dq = s\,dx + t\,dy$, the derivatives p_y, q_x in this example being equal. When any other selection of two independent

variables is made, the new derivatives can be expressed in terms of r, s, t.

Thus since $dx = -\frac{t}{s}dy + \frac{1}{s}dq$; $dp = -\frac{(rt - s^2)}{s}dy + \frac{r}{s}dq$ we have $\left(\frac{\partial x}{\partial y}\right)_q = -\frac{t}{s}$, $\left(\frac{\partial p}{\partial q}\right)_y = \frac{r}{s}$; &c.

The most important relations are, however, those that correspond to $\left(\frac{\partial p}{\partial y}\right)_x = s = \left(\frac{\partial q}{\partial x}\right)_y$.

Thus since $\left(\frac{\partial x}{\partial y}\right)_q = -\frac{t}{s}$; $\left(\frac{\partial x}{\partial q}\right)_y = \frac{1}{s}$; $\left(\frac{\partial p}{\partial y}\right)_q = -\frac{rt - s^2}{s}$;

$$\left(\frac{\partial p}{\partial q}\right)_y = \frac{r}{s},$$

it follows that $\frac{\partial(px)}{\partial(yq)} = 1$. Similarly the other relations can be found and therefore the required results are

$$\text{(i)} \left(\frac{\partial p}{\partial y}\right)_x = \left(\frac{\partial q}{\partial x}\right)_y; \quad \text{(ii)} \left(\frac{\partial x}{\partial q}\right)_p = \left(\frac{\partial y}{\partial p}\right)_q; \quad \text{(iii)} \left(\frac{\partial p}{\partial q}\right)_x = -\left(\frac{\partial y}{\partial x}\right)_q;$$

$$\text{(iv)} \left(\frac{\partial q}{\partial p}\right)_y = -\left(\frac{\partial x}{\partial y}\right)_p; \quad \text{(v)} \frac{\partial(px)}{\partial(yq)} = 1; \quad \text{(vi)} \frac{\partial(qy)}{\partial(xp)} = 1.$$

The quickest method, however, of establishing (i)–(iv) is to note that since (i) $dz = p\,dx + q\,dy$, then

$$\text{(ii)}\ d(px + qy - z) = x\,dp + y\,dq; \quad \text{(iii)}\ d(qy - z) = -p\,dx + y\,dq;$$
$$\text{(iv)}\ d(px - z) = -q\,dy + x\,dp$$

and if we write $z_2 = px + qy - z$, $z_3 = qy - z$, $z_4 = px - z$, we have

$$\text{(ii)}\ x = \left(\frac{\partial z_2}{\partial p}\right)_q; \quad y = \left(\frac{\partial z_2}{\partial q}\right)_p; \quad \text{(iii)}\ p = -\left(\frac{\partial z_3}{\partial x}\right)_q; \quad y = \left(\frac{\partial z_3}{\partial q}\right)_x;$$

$$\text{(iv)}\ q = -\left(\frac{\partial z_4}{\partial y}\right)_p; \quad x = \left(\frac{\partial z_4}{\partial p}\right)_y$$

from which the required results follow.

6·37. *The Thermodynamic Case.* In Thermodynamics the following differential relation occurs

$$dE = \theta\,d\phi - p\,dv$$

where p is the pressure, v the volume, θ the temperature and ϕ the entropy of a gas. The entropy is defined by the relation $dQ = \theta\,d\phi$ where dQ is the heat supplied at temperature θ. When a volume of gas increases by dv at pressure p, the work done (dW) by the gas is $p\,dv$. Thus dE is the increase in internal energy due to an increase of entropy $d\phi$ and an increase in volume dv.

If therefore E is assumed to be a differentiable function of two of the variables (say v, ϕ), dE the differential is given by

$$dE = \frac{\partial E}{\partial \phi}\,d\phi + \frac{\partial E}{\partial v}\,dv$$

so that we may take

$$\theta = \left(\frac{\partial E}{\partial \phi}\right)_v, \quad p = -\left(\frac{\partial E}{\partial v}\right)_\phi$$

and p, v, θ, ϕ are 4 variables connected by 2 relations, the relationship being of the type considered in the previous paragraph. There are therefore 6 relations of the type (i)–(vi). Thus

$$\text{(i)}\quad \left(\frac{\partial \theta}{\partial v}\right)_\phi = -\left(\frac{\partial p}{\partial \phi}\right)_v \left(= \frac{\partial^2 E}{\partial \phi\,\partial v}\right).$$

Taking $\psi = E - \phi\theta$ we have $d\psi = -\phi\,d\theta - p\,dv$ and therefore

$$\text{(ii)}\quad \left(\frac{\partial p}{\partial \theta}\right)_v = \left(\frac{\partial \phi}{\partial v}\right)_\theta \left(= -\frac{\partial^2 \psi}{\partial v\,\partial \theta}\right)$$

Similarly if $\xi = pv + E$, $I = pv - \theta\phi + E$ we obtain

$$\text{(iii)}\quad \left(\frac{\partial v}{\partial \theta}\right)_p = -\left(\frac{\partial \phi}{\partial p}\right)_\theta \left(= \frac{\partial^2 I}{\partial p\,\partial \theta}\right).$$

$$\text{(iv)}\quad \left(\frac{\partial v}{\partial \phi}\right)_p = \left(\frac{\partial \theta}{\partial p}\right)_\phi \left(= \frac{\partial^2 \xi}{\partial p\,\partial \phi}\right).$$

These relations are sometimes called the *Four Thermodynamic Relations*, and the functions E, ψ, ξ, I *Thermodynamic Potentials*.

The other two relations are easily proved to be

$$\frac{\partial(p, v)}{\partial(\theta, \phi)} = \frac{\partial(\theta, \phi)}{\partial(p, v)} = 1.$$

Physical interpretations may be given to some of the partial derivatives. For example

$\theta\left(\dfrac{\partial \phi}{\partial \theta}\right)_v$ is the *specific heat at constant volume* (C_v)

$\theta\left(\dfrac{\partial \phi}{\partial \theta}\right)_p$ is the *specific heat at constant pressure* (C_p).

$\dfrac{1}{v}\left(\dfrac{\partial v}{\partial \theta}\right)_p$ is the *coefficient of cubical expansion at constant pressure* (α_p).

$-\dfrac{1}{v}\left(\dfrac{\partial v}{\partial p}\right)_\theta$ is the *compressibility at constant temperature* (κ)

An effective method of establishing thermodynamic results consists in expressing all the derivatives that occur in terms of those belonging to a particular selection of independent variables.

Thus we could take $dI = d(pv - \theta\phi + E) = v\,dp - \phi\,d\theta$ so that $v = I_p$, $\phi = -I_\theta$ (p, θ being the chosen independent variables and therefore $dv = K\,dp + S\,d\theta$, $d\phi = -S\,dp - T\,d\theta$ where $K = I_{pp}$, $S = I_{p\theta}$, $T = I_{\theta\theta}$.

Examples. (i) Find C_v, C_p, α_p, κ in terms of K, S, T and deduce *Rankine's formula* $\kappa(C_p - C_v) = v\alpha_p^2\theta$.

$$K\,dp = dv - S\,d\theta,\; K\,d\phi = -S\,dv - (KT - S^2)d\theta.$$

Therefore $C_v = -\theta\left(\frac{\partial\phi}{\partial\theta}\right)_v = -\theta\left(T - \frac{S^2}{K}\right)$; and from $d\phi = -S\,dp - T\,d\theta$ we have
$$C_p = -\theta\left(\frac{\partial\phi}{\partial\theta}\right)_p = -\theta T;$$
and from $dv = K\,dp + S\,d\theta$ we have $\alpha_p = \frac{1}{v}\left(\frac{\partial v}{\partial\theta}\right)_p = \frac{S}{v}$ and $\kappa = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_\theta = -\frac{K}{v}$

i.e.
$$C_p - C_v = -\theta\frac{S^2}{K} = \frac{v\alpha_p^2\theta}{\kappa}.$$

(ii) Show that $\dfrac{C_p}{C_v} = \dfrac{\left(\frac{\partial p}{\partial v}\right)_\phi}{\left(\frac{\partial p}{\partial v}\right)_\theta}$.

From $(KT - S^2)\,dp = T\,dv + S\,d\phi$ we have $\left(\frac{\partial p}{\partial v}\right)_\phi = \frac{T}{KT - S^2}$.

From $K\,dp = dv - S\,d\theta$ we have $\left(\frac{\partial p}{\partial v}\right)_\theta = \frac{1}{K}$.

Also, see *Example* (i) above, $C_p = -\theta T$, $C_v = -\dfrac{\theta(KT - S^2)}{K}$

i.e.
$$\frac{C_p}{C_v} = \frac{KT}{KT - S^2} = \frac{\left(\frac{\partial p}{\partial v}\right)_\phi}{\left(\frac{\partial p}{\partial v}\right)_\theta}.$$

(iii) If C_v, C_p are constants, show that a characteristic equation of a gas is of the form $(p - a)(v - b) = R\theta$ where a, b are constants and $R = C_p - C_v$.

We have shown that $C_p = -\theta I_{\theta\theta}$, $C_v = -\theta I_{\theta\theta} + \theta\dfrac{(I_{p\theta})^2}{I_{pp}}$.

From the first, $I = -C_p\theta(\log\theta - 1) - \theta\lambda(p) + \mu(p)$, by integration. Substituting in $RI_{pp} + \theta(I_{p\theta})^2 = 0$ we find $R\{\theta\lambda''(p) + \mu''(p)\} + \theta\{\lambda'(p)\}^2 = 0$ i.e. $\mu''(p) = 0$ so that $\mu(p) = bp + b_1$ where b, b_1 are constants and
$$R\lambda''(p) + \{\lambda'(p)\}^2 = 0.$$

From this last equation we obtain $\lambda'(p) = \dfrac{R}{p + cR}$ and therefore
$$\lambda(p) = R\log(p - a) + c' \text{ where } a = -cR$$
i.e.
$$I = -C_p\theta(\log\theta - 1) + \theta\{R\log(p - a) + c'\} + bp + b_1.$$

Now $v = I_p = \dfrac{\theta R}{p - a} + b$ or $(p - a)(v - b) = R\theta$

6.4. Transformations in General. In the transformations we have met with, three types may be recognized, the second inclusive of the first, and the third inclusive of the first and second.

(i) Let $u_r = u_r(x_1, \ldots, x_n)$ where $x_s = x_s(X_1, X_2, \ldots, X_n)$,
$$\begin{pmatrix} r = 1 \text{ to } m \\ s = 1 \text{ to } n \end{pmatrix}.$$

Then
$$\frac{\partial u_r}{\partial X_t} = \sum_{s=1}^{n}\frac{\partial u_r}{\partial x_s}\cdot\frac{\partial x_s}{\partial X_t} \quad \begin{pmatrix} r = 1 \text{ to } m \\ t = 1 \text{ to } n \end{pmatrix}$$

so that the variables $u_1, \ldots, u_m, x_1, \ldots, x_n, p_{rs}$ where $p_{rs} = \dfrac{\partial u_r}{\partial x_s}$ are transformed into $u_1, \ldots, u_m, X_1, \ldots, X_n, P_{rs}$ where $P_{rs} = \dfrac{\partial u_r}{\partial X_s}$ and we have

$$du_r - \sum_{t=1}^{n} p_{rt}\,dx_t = du_r - \sum_{t=1}^{n} P_{rt}\,dX_t, \ (r = 1 \text{ to } m).$$

Such a transformation may be called *Explicit*.

Note. The number of independent variables X_r is of course not restricted to n. Thus if $V = V(x, y, z)$, $W = W(x, y, z)$ and the independent variables are transformed to u, v by the equation $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, we have $V_u = V_x x_u + V_y y_u + V_z z_u$, &c., and

$$dV = V_u\,du + V_v\,dv, \quad dV = V_x\,dx + V_y\,dy + V_z\,dz$$

with a similar result for dW. When the number of variables X_r is equal to the number of variables x_r, the transformation is called a *point-transformation*. In the example just considered, the transformation is restricted.

(ii) Let $u_r = u_r(x_1, \ldots, x_n)$, $r = 1$ to m, and let $(m + n)$ new variables be taken $U_1, U_2, \ldots, U_m, X_1, X_2, \ldots, X_n$ where $U_r = U_r(u_1, \ldots u_m, x_1, \ldots x_n)$ and $X_s = X_s(u_1, \ldots, u_m, x_1, \ldots x_n)$. Denote $\dfrac{\partial u_r}{\partial x_s}$ by p_{rs} and $\dfrac{\partial U_r}{\partial X_s}$ by P_{rs}, then it is obvious from the method of determining the derivatives P_{rs} (U_r being regarded as a function of $X_1, \ldots, X_n$) that the differential expressions

$$dU_r - \sum_{t=1}^{n} P_{rt}\,dX_t \ (r = 1 \text{ to } n)$$

are *linear* combinations of the expressions $du_s - \sum_{m=1}^{n} p_{sm}\,dx_m$ $(s = 1 \text{ to } n)$.

Thus $u_1, \ldots, u_m, x_1, \ldots, x_n, p_{rs}$ are transformed into $U_1, \ldots, U_m, X_1, \ldots, X_n, P_{rs}$ where

$$dU_r - \sum_{t=1}^{n} P_{rt}\,dX_t = \sum_{s=1}^{n} \kappa_{rs}\Big(du_s - \sum_{m=1}^{n} p_{sm}\,dx_m\Big), \ (r = 1 \text{ to } n).$$

Note. As in case (i) the number of new variables need not be equal to n.

Example. Let $z = z(x, y)$ and let the variables be transformed to X, Y, Z where $X = X(x, y, z)$, $Y = Y(x, y, z)$, $Z = Z(x, y, z)$, so that Z is a function of X, Y possessing derivatives $P\Big(= \dfrac{\partial Z}{\partial X}\Big)$, $Q\Big(= \dfrac{\partial Z}{\partial Y}\Big)$.

To obtain P, Q we have

$$\begin{aligned} 0 = dZ - P\,dX - Q\,dY &= Z_x\,dx + Z_y\,dy + Z_z\,dz \\ &\quad - P(X_x\,dx + X_y\,dy + X_z\,dz) - Q(Y_x\,dx + Y_y\,dy + Y_z\,dz). \end{aligned}$$

Thus $\qquad (Z_x + Z_z p) = P(X_x + X_z p) + Q(Y_x + Y_z p)$

and $\qquad (Z_y + Z_z q) = P(X_y + X_z q) + Q(Y_y + Y_z q)$

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$.

It may be verified that

$$P = \frac{\dfrac{\partial(Z,Y)}{\partial(x,y)} + p\dfrac{\partial(Z,Y)}{\partial(z,y)} + q\dfrac{\partial(Z,Y)}{\partial(x,z)}}{\dfrac{\partial(X,Y)}{\partial(x,y)} + p\dfrac{\partial(X,Y)}{\partial(z,y)} + q\dfrac{\partial(X,Y)}{\partial(x,z)}}, \quad Q = \frac{\dfrac{\partial(X,Z)}{\partial(x,y)} + p\dfrac{\partial(X,Z)}{\partial(z,y)} + q\dfrac{\partial(X,Z)}{\partial(x,z)}}{\dfrac{\partial(X,Y)}{\partial(x,y)} + p\dfrac{\partial(X,Y)}{\partial(z,y)} + q\dfrac{\partial(X,Y)}{\partial(x,z)}}$$

and from the equation giving $dZ - P\,dX - Q\,dY$ above it follows that when P, Q have these values

$$dZ - P\,dX - Q\,dY = (Z_z - PX_z - QY_z)(dz - p\,dx - q\,dy)$$

and by interchanging the variables

$$dz - p\,dx - q\,dy = (z_Z - px_Z - qy_Z)(dZ - P\,dX - Q\,dY).$$

It is easily verified from the above values of P, Q that the multiplier

$$Z_z - PX_z - QY_z = \frac{\dfrac{\partial(X,Y,Z)}{\partial(x,y,z)}}{\dfrac{\partial(X,Y)}{\partial(x,y)} + p\dfrac{\partial(X,Y)}{\partial(z,y)} + q\dfrac{\partial(X,Y)}{\partial(x,z)}}$$

Thus if $X = xyz$, $Y = xy + yz + zx$, $Z = x^2 + y^2 + z^2$, the multiplier is

$$\frac{2(x-y)(y-z)(z-x)(x+y+z)}{z^2(x-y) - px^2(y-z) - qy^2(z-x)}$$

Transformations of this type may be called *Implicit*.

(iii) The transformation in (ii) suggests the possibility of transforming

$$u_1(x_1, x_2, \ldots, x_n), \ldots, u_m(x_1, \ldots, x_n),\ p_{rs}\left(= \frac{\partial u_r}{\partial x_s}\right),\ \binom{r = 1 \text{ to } m}{s = 1 \text{ to } n}$$

into

$$U_1(X_1, \ldots, X_n), \ldots, U_m(X_1, \ldots, X_n),\ P_{rs}\ \binom{r = 1 \text{ to } m}{s = 1 \text{ to } n}$$

where

$$U_r = U_r(u_1, \ldots, u_m, x_1, \ldots, x_n, \ldots, p_{st}, \ldots)\ (r = 1 \text{ to } m)$$
$$X_r = X_r(u_1, \ldots, u_m, x_1, \ldots, x_n, \ldots, p_{st}, \ldots)\ (r = 1 \text{ to } n)$$
$$P_{rq} = P_{rq}(u_1, \ldots, u_m, x_1, \ldots, x_n, \ldots, p_{st}, \ldots)\ \binom{r = 1 \text{ to } m}{q = 1 \text{ to } n}$$

so that the differential expressions $dU_r - \sum_{t=1}^{n} P_{rt}\,dX_t$ ($r = 1$ to m) are linear combinations of the expressions $du_r - \sum_{t=1}^{n} p_{rt}\,dx_t$ ($r = 1$ to m).

Such a transformation, if it is obtained, is called a *Contact Transformation* (or *Tangential Transformation*) since the tangent planes (in n dimensions) of the one set of n-dimensional surfaces u_r transforms into the tangent planes of new set U_r. In the first two types of transformations, *any* new functions may be introduced (subject to the conditions of the existence theorem) but in the third case certain conditions must be satisfied by the functions introduced by the transformation. These conditions are interpreted by Lie in his theory of contact transformations when there is *one* dependent variable, and incomplete interpretations have been given when there are more dependent variables than one. Lie's theory, modified by other writers, indicates how the functions of the transformation may be obtained. *One* of the new functions may be

taken arbitrarily and the others are obtained as solutions of definite differential equations involving that function. Transformations of this type are important in the theory of partial differential equations of the first order. For example, suppose that $P(x, y, z, p, q)$ is any function of its arguments where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$, z being a function of x, y, and let Z, P, Q, X, Y be a contact transformation. Then $P = 0$ may be integrated immediately to give the general solution $Z = \phi(Y)$ where ϕ is arbitrary: i.e. the solution of the equation $P = 0$ is given by

$$Z = \phi(Y),\ Q = \phi'(Y),\ P = 0$$

(three equations from which p, q can be theoretically eliminated).

The following simple examples illustrate this type of transformation.

Examples. (i) Let $X_1 = p_1$, $X_2 = p_2$, . . . , $X_n = p_n$, $Z = p_1x_1 + p_2x_2 + \ldots + p_nx_n - z$ where $z = z(x_1, x_2, \ldots, x_n)$, $p_r = \dfrac{\partial z}{\partial x_r}$.

Then $dZ = x_1\,dp_1 + x_2\,dp_2 + \ldots + x_n\,dp_n$, and if we take $P_r = x_r$ we have

$$dZ - P_1\,dX_1 - \ldots - P_n\,dX_n = -(dz - p_1\,dx_1 - \ldots - p_n\,dx_n).$$

(ii) Let $Z = z(1-p)^2$, $X = p^2x + q^2y - z$, $Y = y(1-q)^2$, $P = \dfrac{p-1}{p}$, $Q = \dfrac{q(p-1)}{p(1-q)}$ where $z = z(x, y)$, $p = z_x$, $q = z_y$ then

$$dZ - P\,dX - Q\,dY = \frac{p-1}{p}(dz - p\,dx - q\,dy)$$

so that $P = \dfrac{\partial Z}{\partial X}$ and $Q = \dfrac{\partial Z}{\partial Y}$ if Z is regarded as a function of X, Y.

Examples VI

1. If $u = 2x^2 + 3y^2 + 4xy + 6x + 2y$, $x = 2X + 3Y - 4$; $y = X - Y + 1$, show that $\dfrac{\partial}{\partial X}u(X, Y) = 12x + 14y + 14$, $\dfrac{\partial}{\partial Y}u(X, Y) = 8x + 6y + 16$.

2. If $u = ae^{xy} + be^{x+y}$; $x = \log(\xi^2 + \eta^2)$, $y = \arctan\left(\dfrac{\eta}{\xi}\right)$, show that $\dfrac{\partial}{\partial \xi}u(\xi, \eta) = ae^{xy-\frac{1}{2}x}(2y\cos y - x\sin y) + be^{y+\frac{1}{2}x}(2\cos y - \sin y)$.

3. If $u = xy + yz + zx$; $x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$, show that $(z\tan\theta)\dfrac{\partial}{\partial\theta}u(\theta, \phi) = 2uz - a^2(x + y)$.

4. If $V = xyz$, $x = a\cos\omega t$, $y = a\sin\omega t$, $z = a\omega t$, prove that

$$\frac{d}{dt}V(t) = a\omega z(x^2 - y^2) + a\omega xy$$

5. If $z = z(x, y)$; $x = u + v$, $y = uv$, prove that

$$\frac{\partial^2}{\partial u\partial v}z(u, v) = z_{xx} + xz_{xy} + yz_{yy} + z_y$$

6. If $V = \psi(u, v)$, $\phi(u, v) = E(x, y)$; $\chi(u, v) = F(x, y)$, prove that

$$\left\{\frac{\partial}{\partial x}V(x, y)\right\}\frac{\partial(\phi, \chi)}{\partial(u, v)} = E_x\frac{\partial(\psi, \chi)}{\partial(u, v)} + F_x\frac{\partial(\phi, \psi)}{\partial(u, v)}.$$

7. If $V = u^3 + v^3 + uv$; $u + v = x^2 + y^2$, $u^2 + v^2 = 2xy$, prove that

$$3\frac{\partial}{\partial y}V(x, y) = \frac{2x(x^2 - y^2)}{(x^2 + y^2)^2} + 8y(x^2 + y^2)$$

8. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that $r^2\theta_x(x, y, z) = x_\theta$; $r^2\theta_y(x, y, z) = y_\theta$; $r^2\theta_z(x, y, z) = z_\theta$.

9. If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$,
$$u + v + w = x^2 + y^2 + z^2$$
prove that

(i) $\frac{\partial}{\partial x}u(x, y, z) = -\frac{9(v + w)x^2 - 12vwx - 2}{6(u - v)(u - w)}$.

(ii) $\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(x - y)(y - z)(z - x)}{(u - v)(v - w)(w - u)}$.

10. If $u^3 - v^3 = x^4 + y^5$, $u^2 + v^2 = x^3 + y^4$, prove that

(i) $\frac{\partial}{\partial x}u(x, y) = \frac{x^2(8x - 9v^3)}{12u^2(u^3 - v^3)}$; (ii) $\frac{\partial(u, v)}{\partial(x, y)} = \frac{2x^2y^3(x - y^2)}{3u^2v^2(u^3 - v^3)}$.

11. If $u^3 + v^3 + 2uvx + y = 0$, $uv - (u + v)y + x^3 = 0$, prove that
$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{uv(u + v) - x}{(u - v)\{(u - v) + y(1 - x)\}}$$

12. If $\phi(x_1, x_2, \ldots, x_n) = 0$ prove that $\frac{\partial x_1}{\partial x_2}\frac{\partial x_2}{\partial x_3}\cdots\frac{\partial x_n}{\partial x_1} = (-1)^n$.

13. If the five variables x_1, x_2, x_3, x_4, x_5 are connected by two functional relations, prove that

(i) $\left(\frac{\partial x_1}{\partial x_2}\right)_{x_3, x_4}\left(\frac{\partial x_2}{\partial x_3}\right)_{x_4, x_5}\left(\frac{\partial x_3}{\partial x_4}\right)_{x_5, x_1}\left(\frac{\partial x_4}{\partial x_5}\right)_{x_1, x_2}\left(\frac{\partial x_5}{\partial x_1}\right)_{x_2, x_3} = -1$

(ii) $\left(\frac{\partial x_2}{\partial x_3}\right)_{x_4, x_5}\left(\frac{\partial x_3}{\partial x_4}\right)_{x_5, x_1}\left(\frac{\partial x_4}{\partial x_5}\right)_{x_1, x_2}\left(\frac{\partial x_5}{\partial x_2}\right)_{x_1, x_4} = 1$

where the suffixes denote the other independent variables during the differentiation

14. If the six variables $x_1, x_2, x_3, x_4, x_5, x_6$ are connected by three relations, show that

(i) $\left\{\frac{\partial(x_1, x_2)}{\partial(x_4, x_5)}\right\}_{x_6}\left\{\frac{\partial(x_3, x_4)}{\partial(x_6, x_1)}\right\}_{x_2}\left\{\frac{\partial(x_5, x_6)}{\partial(x_2, x_3)}\right\}_{x_4} = 1$

(ii) $\left(\frac{\partial x_1}{\partial x_2}\right)_{x_3, x_4}\left(\frac{\partial x_2}{\partial x_3}\right)_{x_4, x_5}\left(\frac{\partial x_3}{\partial x_4}\right)_{x_5, x_6}\left(\frac{\partial x_4}{\partial x_5}\right)_{x_6, x_1}\left(\frac{\partial x_5}{\partial x_1}\right)_{x_6, x_2} = -1.$

15. If $u^3 + uvy + v^3 - x + y = 0$; $2uv + u^2v^2x^2y^2 - x^2 + y^2 = 0$ prove that
$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{2(x - y)(u^2v^2xy - uvx - uvy - 1)}{3(u^2 - v^2)(2x^2y^2u^2v^2 - 2)}.$$

16. If $f(x, y, \alpha, \beta) = 0$, $\alpha = \phi(x, y)$, $\beta = \psi(x, y)$ prove that
$$\frac{d}{dx}y(x) = -\frac{f_x + f_\alpha\phi_x + f_\beta\psi_x}{f_y + f_\alpha\phi_y + f_\beta\psi_y}.$$

17. If $f(x, y, \alpha, \beta) = 0$; $\phi(x, \alpha) = 0 = \psi(y, \beta)$, show that
$$\phi_\alpha\frac{\partial(f, \psi)}{\partial(y, \beta)}\frac{d}{dx}y(x) + \psi_\beta\frac{\partial(f, \phi)}{\partial(x, \alpha)} = 0.$$

18. If $f(u, v, w, x, y) = \phi(u, v, w, x, y) = \psi(u, v, w, x, y) = 0$, prove that
$$\frac{\partial}{\partial x}u(x, y) = -\frac{J_1}{J}, \quad \frac{\partial}{\partial y}u(x, y) = -\frac{J_2}{J}$$
where
$$J_1 = \frac{\partial(f, \phi, \psi)}{\partial(x, v, w)}, \quad J_2 = \frac{\partial(f, \phi, \psi)}{\partial(y, v, w)}, \quad J = \frac{\partial(f, \phi, \psi)}{\partial(u, v, w)}.$$

19. If $u^3 + v^3 + w^3 = x + y^2 + z^3$, $u + v^2 + w^3 = x^3 + y + z^2$,
$$u + v + w^3 = x^2 + y^3 + z$$
prove that
$$\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \frac{1 - 4(xy + yz + zx) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}.$$

20. A curve in the $x - y$ plane is given by $y = f(x, \alpha)$ where $\phi(x, \alpha) = 0$. Prove

(i) $\phi_\alpha \dfrac{dy}{dx} = \dfrac{\partial(f, \phi)}{\partial(x, \alpha)}$

(ii) $\phi_\alpha^3 \dfrac{d^2y}{dx^2} = \phi_\alpha(f_{xx}\phi_\alpha^2 - 2f_{x\alpha}\phi_x\phi_\alpha + f_{\alpha\alpha}\phi_x^2) - f_\alpha(\phi_{xx}\phi_\alpha^2 - 2\phi_{x\alpha}\phi_x\phi_\alpha + \phi_{\alpha\alpha}\phi_x^2)$.

21. A curve in the $x - y$ plane is given by $0 = f(x, y, \alpha) = \dfrac{\partial}{\partial\alpha} f(x, y, \alpha)$. Prove that (i) $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$ (ii) $f_{\alpha\alpha} f_y^3 \dfrac{d^2y}{dx^2} = (f_x f_{y\alpha} - f_y f_{x\alpha})^2 - f_{\alpha\alpha}(f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_x^2 f_{yy})$.

22. If $\phi(x,\ y,\ z,\ \alpha,\ \beta) = 0$, $\phi_\alpha = 0$, $\phi_\beta = 0$, prove that $\dfrac{\partial}{\partial x} z(x,\ y) = -\dfrac{\phi_x}{\phi_z}$, $\dfrac{\partial}{\partial y} z(x,\ y) = -\dfrac{\phi_y}{\phi_z}$.

23. If $u = u(x,\ y,\ z)$ and $z = z(x,\ y)$ prove that
$$\frac{\partial}{\partial y} u(x,\ y) = \frac{\partial}{\partial z} u(x,\ z) \frac{\partial}{\partial y} z(x,\ y)$$

24. If $u = \dfrac{x - y}{z}$, $v = \dfrac{z + y}{x}$, $w = \dfrac{y(x + y + z)}{xz}$ show that $\dfrac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = 0$ and find the functional relation connecting u, v, w.

25. If $u = x + y + z$, $v = xy + yz + zx$, $w = x^3 + y^3 + z^3 - 3xyz$, prove that $\dfrac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = 0$ and find the relation connecting u, v, w.

26. Obtain a functional relation connecting X, Y, Z, U where
$$U = xyz + yzu + zux + uxy,\ Z = x^3 + y^3 + z^3 + u^3,\ Y = x^2 + y^2 + z^2 + u^2,$$
$$X = x + y + z + u.$$

27. If $f(x)$ is defined by the equations $f'(x) = 1/x$, $f(1) = 0$, prove, without assuming the logarithmic function, that $f(x) + f(y) = f(xy)$.

28. If $(1 + x^2)f'(x) = 1$, $f(0) = 0$, and $u = f(x) + f(y)$, $v = \dfrac{x + y}{1 - xy}$, prove that $\dfrac{\partial(u,\ v)}{\partial(x,\ y)} = 0$, and deduce that $f(x) + f(y) = f\left(\dfrac{x + y}{1 - xy}\right)$.

29. If $f'(x) = \dfrac{1}{\sqrt{(1 + x^4)}}$ and $f(0) = 0$, prove that
$$f(x) + f(y) = f\left\{\frac{x\sqrt{(1 + y^4)} + y\sqrt{(1 + x^4)}}{1 - x^2y^2}\right\}$$

30. If $\dfrac{\partial}{\partial x} u(x,\ y) = \left\{\dfrac{\partial}{\partial y} v(x,\ y)\right\}^2$, show that $\left\{\dfrac{\partial y}{\partial v}(u,\ v)\right\} \dfrac{\partial(x,\ y)}{\partial(u,\ v)} = \left\{\dfrac{\partial x}{\partial u}(u,\ v)\right\}^2$.

31. If $z = z(x,\ y)$, $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, prove that when r, s, t are expressed as functions of p, q, then
$$rs_p + ss_q = sr_p + tr_q,\ ss_p + ts_q = rt_p + st_q.$$

32. If $Z = px + qy - z$, where $z = z(x,\ y)$, $p = z_x$, $q = z_y$ show that when Z is expressed as a function of p, q, then $R = \dfrac{t}{rt - s^2}$, $S = -\dfrac{s}{rt - s^2}$, $T = \dfrac{r}{rt - s^2}$ where $R = Z_{pp}$, $S = Z_{pq}$, $T = Z_{qq}$, and $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.

33. If $Z = px - z$, and Z is expressed as a function of p, y, prove that in the notation of *Example 31*, $R = \frac{1}{r}$, $S = -\frac{s}{r}$, $T = -\frac{rt - s^2}{r}$ where $R = Z_{pp}$, $S = Z_{py}$, $T = Z_{yy}$.

34. If in the notation of *Example 31*, $Z = p^2x + qy^2 - z$, $X = x(p-1)^2$, $Y = \frac{1}{y} + \log(qy^2)$, $P = \frac{p}{p-1}$, $Q = qy^2$ and Z is expressed as a function of X, Y, then $P = \partial Z/\partial X$, $Q = \partial Z/\partial Y$ and $dZ - P\,dX - Q\,dY = dz - p\,dx - q\,dy = 0$; show also that Z_{XX}

$$= \frac{(1-2y)qr - y^2(rt-s^2)}{(p-1)^4\{(2y-1)(p-1)q - 2xqr(2y-1) + (p-1)ty^2 - 2xy^2(rt-s^2)\}}.$$

Prove the results given in *Examples 35–41* for the Thermodynamic case:

35. $\left(\frac{\partial C_v}{\partial v}\right)_\theta = \theta\left(\frac{\partial^2 p}{\partial \theta^2}\right)_v$ **36.** $\left(\frac{\partial C_p}{\partial p}\right)_\theta = -\theta\left(\frac{\partial^2 v}{\partial \theta^2}\right)_p$

37. $C_p - C_v = \theta\left(\frac{\partial p}{\partial \theta}\right)_v\left(\frac{\partial v}{\partial \theta}\right)_p$ **38.** $C_p - C_v = -\theta\left(\frac{\partial v}{\partial p}\right)_\theta\left\{\left(\frac{\partial p}{\partial \theta}\right)_v\right\}^2$

39. $dE = C_v\,d\theta + \left\{\theta\left(\frac{\partial p}{\partial \theta}\right)_v - p\right\}dv$

40. The curves of constant entropy (*adiabatics*) are of the form $(p-a)(v-b)^\gamma$ = constant, when C_p, C_v are constant and $\gamma = C_p/C_v$, and a, b are constants.

41. The internal energy $E = C_v\theta$ + constant, in *Example 40*, if a is zero.

42. The equations for plane waves of a gas are given to be

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} = 0, \quad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

where u, ρ are functions of x, t, and c is constant. By taking $\xi\,(= x - ut)$, t as the dependent variables and u, $\sigma\,(= \log \rho)$ as the independent variables, prove that these equations are equivalent to the single equation $\phi_{\sigma\sigma} - c^2\phi_{uu} + \phi_\sigma = 0$ where $\xi = -c^2\phi_u$, $t = \phi_\sigma$.

43. The equation of long waves in a uniform canal with vertical sides is given to be $\xi_{tt}(1+\xi_x)^3 = c^2\xi_{xx}$ where ξ is a function of x, t, and c is constant. Taking the independent variables to be $u\,(= \xi_t - 2c(1+\xi_x)^{-\frac{1}{2}})$, $v\,(= \xi_t + 2c(1+\xi_x)^{-\frac{1}{2}})$, and the dependent variable to be $E\,(= -x\xi_x + t\xi_t - \xi)$, prove that

$$2(u - v)E_{uv} = 3(E_u - E_v).$$

Solutions

24. $(w+1) = ut$ **25.** $w = u(u^2 - 3v)$ **26.** $6U = X^3 - 3XY + 2Z$

CHAPTER VII

INDETERMINATE FORMS. MAXIMA AND MINIMA.

7. Indeterminate Forms. If $f(a) = 0$, $\phi(a) = 0$, the function $f(x)/\phi(x)$ is said to take the '*Indeterminate Form*' $0/0$ at $x = a$, although it may tend to a determinate limit when $x \to a$.

Other indeterminate forms occur such as those indicated by the symbols ∞/∞, $\infty - \infty$, 0^0, $0 \times \infty$, ∞^0, 1^∞, 0^{0^0}, ∞^{0^∞}, &c.

For the cases that usually arise the most practical method of evaluating the limit, when it exists, consists in finding the expansion of the function in the appropriate neighbourhood. Before illustrating this obvious method, however, we shall obtain two allied theorems that are of wider application than the method of expansions.

7·01. Theorems on Indeterminate Forms. Theorem I. Let (i) $f(x)$, $\phi(x)$ be continuous near $x = a$ and possess derivatives $f'(x)$, $\phi'(x)$; (ii) $f(a) = 0 = \phi(a)$; then

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$

if the latter limit exists.

We shall assume that $\phi(x)$ is not zero near a, and therefore that $\phi(a + h) \neq 0$ for sufficiently small values of h.

Let $F(x) = f(x) - \dfrac{f(a + h)}{\phi(a + h)} \cdot \phi(x)$, $(h \neq 0)$.

Then $F(a + h) = 0 = F(a)$ and therefore by Rolle's Theorem

$$F'(a + \theta h) = 0$$

for some value of θ in the interval $0 < \theta < 1$,

i.e.
$$\frac{f(a + h)}{\phi(a + h)} = \frac{f'(a + \theta h)}{\phi'(a + \theta h)}$$

i.e.
$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{h \to 0} \frac{f'(a + \theta h)}{\phi'(a + \theta h)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}.$$

Note. If $\dfrac{f'(x)}{\phi'(x)}$ takes the indeterminate form $\dfrac{0}{0}$, the theorem may be reapplied.

Theorem II (a). Let (i) $f(x)$, $\phi(x)$ be continuous and possess derivatives for all large x; (ii) $\lim\limits_{x \to +\infty} f(x) = +\infty$ (or $-\infty$) and $\lim\limits_{x \to +\infty} \phi(x) = +\infty$ (or $-\infty$), then

$$\lim_{x \to +\infty} \frac{f(x)}{\phi(x)} = \lim_{x \to +\infty} \frac{f'(x)}{\phi'(x)}$$

if the latter limit exists.

It is sufficient to prove the theorem when $f(x)$, $\phi(x)$ both tend to $+\infty$, since the other cases may be reduced to this by changing the sign of f or ϕ or of both f and ϕ.

Let $F(x) = f(x) - f(x_1) - \dfrac{f(x_2) - f(x_1)}{\phi(x_2) - \phi(x_1)}(\phi(x) - \phi(x_1))$ where $x_2 > x_1$ and where, since $\phi(x) \to +\infty$, x_2 can be chosen sufficiently large to ensure that $\phi(x) \neq \phi(x_1)$, $(x \geq x_2)$. Then $F(x_1) = 0 = F(x_2)$ and therefore $F'(x_3) = 0$ for some value x_3 satisfying the inequality $x_2 > x_3 > x_1$.

$$\text{Thus}\qquad \frac{f(x_2) - f(x_1)}{\phi(x_2) - \phi(x_1)} = \frac{f'(x_3)}{\phi'(x_3)}\ (x_2 > x_3 > x_1).$$

Let $\lim\limits_{x \to +\infty} \dfrac{f'(x)}{\phi'(x)} = l$; then x_1 can be chosen sufficiently large to ensure that $\left|\dfrac{f'(x)}{\phi'(x)} - l\right| < \varepsilon$ for all $x > x_1$.

Keeping x_1 fixed and let $x_2 \to +\infty$. x_2 can be taken sufficiently large to ensure that $\left|\dfrac{\phi(x_1)}{\phi(x_2)}\right| < \varepsilon$, $\left|\dfrac{f(x_1)}{f(x_2)}\right| < \varepsilon$, (since $\phi(x)$, $f(x) \to +\infty$).

$$\text{Now}\qquad \frac{f(x_2)}{\phi(x_2)} = \frac{f'(x_3)}{\phi'(x_3)} \left\{\frac{1 - \dfrac{\phi(x_1)}{\phi(x_2)}}{1 - \dfrac{f(x_1)}{f(x_2)}}\right\} = (l + \rho)\left(\frac{1 - \sigma_1}{1 - \sigma_2}\right)$$

where $|\rho| = \left|\dfrac{f'(x_3)}{\phi'(x_3)} - l\right| < \varepsilon$; $|\sigma_1| = \left|\dfrac{\phi(x_1)}{\phi(x_2)}\right| < \varepsilon$; $|\sigma_2| = \left|\dfrac{f(x_1)}{f(x_2)}\right| < \varepsilon$

i.e. $\dfrac{f(x_2)}{\phi(x_2)} - l$ is small when x_2 is large.

i.e. $\lim\limits_{x \to +\infty} \dfrac{f(x)}{\phi(x)} = \lim\limits_{x \to +\infty} \dfrac{f'(x)}{\phi'(x)}$ if the latter limit exists.

Corollary. By a similar proof, we may show that when $f(x) \to +\infty$ (or $-\infty$) and $\phi(x) \to +\infty$ (or $-\infty$), when $x \to -\infty$, then

$$\lim_{x \to -\infty} \frac{f(x)}{\phi(x)} = \lim_{x \to -\infty} \frac{f'(x)}{\phi'(x)}$$

when the latter limit exists.

Note. In some cases $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to -\infty} f(x) = +\infty$ (or $-\infty$) and

$$\lim_{x \to +\infty} \phi(x) = \lim_{x \to -\infty} \phi(x) = +\infty \text{ (or } -\infty)$$

and we may then write $\lim\limits_{|x| \to \infty} \dfrac{f(x)}{\phi(x)} = \lim\limits_{|x| \to \infty} \dfrac{f'(x)}{\phi'(x)}$ when the latter limit exists.

Theorem II (b). Let (i) $f(x)$, $\phi(x)$ be continuous near $x = a$ (but not at $x = a$); (ii) $f'(x)$, $\phi'(x)$ exist near $x = a$ (but not necessarily at $x = a$).

(iii) $\lim_{x \to a} f(x) = +\infty$ (or $-\infty$) and $\lim_{x \to a} \phi(x) = +\infty$ (or $-\infty$), then

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$

if the latter limit exists.

Let $x = a + \dfrac{1}{\xi}$, then

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{|\xi| \to \infty} \frac{f\left(a + \frac{1}{\xi}\right)}{\phi\left(a + \frac{1}{\xi}\right)} = \lim_{|\xi| \to \infty} \left\{ \frac{-\frac{1}{\xi^2} f'\left(a + \frac{1}{\xi}\right)}{-\frac{1}{\xi^2} \phi'\left(a + \frac{1}{\xi}\right)} \right\}$$

if the limit on the right exists.

i.e. $$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$

if the latter limit exists.

Note. If $\dfrac{f'(x)}{\phi'(x)}$ takes the indeterminate form $\dfrac{\infty}{\infty}$, the theorem may be reapplied.

7.02. Other Indeterminate Forms. The form $0/0$ is theoretically equivalent to $\dfrac{\infty}{\infty}$ since we may write $\dfrac{f}{\phi} = \dfrac{\phi^{-1}}{f^{-1}}$, but it will sometimes be found that the application of Theorem I to $\dfrac{\phi^{-1}}{f^{-1}}$ when $|f|, |\phi| \to \infty$ is ineffective. Other indeterminate forms should, if possible, be reduced to $0/0$ or ∞/∞, whichever is more suitable.

Thus, if $f(x) \to 0$, $\phi(x) \to 0$, $\psi(x) \to \infty$, $\chi(x) \to \infty$, $\lambda(x) \to 1$ when $x \to a$,

(i) $f\psi$, $(0 \times \infty)$ may be written $\dfrac{f}{\psi^{-1}}, \left(\dfrac{0}{0}\right)$ or $\dfrac{\psi}{f^{-1}}, \left(\dfrac{\infty}{\infty}\right)$.

(ii) $\psi - \chi$, $(\infty - \infty)$ may be written $\psi\left(1 - \dfrac{\chi}{\psi}\right)$ which if $\dfrac{\chi}{\psi} \to 1$, takes the form $\infty \times 0$ (i).

(iii) f^ϕ, (0^0) is $e^{\phi \log f}$ and $\phi \log f$ is $0 \times \infty$ (i).

(iv) ψ^f, (∞^0) is $e^{f \log \psi}$ and $f \log \psi$ is $0 \times \infty$ (i).

(v) λ^ψ, (1^∞) is $e^{\psi \log \lambda}$ and $\psi \log \lambda$ is $\infty \times 0$ (i).

Suitable modifications may be found for indeterminate forms of a more complex type.

For example, $(f/\phi)^\psi$ is $(0/0)^\infty$ and may be written $e^{\psi \log (f/\phi)}$.

The function $\psi \log (f/\phi)$ takes the form $\infty \times 0$ if $f/\phi \to 1$.

Examples. (i) $\lim_{x \to 1} \dfrac{x^5 - 2x^4 + x^3 + x^2 - 2x + 1}{x^4 - 2x^3 - x^2 + 4x - 2} \left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 1} \frac{5x^4 - 8x^3 + 3x^2 + 2x - 2}{4x^3 - 6x^2 - 2x + 4} \left(\frac{0}{0}\right) = \lim_{x \to 1} \frac{20x^3 - 24x^2 + 6x + 2}{12x^2 - 12x - 2} = -2.$$

(ii) $\lim\limits_{x\to\frac{1}{2}\pi} \dfrac{\log(x-\frac{1}{2}\pi)}{\tan x}\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\frac{1}{2}\pi} \dfrac{(x-\frac{1}{2}\pi)^{-1}}{\sec^2 x}\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\frac{1}{2}\pi} \dfrac{\cos^2 x}{x-\frac{1}{2}\pi}\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to\frac{1}{2}\pi} (-2\cos x \sin x) = 0.$$

(iii) $\lim\limits_{x\to\frac{1}{2}\pi} (\frac{1}{2}\pi - x)\tan x \; (0 \cdot \infty) = \lim \dfrac{\frac{1}{2}\pi - x}{\cot x}\left(\dfrac{0}{0}\right) = \lim \dfrac{-1}{-\operatorname{cosec}^2 x} = 1.$

(iv) $\lim\limits_{x\to 0} \left(\cot x - \dfrac{1}{x}\right)(\infty - \infty) = \lim \left(\dfrac{x\cos x - \sin x}{x \sin x}\right)\left(\dfrac{0}{0}\right)$

$$= \lim \frac{-x\sin x}{\sin x + x\cos x}\left(\frac{0}{0}\right) = \lim \frac{-\sin x - x\cos x}{2\cos x - x\sin x} = 0.$$

(v) $\lim\limits_{x\to 0} x^x \; (0^0) = \exp\lim(x\log x) = \exp\lim \dfrac{\log x}{(1/x)} = 1.$

(vi) $\lim\limits_{x\to 0} (\cot x)^{\sin x} \; (\infty^0) = \exp\lim \sin x \log \cot x$

$$= \exp\lim\left(\frac{\log \cot x}{\operatorname{cosec} x}\right) = \exp\lim\left(\frac{-\tan x \operatorname{cosec}^2 x}{\cot x \operatorname{cosec} x}\right) = 1.$$

(vii) $\lim\limits_{x\to 0} (1+\tan x)^{\operatorname{cosec} x} \; (1^\infty) = \exp\lim\left(\dfrac{\log(1+\tan x)}{\sin x}\right)$

$$= \exp\lim \frac{\sec^2 x}{\cos x + \sin x} = e.$$

(viii) $\lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}} \left(\dfrac{0}{0}\right)^\infty = \exp\lim \dfrac{\log \sin x - \log x}{x^2}.$

Now $\dfrac{\sin x}{x} \to 1$ and therefore $\log \sin x - \log x \to 0$.

Thus the limit is $\exp\lim \dfrac{x\cos x - \sin x}{2x^2 \sin x} = e^{-\frac{1}{6}}$ by repeated applications of Theorem I.

(ix) $\lim\limits_{x\to +0} x^\alpha\left\{\log\left(\dfrac{1}{x}\right)\right\}^\beta$. Let $u = x^\alpha\left\{\log\left(\dfrac{1}{x}\right)\right\}^\beta$.

(a) $\alpha < 0$, $\beta > 0$, then $u \to +\infty$

(b) $\alpha > 0$, $\beta < 0$; then $u \to 0$, these results being obvious.

(c) $\alpha > 0, \beta > 0$, $u = \dfrac{\left\{\log\left(\frac{1}{x}\right)\right\}^\beta}{x^{-\alpha}}$, $\lim u = \lim \dfrac{\beta}{\alpha}\dfrac{\left\{\log\left(\frac{1}{x}\right)\right\}^{\beta-1}}{x^{-\alpha}}$ by Theorem II and therefore by continued application of Theorem II we prove that $\lim u = 0$ by (b).

(d) If $\alpha < 0$, $\beta < 0$, the limit is ∞, since $\lim(1/u) = 0$ by (c).

Thus $u \to 0$ if $\alpha > 0$ and $u \to \infty$ if $\alpha < 0$.

(x) $\lim\limits_{x\to+\infty} \dfrac{(\log x)^\beta}{x^\alpha} = \lim\limits_{y\to+0} y^\alpha\left\{\log\left(\dfrac{1}{y}\right)\right\}^\beta = 0$ if $\alpha > 0$, and ∞ if $\alpha < 0$ by (ix).

(xi) $\lim\limits_{x\to+\infty} \dfrac{e^{mx}}{x^\alpha} = \lim\limits_{x\to\infty} \dfrac{me^{mx}}{\alpha x^{\alpha-1}}$ (Theorem II) and therefore by continued application of Theorem II, the limit is ∞, all α if $m > 0$.

7.03. *Examples of the Use of Expansions. Examples* (i).

Examples. (i) Expand $\tan x$ as far as x^5 (x small).

$$\tan x = \frac{\sin x}{\cos x} = x\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)\left\{1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \left(\frac{x^2}{2} - \frac{x^4}{24}\right)^2 + O(x^6)\right\}$$

$$= x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + O(x^7).$$

(ii) Expand $\sin^3 x$ as far as x^7 when x is small. Either

$$\sin^3 x = x^3\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)^3 = x^3 - \frac{1}{2}x^5 + \frac{13}{120}x^7 + O(x^9)$$

or $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ and expand $\sin x$ and $\sin 3x$.

(iii) Find $\lim\limits_{x\to 0}\left(\cot^2 x - \dfrac{1}{x^2}\right)$.

$$\cot^2 x = \frac{1}{x^2}(1 - x^2 + O(x^4))\left(1 - \frac{1}{3}x^2 + O(x^4)\right)^{-1} = \frac{1}{x^2} - \frac{2}{3} + O(x^2)$$

i.e.

$$\lim\left(\cot^2 x - \frac{1}{x^2}\right) = -\frac{2}{3}.$$

(iv) Evaluate $\lim\limits_{x\to 0}\dfrac{\sin\sin\sin x - x\cos x}{x^5}$.

$$\sin\sin x = \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) - \frac{1}{6}\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)^3 + \frac{x^5}{120} + O(x^7)$$

$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5 + O(x^7)$$

$\sin\sin\sin x = x - \frac{1}{2}x^3 + \frac{11}{40}x^5 + O(x^7)$ similarly.

Also $x\cos x = x - \frac{1}{2}x^3 + \frac{1}{24}x^5 + O(x^7)$. Thus the limit is $\frac{11}{40} - \frac{1}{24} = \frac{7}{30}$.

(v) Find $\lim\limits_{x\to 0}\dfrac{\sin^2 x\sinh^4 x\,(e^x - 1)^3}{x^9}$.

The limit is $\left(\lim\dfrac{\sin x}{x}\right)^2\left(\lim\dfrac{\sinh x}{x}\right)^4\left(\lim\dfrac{e^x - 1}{x}\right)^3 = 1.$

(vi) Find $\lim\limits_{x\to 0}(1 + \sin x)^{\cot x}$; let $u = (1 + \sin x)^{\cot x}$.

Then $\log u = \dfrac{\{1 - \frac{1}{2}x^2 + O(x^4)\}\{\sin x - \frac{1}{2}\sin^2 x + O(x^3)\}}{x - \frac{1}{6}x^3 + O(x^5)} = 1 + O(x)$,

i.e. $u \to e$.

7.1. Maxima and Minima of Functions of One Variable. A function $f(x)$ is said to have a *maximum* (*minimum*) value at $x = a$, if $f(a)$ is algebraically greater (less) than all the values of $f(x)$ near $x = a$. When $f(x)$ is defined for all values near $x = a$ (including $x = a$), then $f(a)$ is a maximum (minimum) if an interval can be found

$$|x - a| < \delta\ (\neq 0)$$

for which $f(x) < f(a)$, $(f(x) > f(a))$. Thus $x = 0$ gives a maximum value to $1 - x^2$ and a minimum to $|x|$.

Note. We omit the case when $f(x) = f(a)$ over an interval, and $f(a)$ a maximum (minimum) in the broad sense. For example, if $f(x) = |x - 1| + |x + 1|$, $f(x)$ has a minimum 2 for all x in $-1 < x < 1$. In many cases $f(x)$ may not be defined for all values of x and may not possess a maximum or minimum. For example, $\dfrac{x^2(x - 2)}{(x - 1)}$ is defined for all x except $x = 1$ and has no maximum nor minimum. The definition we have given of maximum is strictly that of a maximum relative to values in a neighbourhood; and therefore in this sense a function may have an unlimited number of maxima and minima.

Notes. (i) The problem of determining maximum and minimum values is concerned with real variables, and in problems where complex values arise through analytical conditions, the function may possess a maximum or minimum in a less restrictive sense. Thus $\sqrt{\{x(x - 5)(x - 8)\}}$ has a relative maximum 6 when $x = 2$ obtained analytically from the equation $(x - 2)(3x - 20) = 0$. The value $x = \frac{20}{3}$ does not give a real value to the function, which has, however, an obvious minimum 0 when $x = 0$, 5 or 8.

(ii) Values obtained analytically may be inadmissible in practical applications even when they are real.

For example, if a number of spheres are projected at a certain instant from given points under a given law of attraction the distance between the centres of two of them is a function of the interval of time t after the instant of projection. This function may possess maxima or minima when t satisfies some numerical equation obtained by analysis. Such a value of t will, however, be inadmissible when (a) it is complex, (b) it is real and negative, (c) the corresponding distance δ is complex (e.g. when δ^2 is negative), (d) $\delta < r_1 + r_2$ where r_1, r_2 are the radii of the spheres, (e) a collision has taken place before t reaches the value found.

7.11. *Analytical Conditions for Maxima and Minima (One Variable).* The conditions obtained here and in subsequent paragraphs imply the existence of the derivatives that occur. In general, also, we shall obtain, for simplicity, conditions that are *sufficient*.

If $f(x)$ possesses a second derivative near a (including a), then

$$f(a + h) - f(a) = hf'(a) + \frac{h^2}{2}f''(a + \theta h).$$

The *sign* of $f(a + h) - f(a)$ is that of $hf'(a)$ if h is small and therefore cannot be invariable unless $f'(a) = 0$.

Thus if $f'(x)$ exists, a *necessary* condition is $f'(a) = 0$ and the possible values of a are obtained by solving the equation $f'(x) = 0$. If in addition $f''(x)$ is > 0 at and near $x = a$, $f(a + h) - f(a) > 0$ showing that $f(a)$ is minimum; whilst if $f''(x) < 0$, $f(a)$ is a maximum.

In particular if $f''(x)$ is continuous and $f''(a) > 0\ (< 0)$, (where $f'(a) = 0$), then $f(a)$ is a minimum (maximum).

If $f''(a) = 0$ and $f(x)$ possesses higher derivatives, let $f^n(x)$ be the first that does not vanish when $x = a$; then

$$f(a + h) - f(a) = \frac{h^n}{n!}f^{(n)}(a + \theta h),\ (0 < \theta < 1).$$

If $f^{(n)}(x) > 0\ (< 0)$ near a, $f(a)$ is a minimum (maximum) if n is *even*. But if n is odd, $f(a)$ is not a maximum nor minimum.

Notes. (i) If n is odd, $f'(x)$ has a minimum or maximum and the curve $y = f(x)$ has an inflexion at $x = a$.

(ii) It is often simpler to consider the *approximation* to $f'(x)$ near $x = a$ (where $f'(a) = 0$), in order to discriminate between the values. We may assume that $f'(x) = (x - a)^n\phi(x, a)$ where $\phi(a, a) \neq 0$. If $\phi(x, a)$ is of constant sign near $x = a$ (when it is continuous for example), then $f(a)$ is a minimum when $\phi(a, a) > 0$ and n is *odd*, whilst $f(a)$ is a maximum when $\phi(a, a) < 0$ and n is *odd*. Otherwise there is an inflexion.

Examples. (i) $f(x) = 2x^7 - 7x^6 - 21x^4 - 14x^3 + 21x^2 - 28x - 14$.

$f'(x) = 14(x - 1)^3(x + 1)^2(x - 2)$.

Near $x = -1$, $f'(x) = (-)(x + 1)^2$, *inflexion*.

Near $x = 1$, $f'(x) = (+)(x - 1)^3$, *minimum*. Similarly $x = 2$ gives a *maximum*.

(ii) The distance between the centres of two solid spheres of radii a, b is c. A point source of light is placed on the line of centres between the two spheres. Find the position of the source that will illuminate the greatest total surface. In this example $c > a + b$.

Let x be the distance of the source from the centre of the sphere of radius a measured towards the centre of the other sphere and let $a > b$.

The surface illuminated is $2\pi a^2\left(1 - \frac{a}{x}\right) + 2\pi b^2\left(1 - \frac{b}{c - x}\right)$ and $a < x < c - b$.

There is a maximum when $\frac{a^3}{x^2} = \frac{b^3}{(c - x)^2}$, i.e. $x = \frac{ca^{\frac{3}{2}}}{a^{\frac{3}{2}} + b^{\frac{3}{2}}}$ (since the other value of x is greater than c). This value of x is $> a$ but $< c - b$ only if $c > b + \frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}}$.

If $c < b + \frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}}$, the maximum area is obtained by taking $x = c - b$, since the rate of increase is positive if x increases.

Thus if $c > b + \frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}}$, $\quad x = \frac{ca^{\frac{3}{2}}}{a^{\frac{3}{2}} + b^{\frac{3}{2}}}$, $\quad$ Area $= 2\pi(a^2 + b^2) - \frac{2\pi}{c}(a^{\frac{3}{2}} + b^{\frac{3}{2}})^2$

and $\quad$ if $c < b + \frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}}$, $x = c - b$, $\quad$ Area $= \frac{2\pi a^2(c - a - b)}{c - b}$.

7.2. Maxima and Minima of Functions of Two Variables. If a function $f(x, y)$ is defined at all points near (a, b) (including (a, b)), then $f(a, b)$ is called a relative *maximum* (*minimum*) of $f(x, y)$ if

$$f(a, b) > f(x, y) \; (< f(x, y))$$

for all (x, y) in the neighbourhood.

7.21. Analytical Conditions for a Maximum or Minimum of $f(x, y)$. Taylor's Expansion gives

$$f(a + h, b + k) - f(a, b)$$
$$= hf_a + kf_b + \tfrac{1}{2}(h^2 f_{aa} + 2hkf_{ab} + k^2 f_{bb}) + O(\rho^3)$$

where $\rho^2 = h^2 + k^2$.

The sign of $f(a + h, b + k) - f(a, b)$ is the same as that of $hf_a + kf_b$ when h, k are small and therefore cannot be invariable unless f_a, f_b both vanish.

Note. That the conditions $f_a = 0 = f_b$ are *necessary* follows also from the fact that $f(x, y)$ must be a maximum (or minimum) when y is fixed or when x is fixed.

When $f_a = 0 = f_b$, the sign of $f(a + h, b + k) - f(a, b)$ is the same as that of $h^2 f_{aa} + 2hkf_{ab} + k^2 f_{bb}$ when h, k are small.

Suppose that the second derivatives do not *all* vanish. Then this quadratic in (h, k) can be of invariable sign only when its factors are not real, i.e. only when $f_{ab}^2 < f_{aa} f_{bb}$. But if the factors are real the quadratic is positive for some displacements and negative for others, for it may be written $(\lambda_1 h - \mu_1 k)(\lambda_2 h - \mu_2 k)$ and this has one sign when h/k lies between μ_2/λ_2, μ_1/λ_1 and the other sign when h/k lies outside these limits.

In this case $f(x, y)$ is said to have a *saddle point* (or *minimax*) at (a, b).

Finally, if the quadratic is a complete square, it may be written $\pm(\lambda_1 h - \mu_1 k)^2$; and is therefore of invariable sign for all displacements except those that satisfy the equation $\lambda_1 h = \mu_1 k$. Since further investigation is necessary to determine the nature of the point (a, b), this is sometimes called the 'Doubtful Case'. No useful purpose, however, is served by elaborating the analytical conditions that discriminate between maxima and minima in the doubtful case. For a case arising in practice it is sufficient to draw the contour $f(x, y) = f(a, b)$ (see next paragraph).

in the neighbourhood of (a, b), making use of the Analytical Polygon for that point.

Summarizing: (i) Values a, b are determined by solving the equations $f_x = 0 = f_y$.

(ii) If $f_{aa} > 0\ (< 0)$, and $f_{ab}^2 < f_{aa}f_{bb}$, $f(a, b)$ is a *minimum* (*maximum*).

(iii) If $f_{ab}^2 > f_{aa}f_{bb}$, the point (a, b) is a *saddle point* (i.e. $f(a, b)$ is neither a maximum nor a minimum).

(iv) If $f_{ab}^2 = f_{aa}f_{bb}$, the case is *doubtful*. Draw the contour

$$f(x, y) = f(a, b).$$

7.22. *The Use of Contours.* When $f_a = 0 = f_b$, the contour

$$f(x, y) = f(a, b)$$

has a *singular point* at (a, b). If this contour has *real* branches at (a, b), then $f(x, y) = f(a, b)$ not only at (a, b) but also at *real* points near (a, b), and therefore $f(a, b)$ cannot be a true maximum nor minimum in the *strict* sense. It will usually happen that $f(x, y) - f(a, b)$ is positive on one side of a branch and negative on the other; and in that case $f(a, b)$ is neither a maximum nor a minimum, the point (a, b) being a saddle point. The saddle point in the general case may be of a multiple type such as that given by $(x^2 - y^2)(x^2 - 4y^2) + x^4 + y^4$ at $(0, 0)$.

It is, however, possible for $f(x, y) - f(a, b)$ to have the same sign on both sides of every branch so that $f(a, b)$ is a maximum (or minimum) in the *broad* sense. Thus in this sense $(x^2 + y^2 - x)^2$ has a minimum at $(0, 0)$ although the critical contour $x^2 + y^2 - x$ is a *real* circle.

If, however, $f(x, y) - f(a, b) = 0$ has *no* real branch we infer that (a, b) gives a true maximum or minimum; for let

$$f(x, y) - f(a, b) = F(x - a, y - b) + R$$

where $F(x - a, y - b)$ are those terms that give the first approximation to the curve at (a, b); and let $(0, 0)$ be isolated for the curve $F(\xi, \eta) = 0$. If possible let $F(\xi_1, \eta_1)$ be > 0 and $F(\xi_2, \eta_2) < 0$, the points (ξ_1, η_1), (ξ_2, η_2) being anywhere within a small circle of radius > 0 and of centre 0. The function $\phi(t) = F(t\xi_1 + (1 - t)\xi_2, t\eta_1 + (1 - t)\eta_2)$ has opposite signs for $t = 1$, $t = 0$; and if, as is normally the case, F is continuous it follows that $\phi(t)$ must vanish for some point joining (ξ_1, η_1), (ξ_2, η_2). This contradicts the hypothesis that $(0, 0)$ is isolated for $F(\xi, \eta) = 0$.

Examples. (i) $z = ax^2 + 2hxy + by^2 + 2gx + 2fy + c\ (= f(x, y))$. The possible values are x_0, y_0 where

$$ax_0 + hy_0 + g = 0 = hx_0 + by_0 + f.$$

Denote the co-factors of $\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ by the corresponding capital letters.

(i) Let $C\ (= ab - h^2) \neq 0$: then $x_0 = G/C$, $y_0 = F/C$ and the corresponding value of z is Δ/C.

Therefore $\quad z - \Delta/C = a(x - x_0)^2 + 2h(x - x_0)(y - y_0) + b(y - y_0)^2.$

(a) If $C > 0$, $a > 0$, $z = \Delta/C$ is a *minimum*.

(b) If $C > 0$, $a < 0$, $z = \Delta/C$ is a *maximum*.

Since $C > 0$, a, b cannot be zero and must have the same sign. The neighbouring contours are ellipses and (x_0, y_0) is isolated.

(c) If $C < 0$, (x_0, y_0) is a *saddle point*. The critical contour consists of two straight lines.

(i) Let $C(= ab - h^2) = 0$. Take $a = \alpha^2$, $h = \alpha\beta$, $b = \beta^2$ $(\alpha \neq 0)$, for definiteness, the other possibilities being of a similar type. The equations giving (x_0, y_0) are

$$\alpha^2 x_0 + \alpha\beta y_0 + g = 0 = \alpha\beta x_0 + \beta^2 y_0 + f$$

which have no solution unless $f = \beta g/\alpha$. In the latter case there is a line of minima $\alpha x + \beta y + g/\alpha = 0$, i.e. there is a minimum for any displacement away from the line, whilst z is stationary for displacements along the line.

(ii) $z = 3x^2 - y^2 + x^3 \,(= f(x, y))$.

$f_x = 6x + 3x^2$, $f_y = -2y$ so that the possible points are $(0, 0)$, $(-2, 0)$. The contour for $(0, 0)$ is $3x^2 - y^2 + x^3 = 0$. *Saddle point.*

The contour for $(-2, 0)$ is $y^2 = (x + 2)^2(x - 1)$ and near $(-2, 0)$ on this contour $y^2 = -3(x + 2)^2$ which gives an isolated point $(-2, 0)$.

Here $z - 4 = -3(x + 2)^2 - y^2 + (x + 2)^3$ so that $z = 4$ is a maximum. (*See Fig.* 27 (a), *Chap. III.*)

(iii) $z = f(x, y) = x^2 + y^2 - \frac{1}{2}x^4$.

$f_x = 2x - 2x^3$, $f_y = 2y$, possible points are $(0, 0)$, $(\pm 1, 0)$.

x	y	z	f_{xx}	f_{xy}	f_{yy}	$f_{xy}^2 - f_{xx}f_{yy}$	Result
			$2 - 6x^2$	0	2		
0	0	0	2	0	2	-4	minimum
1	0	$\frac{1}{2}$	-4	0	2	8	minimax
-1	0	$\frac{1}{2}$	-4	0	2	8	minimax

(*See Fig.* 27 (b), *Chap. III.*)

(iv) $z = (y - x^2)^2 + x^4$, $(0, 0)$ is the only point. (*Doubtful Case.*)

The contour $(y - x^2)^2 + x^4 = 0$ is isolated at $(0, 0)$ (*minimum*). (*Fig.* 28 (*a*), *Chap. III.*)

(v) $z = (y - x^2)^2 - x^5$: saddle point at $(0, 0)$. (*Doubtful Case.*) (*See Fig.* 28 (b), *Chap. III.*)

(vi) Find the shortest distance between the two curves $y^2 = 4ax$, $y^2 = 2a(x - c)$, $(a, c > 0)$.

Let a point on the first be taken as $(at^2, 2at)$ and a point on the second as $(c + 2au^2, 2au)$. Then $F(t, u)$, the square of the distance between these points, is given by

$$F(t, u) = (at^2 - 2au^2 - c)^2 + 4a^2(t - u)^2.$$

$$F_t = 4at(at^2 - 2au^2 - c) + 8a^2(t - u),$$

$$F_u = -8au(at^2 - 2au^2 - c) - 8a^2(t - u).$$

Therefore for a maximum or minimum we must have (i) $at^2 - 2au^2 - c = 0 = t - u$, or (ii) $t = 2u$.

(i) leads to complex values (the curves do not intersect).

(ii) leads to $2au^2 = c - a$ or $u = 0$.

If (*a*) $c < a$, there is one real solution given by $t = u = 0$.

If (*b*) $c > a$, there are 3 solutions $t = u = 0$, or $t = 2u$, $u = \pm\sqrt{\left(\dfrac{c - a}{2a}\right)}$.

It will be found that for $t = u = 0$, $F_{tt} = 8a^2 - 4ac$, $F_{tu} = -8a^2$, $F_{uu} = 8a^2 + 8ac$ and therefore $F_{tu}^2 - F_{tt}F_{uu} = 32a^2c(c - a)$.

When $c < a$, the one solution $t = u = 0$ gives a *minimum* (since $F_{tt} > 0$).

When $c > a$, this solution gives a minimax.

For the other solutions, $F_{tt} = 4a(4c - 3a)$, $F_{tu} = 8a(a - 2c)$, $F_{uu} = 16ac$ and $F_{tu}^2 - F_{tt}F_{uu} = 64a^2(a - c) < 0$ and therefore these give minima since $4c > 3a$.

When $c = a$, the shortest distance is a, and this is an example of the *doubtful case*. The distance must be a minimum since the function is unbounded above and is positive. Actually $F(t, u)$ may be written $a^2\{1 + 2(t - 2u)^2 + (t^2 - 2u^2)^2\}$ from which it is also obvious that $t = u = 0$ gives a minimum.

Thus the shortest distance is (i) c when $c \leq a$, (ii) $\sqrt{\{a(2c - a)\}}$ when $c > a$.

7.3. Maxima and Minima of Functions of Several Variables.

By an obvious extension of the method for two variables we find that

the possible values of $x_1, x_2, \ldots, x_m$ that will give a maximum or minimum value to $f(x_1, x_2, \ldots, x_m)$ are obtained by solving the equations

$$\frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_2} = \ldots = \frac{\partial f}{\partial x_m}.$$

When $(a_1, a_2, \ldots, a_m)$ is a solution of these equations, the nature of the solution is determined by considering the sign of the expression

$$\sum_2^n \left(h_1 \frac{\partial}{\partial a_1} + \ldots + h_m \frac{\partial}{\partial a_m}\right)^r f + O(\rho^{n+1})$$

where $\rho = (h_1^2 + h_2^2 + \ldots + h_m^2)^{\frac{1}{2}}$.

When the second derivatives do not all vanish at $(a_1, \ldots, a_m)$ the nature of the solution (except in the doubtful case) may be determined by finding the conditions under which the quadratic form

$$\left(h_1 \frac{\partial}{\partial a_1} + \ldots + h_m \frac{\partial}{\partial a_m}\right)^2 f$$

is of constant sign.

An indication of the character of the results to be expected is obtained by considering the case $m = 3$.

For a discussion of the general case, see Bromwich, '*Quadratic Forms and their Classification by Means of Invariant Factors*,' *Cambridge Tract No. 3*. In the general case the results are more quickly obtained by the use of invariants, but here we shall deal with the question directly.

7·31. *The Sign of* $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ $(= E)$. The numbers a, b, c, f, g, h are real constants and x, y, z are real variables.

Let A, B, C, F, G, H be the co-factors of a, b, c, f, g, h in

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

If a, b, c are not all zero, we may without loss of generality assume that $a \neq 0$.

Then $aE = (ax + hy + gz)^2 + Cy^2 - 2Fyz + Bz^2$.

If B, C are not both zero, we may without loss of generality assume $C \neq 0$; then

$$E = \frac{1}{a}(ax + hy + gz)^2 + \frac{1}{aC}(Cy - Fz)^2 + \frac{\Delta}{C}z^2.$$

If $a \neq 0$, $B = C = 0$, $E = \frac{1}{a}(ax + hy + gz)^2 - 2Fyz$.

If $a = b = c = 0$, $E = 2hxy + 2fyz + 2gzx$ (f, g, h not all zero). Then

(i) $a = b = c = 0$, $(\Delta = 2fgh)$; E is *not* invariable in sign whether Δ vanishes or not, since (when $h \neq 0$) $(x_0, y_0, 0)$, $(x_0, -y_0, 0)$ give *opposite* signs to E.

(ii) $a \neq 0$, $B = C = 0$, $\Delta \neq 0$ (so that $F \neq 0$); E is *not* invariable, since (x_0, y_0, z_0), $(x_0, -y_0, z_0)$ where $ax_0 + hy_0 + gz_0 = 0$ give *opposite* signs to E.

$a \neq 0$, $B = C = 0$, $\Delta = 0$ (so that $F = 0$), E is invariable for all displacements *except* those in the *plane* $ax + hy + gz = 0$ where $E = 0$.

(iii) $a \neq 0$, $C \neq 0$, $\Delta \neq 0$; E is invariable if *either* $a > 0$, $C > 0$, $\Delta > 0$ or $a < 0$, $C > 0$, $\Delta < 0$ but not otherwise.

$a \neq 0$, $C \neq 0$, $\Delta = 0$; E is *not* invariable if $C < 0$, since E has *real* factors; but E is invariable if $C > 0$ for all displacements *except* those along the line $ax + hy + gz = 0 = Cy - Fz$.

7.32. *Conditions for a Maximum or Minimum of $f(x, y, z)$.* Let (a, b, c) be a solution of the equations $f_x = 0 = f_y = f_z$. Then if the second derivatives of $f(x, y, z)$ do not all vanish at (a, b, c), the sign of

$$f(a + h,\ b + k,\ c + l) - f(a,\ b,\ c)$$

is that of the quadratic form.

$h^2 f_{aa} + k^2 f_{bb} + l^2 f_{cc} + 2hk f_{ab} + 2kl f_{bc} + 2lh f_{ca}$ when h, k, l are small.

Whether this is invariable in sign or not can be determined by the results of the last paragraph. The only cases that are doubtful so far as the terms of the second degree are concerned are (i) $\Delta = 0$, $C > 0$ (ii) $\Delta = 0$ and all its first minors zero where

$$C = \begin{vmatrix} f_{aa} & f_{ab} \\ f_{ab} & f_{bb} \end{vmatrix},\quad \Delta = \begin{vmatrix} f_{aa} & f_{ab} & f_{ac} \\ f_{ab} & f_{bb} & f_{bc} \\ f_{ac} & f_{bc} & f_{cc} \end{vmatrix}$$

(with the appropriate modifications in (i) when the first minors do not *all* vanish).

In the doubtful case, the terms of higher order must be considered.

7.33. *The Conditions for a Function of m Variables.* By similar reasoning to the above, if $(a_1, \ldots, a_m)$ is a solution of the equations $\dfrac{\partial f}{\partial x_1} = 0 = \dfrac{\partial f}{\partial x_2} = \ldots = \dfrac{\partial f}{\partial x_m}$ where $f(x_1, x_2, \ldots, x_m)$ is a given function of m variables and if

$$\Delta = \begin{vmatrix} \dfrac{\partial^2 f}{\partial a_1^2} & \dfrac{\partial^2 f}{\partial a_1 \partial a_2} & \cdots & \dfrac{\partial^2 f}{\partial a_1 \partial a_m} \\ \dfrac{\partial^2 f}{\partial a_1 \partial a_2} & \dfrac{\partial^2 f}{\partial a_2^2} & \cdots & \dfrac{\partial^2 f}{\partial a_2 \partial a_m} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 f}{\partial a_1 \partial a_m} & \dfrac{\partial^2 f}{\partial a_2 \partial a_m} & \cdots & \dfrac{\partial^2 f}{\partial a_m^2} \end{vmatrix}$$

is not zero, it can definitely be established that $(a_1, \ldots, a_m)$ gives either (i) a maximum, (ii) a minimum or (iii) a minimax; whilst if $\Delta = 0$, it can be established that $(a_1, \ldots, a_m)$ gives a minimax or that the case is *doubtful*.

Examples.

(i) Let $f(x, y, z) = 2xyz - 4xz - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$.

$f_x = 2yz - 4z + 2x - 2$; $f_y = 2xz - 2z + 2y - 4$; $f_z = 2xy - 4x - 2y + 2z + 4$;

$f_{xx} = 2$; $f_{xy} = 2z$; $f_{xz} = 2y - 4$; $f_{yy} = 2$; $f_{yz} = 2x - 2$; $f_{zz} = 2$.

Thus

x	y	z	V	f_{xx}	f_{yy}	f_{zz}	f_{yz}	f_{zx}	f_{xy}	Result
1	3	0	5	2	2	2	0	0	0	minimum
0	3	1	4	2	2	2	2	2	2	minimax
2	1	1	-4	2	2	2	2	2	2	minimax
0	1	1	4	2	2	2	2	2	2	minimax
2	3	1	4	2	2	2	2	2	-2	minimax

The quadratic for (1, 3, 0) is $h^2 + k^2 + l^2$ (> 0).

The quadratic for (0, 3, 1) is

$$h^2 + k^2 + l^2 - 2kl + 2lh + 2hk = (h + k + l)^2 + (k - l)^2 - (k + l)^2, \ (\lessgtr 0)$$

and similarly for the others

(ii) $V = x^3y^2z^4(20 - 2x - 6y - 4z)$

The co-ordinates of any point on one of the planes $x = 0$, $y = 0$, $z = 0$ satisfies the equations $V_x = V_y = V_z = 0$ and V is zero throughout each of these planes. But no point on any of them can be a maximum or minimum; for any displacement in one of them does not alter the value of V, and points exist in each of them, in the neighbourhood of which V is not invariable in sign for displacements in three dimensions.

$$V_x = 2xy^3z^4(20 - 3x + 6y - 4z); \quad V_y = 6x^2y^2z^4(10 - x + 4y - 2z);$$
$$V_z = 4x^2y^3z^3(20 - 2x + 6y - 5z)$$

and these vanish when $x = 2$, $y = -1$, $z = 2$.

Take $x = 2 + h$, $y = -1 + k$, $z = 2 + l$ and we find that

$$V = -128 + \tfrac{32}{3}\{(3h - 6k - 4l)^2 - (6k - 2l)^2 + 10l^2\} + O\{(h^2 + k^2 + l^2)^{\frac{3}{2}}\}.$$

Thus V has a relative minimum value -128.

(iii) $V = x^4y^4z^4u^4 - 4(x + y + z + u)$.

$$V_x = 0 = V_y = V_z = V_u \text{ when } x = y = z = u = 1.$$

Take $x = 1 + h$, $y = 1 + k$, $z = 1 + l$, $u = 1 + p$ and we find that

$$V = -15 + (6h^2 + \ldots + 16hk + \ldots) + O(\rho^3), \ (\rho^2 = h^2 + k^2 + l^2 + p^2).$$

The quadratic terms may be written

$$6(h + \tfrac{4}{3}k + \tfrac{4}{3}l + \tfrac{4}{3}p)^2 - \tfrac{14}{3}(k + \tfrac{2}{7}l + \tfrac{2}{7}p)^2 - \tfrac{30}{7}(l + \tfrac{4}{15}p)^2 - \tfrac{24}{5}p^2$$

and therefore $V = -15$ is a minimax.

7.4. Restricted Maxima and Minima.

A problem of a more difficult type arises when we wish to determine the maxima and minima of a function V of m variables $x_1, x_2, \ldots, x_m$, these variables being connected by the s ($< m$) relations

$$\phi_1(x_1, \ldots, x_m) = 0, \ \phi_2(x_1, \ldots, x_m) = 0, \ \ldots, \phi_s(x_1, \ldots, x_m) = 0.$$

Even when it is possible to express V as a function of $(m - s)$ variables, the maximum and minimum values of V may not always be correctly obtained by this procedure, since the $(m - s)$ variables are not independent variables in the ordinary sense. They may not be free, owing to the restricting equations, to take all real numbers for their values. Also when an arbitrary choice is made of the so-called independent variables, the determination of the others may not be unique and in some cases may be impossible.

Example Let $V = 4x + y + y^2$ where (x, y) lies on the circle

$$x^2 + y^2 + 2x + y = 1.$$

Actually $V = 1 + 2x - x^2$ and if we regard this simply as a function of x, we find that V has apparently a maximum value 2 when $x = 1$. But when $x = 1$, $y^2 + y + 2 = 0$ and there is no real value of y to correspond. Thus V does not appear to have a maximum or minimum from this point of view, although actually, as we shall show below, V has the maximum value $\frac{7}{4}$ when $x = \frac{1}{2}$, $y = -\frac{1}{2}$.

7.41 The Method of Lagrange. In this method, equal importance is given to the variables $x_1, \ldots, x_n$, but to obtain the required result we assume that a correct choice is made of the independent variables. The method will be sufficiently indicated if we consider the problem of determining the maxima and minima of $V(x, y, z, u)$ subject to the conditions

$$\phi(x, y, z, u) = 0, \quad \psi(x, y, z, u) = 0.$$

If the six Jacobians $\dfrac{\partial(\phi, \psi)}{\partial(x, y)}, \dfrac{\partial(\phi, \psi)}{\partial(x, z)}, \dfrac{\partial(\phi, \psi)}{\partial(x, u)}, \dfrac{\partial(\phi, \psi)}{\partial(y, z)}, \dfrac{\partial(\phi, \psi)}{\partial(y, u)}, \dfrac{\partial(\phi, \psi)}{\partial(z, u)}$ all vanish *identically*, ϕ is not functionally distinct from ψ, and we may therefore assume that at least one of them, say $\dfrac{\partial(\phi, \psi)}{\partial(z, u)}$, is not zero.

Note. The six Jacobians may all vanish for a particular value (x_0, y_0, z_0, u_0) which is therefore a 'singular point' for the relation $\phi = 0 = \psi$. It will be assumed in what follows that the point determining the stationary values of V is not singular. If such points do exist, however, in a particular case, there is no reason to suppose that they do not provide actual maxima or minima, although the ordinary analytical conditions are not satisfied there.

Taking therefore x, y as the independent variables, the differentials dV, dz, du that correspond to dx, dy are given by

$$dV = V_x\,dx + V_y\,dy + V_z\,dz + V_u\,du,$$
$$0 = \phi_x\,dx + \phi_y\,dy + \phi_z\,dz + \phi_u\,du,$$
$$0 = \psi_x\,dx + \psi_y\,dy + \psi_z\,dz + \psi_u\,du.$$

For a stationary value of V we must have $dV = 0$ for arbitrary dx, dy (regarding V for the moment as a function of x, y). But since $\dfrac{\partial(\phi, \psi)}{\partial(z, u)} \neq 0$, numbers λ, μ can be found such that

$$V_z + \lambda\phi_z + \mu\psi_z = 0, \quad V_u + \lambda\phi_u + \mu\psi_u = 0$$

from which it follows that

$$0 = dV = (V_x + \lambda\phi_x + \mu\psi_x)dx + (V_y + \lambda\phi_y + \mu\psi_y)dy$$

i.e.
$$V_x + \lambda\phi_x + \mu\psi_x = 0, \quad V_y + \lambda\phi_y + \mu\psi_y = 0$$

The equations

$$\text{(i) } V_x + \lambda\phi_x + \mu\psi_x = 0; \quad \text{(ii) } V_y + \lambda\phi_y + \mu\psi_y = 0;$$
$$\text{(iii) } V_z + \lambda\phi_z + \mu\psi_z = 0; \quad \text{(iv) } V_u + \lambda\phi_u + \mu\psi_u = 0;$$
$$\text{(v) } \phi = 0, \quad \text{(vi) } \psi = 0$$

determine the possible values of x, y, z, u, λ, μ. They are the same equations that would be obtained if any other (correct) selection of independent variables had been made. Thus if a solution of the above equations gives a point for which $\dfrac{\partial(\phi, \psi)}{\partial(z, u)} = 0$, there must be some other Jacobian that does not vanish (since we have assumed that they do not all vanish at the same point).

Note. The elimination of λ, μ from these equations (three at a time) gives

$$\frac{\partial(V, \phi, \psi)}{\partial(x, y, z)} = 0, \quad \frac{\partial(V, \phi, \psi)}{\partial(x, y, u)} = 0, \quad \frac{\partial(V, \phi, \psi)}{\partial(x, z, u)} = 0, \quad \frac{\partial(V, \phi, \psi)}{\partial(y, z, u)} = 0.$$

These Jacobians taken in pairs correspond to the six possible selections of independent variables; and they should all vanish at a stationary value. But although

it is *usually* the case that the vanishing of two of them implies the vanishing of the remaining two, this is not always true. For example, if $V_x = 0$, $\frac{\partial(\phi, \psi)}{\partial(x, y)} = 0$, $\frac{\partial(\phi, \psi)}{\partial(x, z)} = 0$, $\frac{\partial(\phi, \psi)}{\partial(x, u)} = 0$, the first three Jacobians vanish but not the last unless some other condition is satisfied. Thus any method that involves a particular selection of independent variables only may not provide the correct solution.

Example. Let $V = x^2 + y^2$ where $\phi = x^2 + z^2 + u^2 - a^2 = 0$,

$$\psi = y^2 - 2z^2 - 3u^2 - b^2 = 0.$$

If we regarded V as a function of x, y merely we should obtain only the solution $x = 0$, $y = 0$, $u = \pm\sqrt{(b^2 - 2a^2)}$, $z = \pm\sqrt{(3a^2 - b^2)}$, but this does not give all the possibilities. By Lagrange's method we find

$$2x + 2\lambda x = 0,\ 2y + 2\mu y = 0,\ 2z(\lambda + 2\mu) = 0,\ 2u(\lambda + 3\mu) = 0$$

from which we find

(i) $x = y = 0$, $\lambda = \mu = 0$, $u = \pm\sqrt{(b^2 - 2a^2)}$, $z = \pm\sqrt{(3a^2 - b^2)}$,
(ii) $x = u = 0$, $\lambda = 2$, $\mu = -1$, $z = \pm a$, $y = \pm\sqrt{(b^2 - 2a^2)}$,
(iii) $y = z = 0$, $\lambda = -1$, $\mu = \frac{1}{3}$, $x = \pm\sqrt{(a^2 - \frac{1}{3}b^2)}$, $u = \pm b/\sqrt{3}$,
(iv) $z = u = 0$, $\lambda = -1$, $\mu = -1$, $x = \pm a$, $y = \pm b$

where we assume $3a^2 > b^2 > 2a^2$.

The other two possibilities $x = z = 0$, $y = u = 0$, lead to imaginary values of y, x respectively.

In this case $\frac{\partial(V, \phi, \psi)}{\partial(x, y, z)} = 24xyz$, $\frac{\partial(V, \phi, \psi)}{\partial(x, y, u)} = 32xyu$, $\frac{\partial(V, \phi, \psi)}{\partial(x, z, u)} = 8xzu$, $\frac{\partial(V, \phi, \psi)}{\partial(y, z, u)} = 8yzu$ and we can make the first three Jacobians vanish (taking $x = 0$) without making the last vanish. By assuming that they all vanish, we obtain the six possible solutions.

7.42. Lagrange's Method for the General Case. To determine the stationary values of $V(x_1, x_2, \ldots, x_m)$ when

$$\phi_1(x_1, \ldots, x_m) = \phi_2(x_1, \ldots, x_m) = \ldots = \phi_s(x_1, \ldots, x_m) = 0$$
$$(s < m),$$

form the function $E = V + \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_s\phi_s$, and write down the m equations for determining the maxima and minima of E, as if it were a function of m independent variables (and λ_r constant).

We obtain $\frac{\partial V}{\partial x_r} + \lambda_1\frac{\partial\phi_1}{\partial x_r} + \ldots + \lambda_s\frac{\partial\phi_s}{\partial x_r} = 0$ ($r = 1$ to m).

These m equations together with the s equations $\phi_t = 0$ ($t = 1$ to s), determine the possible values of $\lambda_1, \lambda_2, \ldots, \lambda_s, x_1, x_2, \ldots, x_m$.

7.43. Discrimination between the Stationary Values. The quadratic form that gives the first approximation to

$$V(a_1 + h_1, \ldots, a_m + h_m) - V(a_1, \ldots, a_m)$$

near a stationary value $(a_1, \ldots, a_m)$ is most simply obtained by expanding $E = V + \sum_{t=1}^{s}\lambda_t\phi_t$ which of course has the same value as V.

This quadratic form is $\frac{1}{2}\left(h_1\frac{\partial}{\partial a_1} + \ldots + h_m\frac{\partial}{\partial a_m}\right)^2 E$, but in this case the displacements h_r are not independent but are subject to the conditions

$$\sum_{r=1}^{m} h_r\frac{\partial\phi_t}{\partial a_r} = 0 \ (t = 1 \text{ to } s).$$

Sometimes the quadratic form is of invariable sign (positive or negative definite) whether the restricting equations are satisfied or not; but we cannot conclude that the point does not give a true maximum or minimum when the form is not definite.

Examples. Let the quadratic be $Ah_1^2 + Bh_2^2 + Ch_3^2$ with one condition $uh_1 + vh_2 + wh_3 = 0$; and suppose that $ABC \neq 0$, $w \neq 0$.

(i) If $A, B, C > 0$ there is a minimum, and there is a maximum if $A, B, C < 0$.

(ii) If $A, B > 0$, $C < 0$, the quadratic is

$$\frac{1}{w^2}\{h_1^2(Aw^2 + Cu^2) + 2Cuvh_1h_2 + h_2^2(Bw^2 + Cv^2)\}.$$

There is, therefore, a maximum (or minimum) if $\dfrac{u^2}{A} + \dfrac{v^2}{B} + \dfrac{w^2}{C} < 0$. There is a minimax if $\dfrac{u^2}{A} + \dfrac{v^2}{B} + \dfrac{w^2}{C} > 0$ and the case is doubtful if $\dfrac{u^2}{A} + \dfrac{v^2}{B} + \dfrac{w^2}{C} = 0$.

(iii) Find the maximum value of $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$ where

$$x_1 + x_2 + \ldots + x_m = c,$$

the numbers $x_1, \ldots, x_m, c$ being, for simplicity, assumed > 0.

If $E = \log(x_1^{\alpha_1} \ldots x_m^{\alpha_m}) + \lambda(x_1 + \ldots + x_m - c)$ we find $\dfrac{\partial E}{\partial x_r} = \dfrac{\alpha_r}{x_r} + \lambda = 0$ $(r = 1 \text{ to } m)$.

The only solution is therefore given by $x_r = c\dfrac{\alpha_r}{\alpha_1 + \ldots + \alpha_m}$.

Again $\dfrac{\partial^2 E}{\partial x_r^2} = -\dfrac{\alpha_r}{x_r^2}$, $\dfrac{\partial^2 E}{\partial x_r \partial x_s} = 0$. The quadratic is therefore negative (without reference to the restricting condition), and the value obtained gives a maximum.

We therefore find that

$$x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m} < \frac{(x_1 + x_2 + \ldots + x_m)^s}{(\alpha_1 + \alpha_2 + \ldots + \alpha_m)^s}\alpha_1^{\alpha_1}\alpha_2^{\alpha_2} \ldots \alpha_m^{\alpha_m}$$

where $s = \alpha_1 + \alpha_2 + \ldots + \alpha_m$.

This result may also be written

$$p_1X_1 + p_2X_2 + \ldots + p_mX_m > X_1^{p_1}X_2^{p_2} \ldots X_m^{p_m}$$

where $\sum_1^m p_r = 1$, (by writing $x_r = \alpha_r X_r$ and $p_r(\sum_1^m \alpha_s) = \alpha_r$).

(iv) Discuss the stationary values of $V = 4x + y + y^2$ where

$$x^2 + y^2 + 2x + y - 1 = 0.$$

$$E = 4x + y + y^2 + \lambda(x^2 + y^2 + 2x + y - 1).$$

$$E_x = 4 + \lambda(2x + 2) = 0; \quad E_y = 1 + 2y + \lambda(2y + 1) = 0.$$

$\lambda = -1$ gives $x = 1$ and the value of y is imaginary.

$y = -\frac{1}{2}$ gives $x = -\frac{5}{2}$, $\lambda = \frac{4}{3}$, $V = -\frac{41}{4}$ or $x = \frac{1}{2}$, $\lambda = -\frac{4}{3}$, $V = \frac{7}{4}$.

The quadratic is $\lambda h^2 + (1 + \lambda)k^2$ where $h = 0$.

Thus $(\lambda = \frac{4}{3})$, V is a minimum when $x = -\frac{5}{2}$, $y = -\frac{1}{2}$

and $(\lambda = -\frac{4}{3})$, V is a maximum when $x = \frac{1}{2}$, $y = -\frac{1}{2}$.

This example illustrates how it is not sufficient to express V as a function of x, viz. $2x - x^2 + 1$.

(v) Find the stationary values of $V = x^2 + y^2 + z^2$ when $lx + my + nz = 0$ and $ax^2 + by^2 + cz^2 = 1$.

(V in this example is R^2 where R is the length of a semi-diameter of the section of $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = 0$. The stationary values of V determine the principal axes of the section.)

$$E = V + \lambda(lx + my + nz) + \mu(ax^2 + by^2 + cz^2 - 1).$$

$$E_x = 2x + \lambda l + 2\mu ax = 0; \quad E_y = 2y + \lambda m + 2\mu by = 0;$$

$$E_z = 2z + \lambda n + 2\mu cz = 0$$

Therefore $2V - 2\mu = 0$, $2x(1 + \mu a) = l\lambda$, $2y(1 + \mu b) = m\lambda$; $2z(1 + \mu c) = n\lambda$.

Substituting in $lx + my + nz = 0$ we find

$$\frac{l^2}{1 + \mu a} + \frac{m^2}{1 + \mu b} + \frac{n^2}{1 + \mu c} = 0$$

i.e. the stationary values of V are given by the quadratic

$$\frac{l^2}{1 - aV} + \frac{m^2}{1 - bV} + \frac{n^2}{1 - cV} = 0.$$

(vi) A closed rectangular box is to be made from 300 sq. cm. of sheet metal in such a way that the perimeter of its base is 25 times the height of the box. Find the dimensions that give maximum capacity.

If x, y are the dimensions of the base and z the height of the box, we require the maximum value of $V = xyz$ where $xy + yz + zx = 150$, $2x + 2y = 25z$.

In this example V may be expressed easily as a function of z, but if this is done only the absolute maximum (or minimum) is obtained.

Thus $x + y = \frac{25}{2}z$, $xy = 150 - \frac{25}{2}z^2$ giving $V = 150z - \frac{25}{2}z^3$. The stationary values are $z = \pm 2$, so that $V = 200$ (maximum): whilst $z = -2$ gives the analytical minimum -200.

When $z = 2$, $x = 5$ (or 20), $y = 20$ (or 5).

However, by the method of Lagrange

$$F = xyz + \lambda(yz + zx + xy - 150) + \mu(2x + 2y - 25z), \text{ giving}$$

(i) $yz + \lambda(y + z) + 2\mu = 0$; (ii) $zx + \lambda(z + x) + 2\mu = 0$,

(iii) $xy + \lambda(x + y) - 25\mu = 0$

From (i) and (ii) we obtain $(y - x)(z + \lambda) = 0$.

(a) $z = -\lambda$ gives $z = 2$, $\mu = 2$, $xy = 100$, $x + y = 25$ (ignoring negative z) which is the solution obtained above.

Here the quadratic is $(x_0 + \lambda)kl + (y_0 + \lambda)lh + (z_0 + \lambda)hk$ where

$$(y_0 + z_0)h + (z_0 + x_0)k + (x_0 + y_0)l = 0, \quad 2h + 2k - 25l = 0$$

Thus at (5, 20, 2), the quadratic is $3kl + 18lh$ where $22h + 7k + 25l = 0$ and $2h + 2k - 25l = 0$. From these we find $h/15 = k/-40 = l/-2$, giving the value $-75l^2$ to the quadratic and verifying that the stationary value is a maximum.

(b) $y = x$, $4x = 25z$, $x^2 + 2xz = 150$; and if we consider positive values only, we obtain

$$x = 25\sqrt{\left(\frac{2}{11}\right)} = y, \quad z = 4\sqrt{\left(\frac{2}{11}\right)}, \quad \lambda = -\frac{50}{11}\sqrt{\left(\frac{2}{11}\right)}.$$

In the quadratic $l = 0$ and $k = -h$, and its value becomes $\frac{6}{11}\sqrt{\left(\frac{2}{11}\right)}h^2$, showing that the stationary value (193·8 c.c. approx.) is a (relative) minimum.

Examples VII

Evaluate the limits in *Examples 1–24*

1. $\lim_{x \to 0} \frac{\arctan x}{x}$

2. $\lim_{x \to 0} \frac{e^x - e^{-x} - 2\tan x}{\sin^3 x}$

3. $\lim_{x \to 2} \frac{x^5 - 5x^3 + 6x^2 - 4x - 8}{x^4 - 7x^3 + 18x^2 - 20x + 8}$

4. $\lim_{x \to \frac{1}{2}\pi} (2x \tan x - \pi \sec x)$

5. $\lim_{x \to 0} \left(\cot x - \frac{1}{x}\right)$

6. $\lim_{x \to 2} \frac{2 - x - \log(x - 1)}{1 - \sqrt{\{(x - 1)(3 - x)\}}}$

7. $\lim_{x \to 1+} \frac{x - 1 + \sqrt{(x - 1)}}{\sqrt{(x^2 - 1)}}$

8. $\lim_{x \to 0} \frac{\tan x(\cos x + \sin x - 1)}{1 - \cos x}$

9. $\lim_{x \to \frac{1}{2}\pi} (\tan x - \sec x)$

10. $\lim_{x \to 0+} x^4(\log x)^{10}$

11. $\lim_{x \to +\infty} x^{-4}(\log x)^{10}$

12. $\lim\limits_{x\to 1} \dfrac{x^3\sqrt{(5x-4)} - x^{6/5}}{1-x^2}$ **13.** $\lim\limits_{x\to 2} (2-\tfrac{1}{2}x)^{\tan\frac{\pi x}{4}}$ **14.** $\lim\limits_{x\to 0+} x^{-\sin x}$

15. $\lim\limits_{x\to 0} (\cos x + 2\sin x)^{\cot x}$ **16.** $\lim\limits_{x\to 0} (\cos 2x)^{3/x^2}$

17. $\lim\limits_{a\to 0} \dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$ **18.** $\lim\limits_{x\to 0+} \dfrac{e^{-1/x}}{x}$ **19.** $\lim\limits_{x\to 0-} \dfrac{e^{1/x}}{x}$

20. $\lim\limits_{x\to 1} \dfrac{A(x^p-1)+B(x^q-1)^m}{C(x^p-1)+D(x^q-1)^n}$, $(ABCDpqmn \neq 0)$

21. $\lim\limits_{x\to 0} (\operatorname{cosec}^4 x - \cot x \operatorname{cosec}^3 x - \tfrac{1}{2}x^{-4}\tan^2 x)$

22. $\lim\limits_{x\to 0} \dfrac{1}{2x}\sqrt{(1+8x+4x^2-4x^3)} - \dfrac{1}{2x}$

23. $\lim\limits_{x\to\infty} [(x^4+x^3+c_1x^2+c_2x+c_3)^{\frac{1}{4}} - (x^4+x^3+c_4x^2+c_5x+c_6)^{\frac{1}{4}}]$

24. $\lim\limits_{x\to\infty} [(x^6+6x^5+12x^4+1)^{\frac{1}{6}} - (x^6+4x^5+6x^3+1)^{\frac{1}{6}}]$

Find the expansions near $x = 0$ of the functions given in *Examples 25–9*.

25. $e^{4x}(3-2x) - 8xe^{2x}$ as far as x^4.

26. $\log \tfrac{1}{2}(1+e^x)$ as far as x^3.

27. $e^{x\sin x}$ as far as x^4 **28.** $e^{\sin x} - 1$ as far as x^3.

29. $e^{\arctan x} + \log(1+\sin x)$ as far as x^3.

Discuss the stationary values of the functions given in *Examples 30–6*.

30. $\dfrac{x^5(x-2)^2}{(3x+1)^4}$ **31.** $x^7 + x^6 - 2x^5 - 2x^4 + x^3 + x^2 + 2$

32. $3e^{4x} - 4e^{3x}$ **33.** $\dfrac{x^4}{\log x}$ **34.** x^3e^{-x}

35. $\sin^3 x\cos^2 x$ **36.** $2x - \tan x$

37. A top shaped solid consists of a hemisphere and a right circular cone, the base of the latter coinciding with the plane face of the hemisphere. If the volume V of the top is given, find the radius of the spherical part when the surface of the solid is a minimum.

38. On a triangular piece of cardboard ABC lines are drawn parallel to the sides at distances x from them so as to form within ABC a triangle $A_1B_1C_1$ similar to ABC. From A_1 perpendiculars A_1L, A_1M are drawn to AB, AC respectively and the quadrilateral AMA_1L is cut away. Quadrilaterals formed in a similar way are cut away from the other corners. The remainder is folded along A_1B_1, B_1C_1, C_1A_1 so as to form an open triangular box of base $A_1B_1C_1$ and of height x. Show that the capacity of the box is a maximum when $x = \frac{1}{3}r$ where r is the radius of the in-circle of ABC, and that the maximum capacity is $\frac{8}{27}\Delta^2/(a+b+c)$ where Δ is the area of ABC and a, b, c the lengths of its sides.

39. A particle P is allowed to slide from a point O down a smooth inclined plane making an angle α with the horizontal and at the same instant a second particle Q is projected horizontally from O with initial velocity V in the vertical plane through the line of motion of P. Prove that the distance between the particles increases steadily if $\tan\alpha < 2\sqrt{2}$ and find the instant when the distance is a relative minimum when $\tan\alpha > 2\sqrt{2}$.

40. $ABCD$ is the plan of a smooth horizontal table in which $AB = 3$ ft., $AD = 9$ ft. From the midpoint of AD a particle is projected along the table at right angles to AD with a speed of 3 ft./sec. and at the same instant a second particle is projected with a speed of 2 ft./sec. at right angles to AB from the midpoint of AB. Find the minimum distance between the particles whilst they are on the table.

41. The centres A, B of two circles of radii $18a$, $9a$ respectively are $5a$ apart. A particle P describes the larger circle with angular velocity ω, and a particle Q describes the smaller circle with angular velocity 2ω in the same sense. Initially

P is at the point C where AB meets the larger circle, and Q is at the point D where the perpendicular to AB from B meets the smaller circle. The change of direction from BC to BD is the same as that of the angular velocities. Find the maximum and minimum distance between the particles.

42. A window consists of a rectangle $a \times b$ and a semicircle on the side of length a as diameter. Show that, for a fixed perimeter, the maximum amount of light is admitted when $a = 2b$. Find also the corresponding results when

(i) there are *two* semicircles, one on each of the sides a;
(ii) there are *two* semicircles, one on a side a and one on a side b,
(iii) there are *three* semicircles, two on the sides b and one on a side a;
(iv) there are *four* semicircles, one on each side.

43. A picture of length a has its lower edge in contact with a vertical wall at a height h above the ground. The picture is inclined at an angle α ($< \frac{1}{2}\pi$) to the wall. A man, whose eye is at a distance c ($< h$) from the floor, stands opposite the middle of the picture at such a distance that the angle subtended at his eye by the picture lengthwise is as large as possible. Prove that his distance from the wall is

$$\sec\alpha\{(h-c)^2 + a(h-c)\cos\alpha\}^{\frac{1}{2}} - (h-c)\tan\alpha$$

44. Show that the overflow y caused by lowering a sphere of radius x into a hollow circular cone filled with water, the cone being held with its axis vertical and vertex downwards is given by $y = \frac{4}{3}\pi x^3$, $(0 \le x < \frac{1}{3})$; $\frac{1}{3}\pi(2x-1)^2(5x-1)$, $(\frac{1}{3} < x \le 1)$; $\frac{1}{3}\pi\{2x^3 - (x^2-\frac{1}{4})^{\frac{1}{2}}(2x^2+\frac{1}{4})\}$, $(x > 1)$, where unity is the height of the cone, and its semi-vertical angle is $\arcsin\frac{1}{3}$. Show that y and $\dfrac{dy}{dx}$ are continuous for all x (≥ 0). Find the value of x that gives maximum overflow.

Discuss the stationary values of the functions given in *Examples 45–71.*

45. $34x^2 - 24xy + 41y^2$ **46.** $3x^2 + 4xy - 4y^2$ **47.** $x^2y - 4x^2 - y^2$

48. $x^2y + 2x^2 - 2xy + 3y^2 - 4x - 7y$ **49.** $2x^2y - x^2 - y^2 + 2y$

50. $xy^2 - 2xy - 2x^2 - 3x$ **51.** $6x - 3y - 3x^2 - 2xy + 2y^2 - x^2y$

52. $x^3 - 4x^2 + 3y^2 - 5x - 6y$ **53.** $x^2y(x + 2y - 4)$

54. $x^3 - 4x^2 + 4x - 3y^2 - 6y$ **55.** $y^3 - x^2y - 2x^2 + 3y^2 + 2xy - 4x - 2y$

56. $x^4 - x^2 - 2x^2y + y^2$ **57.** $x^3 - 4x^2 - 4x^2y - y^2$

58. $x^4 - x^2 - 2x^2y - y^2$ **59.** $2\sin(x + 2y) + 3\cos(2x - y)$

60. $\sin^2 x\cos y + \sin^2 y\cos x$ **61.** $x^4y^3(12 - 4x - 3y)$ **62.** $\dfrac{x^4 + y^4}{x^2 + y^2}$, $f(0,0) = 0$.

63. $\log(x^2 + y^2) - x - 2y$ **64.** $2xyz + x^2 + y^2 + z^2$

65. $x^2y^3z^4(20 - 2x + 6y - 4z)$ **66.** $\dfrac{e^{x+y+z}}{xyz}$

67. $x^2yz - 2xyz - x^2z - x^2 + y^2 + z^2 - yz - 2xz - 2x + 2y - z$

68. $\dfrac{e^{2x+y-3z}}{x^2 + y^2 + z^2}$ **69.** $3\log(x^2 + y^2 + z^2) - 2x^3 - 2y^3 - 2z^3$

70. $xyzu - x + 4y + 6z - 9u$

71. $8xyz^2 - 32xyz - 8yz^2 + 8xz^2 - x^2 + y^2 - 7z^2 - u^2 + 32xy - 32xz$
$+ 32yz + 30x - 30y + 28z + 2u$

72. Find the minimum value of $x^2 + y^2 + z^2$, (i) when $x + y + z = 3a$, (ii) when $xy + yz + zx = 3a^2$, (iii) when $xyz = a^3$.

73. Find the maximum and minimum values of $x^2 - y^2 - 3x + 15y$ when $(x + y)^2 = 4(x - y)$.

74. Prove that the distance from a fixed point $P_0(x_0, y_0)$ to a variable point P on the curve $F(x, y) = 0$ is stationary when PP_0 is normal to the curve at P.

75. If (x_1, y_1, z_1), (x_2, y_2, z_2) are two points on the curve of intersection of $lx + my + nz = 0$ and $ax^2 + by^2 + cz^2 = 1$, the distance r between these points is stationary when

$$\frac{l^2}{1 - ar^2} + \frac{m^2}{1 - br^2} + \frac{n^2}{1 - cr^2} = 0.$$

76. If (x_1, y_1, z_1), (x_2, y_2, z_2) are two points on the curve of intersection of $lx + my + nz = p$ and $ax^2 + by^2 + cz^2 = 1$, the distance r between these points is stationary when

$$\frac{l^2}{1 - akr^2} + \frac{m^2}{1 - bkr^2} + \frac{n^2}{1 - ckr^2} = 0$$

where $k = \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)\Big/\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2\right)$

77. Find the maximum and minimum values of xy when $x^2 + xy + y^2 = a^2$.

78. Find the point within the triangle ABC, the sum of the squares of whose distances from the sides is a minimum.

79. If $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx \equiv \phi(x, y, z) = 1$, $lx + my + nz = 0$, prove that the stationary values of $V = x^2 + y^2 + z^2$ are given by

$$V^2 \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} + V\{(a + b + c)(l^2 + m^2 + n^2) - \phi(l, m, n)\} - (l^2 + m^2 + n^2) = 0$$

Solutions.

1. 1 **2.** $\frac{1}{2}$ **3.** 3 **4.** -2 **5.** 0 **6.** $\frac{1}{2}$

7. $\frac{1}{\sqrt{2}}$ **8.** 2 **9.** 0 **10.** 0 **11.** 0 **12.** $\frac{59}{30}$

13. $e^{2/\pi}$ **14.** 1 **15.** e^2 **16.** e^{-1} **17.** c/b **18.** 0

19. r **20.** $\frac{aA}{cC}$ $(m = 1)$, $\frac{Bp^m}{Dq^m}$ $(m > 1)$; $\frac{(aA + pB)}{(cC + qD)}$ $(m < 1)$, if not indeterminate and $aA(a - p)/cC(c - q)$ if $aA + pB = 0 = cC + qD$.

21. $-\frac{1}{24}$ **22.** -2 **23.** $\frac{1}{2}(c_1 - c_2)$ **24.** -1

25. $2x + 3 + O(x^5)$ **26.** $\frac{1}{2}x + \frac{1}{3}x^3 + O(x^4)$ **27.** $1 + x^2 + \frac{1}{3}x^4 + O(x^6)$

28. $1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 + O(x^6)$ **29.** $1 + 2x + O(x^4)$

30. $x = 0$ (inflex.); $x = 2$ (min.), $x = 1$ (max.); $x = -\frac{10}{9}$ (max.)

31. $x = 0, 1$ (min.); $x = 0{\cdot}72, -0{\cdot}61$ (approx.), (max.); $x = -1$ (inflex.)

32. $x = -\log 2$ (max.) **33.** $x = e^{-1}$ (min.), $x = 0+$ (max.)

34. $x = 5$ (max.); $x = 0$ (inflex.)

35. $2n\pi$, $(2n + 1)\pi \pm \arctan\sqrt{(\frac{3}{5})}$ (min.); $(2n + 1)\pi$, $2n\pi \pm \arctan\sqrt{(\frac{3}{5})}$ (max.); $(2n + 1)\frac{\pi}{2}$ (inflex.).

36. $(n + \frac{1}{2})\pi$ (max.); $(n + \frac{3}{2})\pi$ (min.).

37. $(0{\cdot}67)V1$, approx.

38. Incentres of ABC, $A_1B_1C_1$ are the same. Area of $A_1B_1C_1$ is $\Delta(r - x)^2/r^2$.

39. $\frac{V^2}{2g}(3\tan\alpha + \sqrt{(\tan^2\alpha - 8)})\left(< \frac{V^2}{g}2\tan\alpha, \text{ when } Q \text{ reaches plane}\right)$.

40. Analytical minimum when $t = \frac{11}{12}$ but particle leaves table when $t = 1$; actual minimum is $\sqrt{(\frac{1}{2})}$ ft.

41. $10{\cdot}3s$ (min.); $20{\cdot}7s$ (max.).

42. (i) $b = 0$. (ii) $a = b$. (iii) $a = 0$. (iv) $a = b$. **44.** $x = 0{\cdot}3$.

45. $(0, 0)$ (min.). **46.** $(0, 0)$ (minimax).

47. $(0, 0)$ (max.); $(\pm 2\sqrt{2}, 4)$ (minimax).

48. $(1, -1)$ (min.); $(1 \pm \sqrt{6}, -2)$ (minimax).

49. $(0, 1)$ (minimax) **50.** $(1, 1)$ (min.); $(0, 3)$, $(0, -1)$ (minimax).

51. $(-1, -1)$ (min.), $(3, 3)$, $(-5, 3)$ (minimax).

52. $(-1, 1)$ (min.), $(-\frac{1}{3}, 1)$ (minimax).

53. $(2, \frac{1}{2})$ (min.). **54.** $(1, 1)$ (max.); $(0, 1)$, $(2, 1)$ (minimax).

55. $(1, -1)$, $(1 \pm \sqrt{3}, -2)$ (minimax). **56.** $(0, 0)$ (min.).

57. $(0, 0)$ (minimax). **58.** $(0, 0)$ (minimax).

59. $\frac{1}{16}\pi(1 + 4m + 12n)$, $\frac{1}{4}\pi(1 + 4m + 2n)$ (max.).
$\frac{1}{16}\pi(7 + 4m + 12n)$, $\frac{1}{2}\pi(1 + 2m + n)$ (min.);
$\frac{1}{16}\pi(3 + 4m + 6n)$, $\frac{1}{4}\pi(3 + 4m - n)$ (minimax.)

60. $(\frac{1}{2}m + n)\pi$, $\frac{1}{2}m\pi$ (minimax);
$(2m - \frac{1}{2}(n - 1))\pi$, $\frac{1}{2}n\pi$ (max.);
$(2m - \frac{1}{2}(n - 3))\pi$, $\frac{1}{2}n\pi$ (min.).

61. $(1, 2)$ (max.). **62.** $(0, 0)$ (min.) **63.** $(\frac{1}{2}, \frac{1}{2})$ (minimax).

64. $(0, 0, 0)$ (min.); $(1, -1, 1)$, $(-1, 1, 1)$, $(1, 1, -1)$, $(-1, -1, -1)$ (minimax).

65. $(2, -1, 2)$ (min.). **66.** $(1, 1, 1)$ (min.).

67. $(1, -1, 0)$ (min.). $(1 \pm \sqrt{2}, -2, 1)$, $(1 \mp \sqrt{2}, 0, -1)$ (minimax).

68. $(\frac{3}{7}, 1, -\frac{2}{7})$ (minimax).

69. $(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 2^{-1/2}, 2^{-1/2}), (2^{-1/2}, 0, 2^{-1/2}), (2^{-1/2}, 2^{-1/2}, 0)$ (minimax); $(3^{-1/2}, 3^{-1/2}, 3^{-1/2})$ (max.).

70. $(-6, -\frac{3}{2}, -1, \frac{3}{2})$ (minimax).

71. $(1, -1, 2, -1)$, $(1 \pm \frac{1}{2}\sqrt{2}, -1 \mp \frac{1}{2}\sqrt{2}, \frac{3}{2}$ or $\frac{1}{2}, -1)$ (minimax).

72. Min. $3a^2$ for (i) (a, a, a); (ii) (a, a, a), $(-a, -a, -a)$; (iii) (a, a, a), $(-a, -a, a)$, $(-a, a, -a)$, $(a, -a, -a)$.

73. -58.5 at $(1.5, -7.5)$ (min.); 5.5 at $(1.5, 0.5)$ (max.); 4 at $(4, 0)$ (min.).

77. $-a^3$ (min.) at $(a, -a)$, $(-a, a)$, $\frac{1}{3}a^3$ (max.) at $(a/\sqrt{3}, a/\sqrt{3})$ $(-a/\sqrt{3}, -a/\sqrt{3})$.

78. The circumcentre.

CHAPTER VIII

VECTORS. TWISTED CURVES. TENSORS.

8. Displacements and Vectors. If P, Q are two points in Euclidean space, the position of Q relative to P is assumed to be known if we are given the *absolute* magnitude of PQ and the *direction* $\overrightarrow{PQ}$. This relative quantity is called a *displacement* and may conveniently be written $\overrightarrow{PQ}$. The absolute magnitude of PQ is called the *modulus* (or module) of $\overrightarrow{PQ}$. If P, Q, R are three non-collinear points (*Fig. 1*), it is assumed in this space that there is only one parallelogram $PQSR$ whose coterminous sides are $\overrightarrow{PQ}$, $\overrightarrow{PR}$; and since $\overrightarrow{PQ}$, $\overrightarrow{RS}$ have the same modulus and direction, $\overrightarrow{PQ} = \overrightarrow{RS}$. Similarly $\overrightarrow{PR} = \overrightarrow{QS}$.

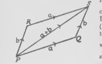


FIG. 1

The operation indicated by $\overrightarrow{PQ} + \overrightarrow{QS}$ is defined to be $\overrightarrow{PS}$ and is therefore equal to $\overrightarrow{PR} + \overrightarrow{RS}$ which is the same as $\overrightarrow{QS} + \overrightarrow{PQ}$. This operation defines the *sum* of two displacements and implies that the *commutative law* of addition is obeyed.

Any quantity that has direction and magnitude and obeys this law of summation is called a *vector*; and the displacement vector provides a suitable geometrical representation of other vectors (such as relative velocity or force).

Note. It is not implied in the above that all physical vectors are completely specified by their magnitudes and directions. Thus the effect of a force on a rigid body, for example, depends on the line of application of the force as well as its magnitude and direction.

In printed work it is customary to use Clarendon type to represent vectors such as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{F}$, but in manuscript it will be found more convenient to place a bar or arrow over the symbol representing the vector.

The definition of the sum of two vectors $\mathbf{a}$, $\mathbf{b}$ implies that

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

and by repeated applications of this result it follows that

$$\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n$$

is independent of the order of summation.

8.01. Subtraction of Vectors. The vector whose modulus is the same as that of $\mathbf{a}$ but whose direction is *opposite* to that of $\mathbf{a}$ is written $-\mathbf{a}$; and the expression $\mathbf{a} - \mathbf{b}$, which defines *subtraction* is defined to mean $\mathbf{a} + (-\mathbf{b})$.

8.02. Scalars and Scalar Multiplication. A quantity (such as mass or modulus) which is completely specified by its magnitude only, is called a *scalar* (or *invariant*) and if k is a scalar, $k\mathbf{a}$ is defined to be a vector whose magnitude is k times the modulus of $\mathbf{a}$ and whose direction is that of $\mathbf{a}$. It is convenient in practice to use both *positive* and *negative* scalars although these are strictly *directed* quantities. Thus whilst such a vector as $-2\mathbf{a}$ is strictly $2(-\mathbf{a})$ where k $(=2)$ is a signless number, it is more useful to regard it as $(-2)\mathbf{a}$ where $k\,(=-2)$ is an *algebraic* quantity. In using the term magnitude, therefore, we shall assume that this is an algebraic magnitude and may be positive or negative. When, for example, the modulus of $\mathbf{a}$ is a, the magnitude of

$$k_1\mathbf{a} + k_2\mathbf{a} + \ldots + k_n\mathbf{a}$$

is $(k_1 + k_2 + \ldots + k_n)a$.

8.03. Position-vectors. Co-ordinates. Components. Let three mutually perpendicular lines $X'OX$, $Y'OY$, $Z'OZ$ be drawn through a point O (the *origin* of reference) and let the directions of these lines $\overrightarrow{X'OX}$, $\overrightarrow{Y'OY}$,

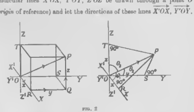


FIG. 2

$\overrightarrow{Z'OZ}$ (*axes of reference*) be specified by the *unit* vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively. Let Q be the projection of any point P on the plane XOY and S the projection of Q on OY. (*Fig. 2.*) Then $\overrightarrow{OS} = x\mathbf{i}$, $\overrightarrow{SQ} = y\mathbf{j}$, $\overrightarrow{QP} = z\mathbf{k}$, where x, y, z are scalars (any real numbers positive or negative). If the vector $\overrightarrow{OP}$ be denoted by $\mathbf{r}$, we have

$$\overrightarrow{OP} = \mathbf{r} = \overrightarrow{OS} + \overrightarrow{SQ} + \overrightarrow{QP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

The numbers x, y, z are called the *Rectangular Cartesian Co-ordinates* of the point P, referred to these axes; and $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ is called the *position-vector* of the point P. The planes XOY, YOZ, ZOX are called the *co-ordinate planes*.

In *Fig. 2* a point P is shown not on a co-ordinate plane and the rectangular parallelopiped is drawn, one corner of which is formed by the co-ordinate planes at O, the opposite corner being P. The dimensions of this solid are $|x|$, $|y|$, $|z|$.

Notes. (i) In the above formula for the position-vector the axes need not be rectangular, but the assumption of rectangular axes will usually simplify metrical results. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors whose directions are not all parallel to the same plane; then the position vector $\overrightarrow{OP}$ can be represented by $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$, the axes $\overrightarrow{X'OX}$, $\overrightarrow{Y'OY}$, $\overrightarrow{Z'OZ}$ being specified by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively and $|x|, |y|, |z|$ being the distances of P from the planes YOZ, ZOX, XOY respectively, measured parallel to the axes OX, OY, OZ.

(ii) If P_1, P_2 have co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2) respectively, the displacement $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$. Thus $\overrightarrow{P_1P_2}$ is the position-vector of the point $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. These co-ordinates $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$ are called the *components* of $\overrightarrow{P_1P_2}$; thus any vector is specified if its components are given.

(iii) In n dimensions, a vector may be specified similarly by means of n co-ordinates (or components) $x_1, x_2, \ldots, x_n$ and may be indicated by the comprehensive symbol x_r (where it is understood that r takes the values 1 to n). The *sum* of two such vectors x_r, y_r is then $x_r + y_r$.

8.04. *Direction Cosines.* Let the angles between $\overrightarrow{OP}$ and $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ (*Fig. 2*) be $\theta_1, \theta_2, \theta_3$ respectively; then $\cos\theta_1$, $\cos\theta_2$, $\cos\theta_3$ (which are not ambiguous) are called the *direction cosines* of $\overrightarrow{OP}$ with respect to the rectangular axes $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$.

Obviously $x = r\cos\theta_1$, $y = r\cos\theta_2$, $z = r\cos\theta_3$ where (x, y, z) are the co-ordinates of P and r is the modulus of $\overrightarrow{OP}$,

i.e. $$\overrightarrow{OP} = r(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$$

where (l, m, n) are the direction cosines of $\overrightarrow{OP}$.

Since $OP^2 = OR^2 + RQ^2 + QP^2$, we have

$$r^2 = r^2(\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3), \text{ i.e. } l^2 + m^2 + n^2 = 1,$$

an identical relation connecting l, m, n.

Again, if P_1, P_2 have co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2) respectively

$$\begin{aligned}\overrightarrow{P_1P_2} &= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}\\ &= r_{12}(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})\end{aligned}$$

where r_{12} is the modulus of $\overrightarrow{P_1P_2}$ and l, m, n the direction cosines of $\overrightarrow{P_1P_2}$, i.e. the direction cosines of the line joining P_1, P_2 are *proportional* to $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$.

8.05. *Vector-projection.* Let a given direction be specified by a *unit* vector $\mathbf{c}$. The *vector projection* of $\mathbf{a}$ in the direction $\mathbf{c}$ is defined to be $(a\cos\theta)\mathbf{c}$, where a is the modulus of $\mathbf{a}$ and θ the angle between $\mathbf{a}$ and $\mathbf{c}$. This definition is not ambiguous since $\cos(-\theta) = \cos\theta$. The magnitude of the projection is $a\cos\theta$ which may be positive or negative, the

positive direction of the line c being that of **c**. The projection of a *point* on a line is defined to be the foot of the perpendicular from the point to the line and therefore if $\overrightarrow{P_1P_2}$ (*Fig* 3) is denoted by **a**, and Q_1, Q_2 are the projections of P_1, P_2 respectively on a line of direction **c**, then the projection of $\overrightarrow{P_1P_2}$ on **c** is $\overrightarrow{Q_1Q_2}$ ($=(a\cos\theta)\mathbf{c}$).

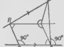


FIG. 3

Again if $\overrightarrow{P_0P_1}, \overrightarrow{P_1P_2}, \ldots, \overrightarrow{P_{n-1}P_n}$ are denoted by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and if $Q_0, Q_1, \ldots, Q_n$ are the projections of $P_0, P_1, \ldots, P_n$ on **c** then the projection of

$$\overrightarrow{P_0P_n} = \overrightarrow{Q_0Q_n} = \overrightarrow{Q_0Q_1} + \overrightarrow{Q_1Q_2} + \ldots + \overrightarrow{Q_{n-1}Q_n}.$$

But $\overrightarrow{P_0P_n} = \overrightarrow{P_0P_1} + \overrightarrow{P_1P_2} + \ldots + \overrightarrow{P_{n-1}P_n}$, i.e. the projection of $(\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n)$ is equal to sum of the projections of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

Thus the projection of the sum is the sum of the projections.

Note. The vector projections of $\overrightarrow{OP}$ on the co-ordinate axes are $x\mathbf{i}$, $y\mathbf{j}$, $z\mathbf{k}$ where (x, y, z) are the co-ordinates of P.

8.06. The Equations of a Straight Line. A straight line is specified if we are given its *direction* and the position vector of a point A on it. Let **c** denote a *unit* vector in one direction of a line and let **a** be the

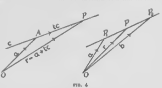


FIG. 4

position vector of A, a point on it. (*Fig. 4.*) If P is any point on the line, $\overrightarrow{AP} = t\mathbf{c}$ where t is a scalar ($+$ or $-$). Thus if $\mathbf{r} = \overrightarrow{OP}$, we have $\mathbf{r} = \mathbf{a} + t\mathbf{c}$, the vector equation of the line.

If (x_0, y_0, z_0) are the co-ordinates of A and (l, m, n) are the direction cosines of the line (in the direction of **c**)

$$\mathbf{a} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}, \quad \mathbf{c} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}.$$

so that the co-ordinates (x, y, z) of the variable point P are given by

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x_0 + lt)\mathbf{i} + (y_0 + mt)\mathbf{j} + (z_0 + nt)\mathbf{k}$$

Thus $x = x_0 + lt$, $y = y_0 + mt$, $z = z_0 + nt$ (these being the Cartesian equations of the line).

The variable scalar t is the magnitude of AP, and may be positive or negative, the positive direction of the line being that of **c**.

Conversely, the equations $x = a_1 + b_1\theta$, $y = a_2 + b_2\theta$, $z = a_3 + b_3\theta$, where θ is a variable parameter, represent a straight line through (a_1, a_2, a_3) whose direction cosines are *proportional* to (b_1, b_2, b_3).

Notes (i) Let P_1, P_2 be two points on the line, where $\overrightarrow{OP_1} = \mathbf{a}$, $\overrightarrow{OP_2} = \mathbf{b}$. Then $\overrightarrow{P_1P_2} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = \mathbf{a} + \theta(\mathbf{b} - \mathbf{a})$ where θ is a variable scalar. Thus the equation of the line joining P_1P_2 is given by $\mathbf{r} = \theta_1\mathbf{a} + \theta_2\mathbf{b}$ where $\theta_1 + \theta_2 = 1$

If the co-ordinates of P_1, P_2 are (x_1, y_1, z_1), (x_2, y_2, z_2) respectively, then $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ so that

$$x = \theta x_1 + (1 - \theta)x_2,\ y = \theta y_1 + (1 - \theta)y_2,\ z = \theta z_1 + (1 - \theta)z_2$$

or $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$ give the equations of the line joining P_1, P_2 (as is otherwise obvious)

(ii) Since $\overrightarrow{P_1P} = \theta(\mathbf{b} - \mathbf{a})$, $\overrightarrow{P_1P_2} = \mathbf{b} - \mathbf{a}$, the position vector of the point dividing $\overrightarrow{P_1P_2}$ in the ratio $\theta : (1 - \theta)$ is $(1 - \theta)\mathbf{a} + \theta\mathbf{b}$ Thus the point dividing $\overrightarrow{P_1P_2}$ in the ratio $k : 1$ is $\dfrac{\mathbf{a} + k\mathbf{b}}{k + 1}$ $\left(\text{where } \theta = \dfrac{k}{k + 1}\right)$

8.07. The Equation of a Plane A plane may be specified by three non-collinear points P_1, P_2, P_3 on it. (*Fig. 5.*)

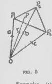


FIG. 5

Let the position vectors of P_1, P_2, P_3 be **a**, **b**, **c** respectively and let **r** be the position vector of a variable point P on the plane.

Since the displacement $\overrightarrow{P_1P}$ is in the same plane as $\overrightarrow{P_1P_2}$, $\overrightarrow{P_1P_3}$, it follows that $\overrightarrow{P_1P} = t\overrightarrow{P_1P_2} + u\overrightarrow{P_1P_3}$ where t, u are variable scalars (positive or negative), i e,

$$\mathbf{r} - \mathbf{a} = t(\mathbf{b} - \mathbf{a}) + u(\mathbf{c} - \mathbf{a})$$

or $\mathbf{r} = t_1\mathbf{a} + t_2\mathbf{b} + t_3\mathbf{c}$ (where $t_1 + t_2 + t_3 = 1$) is the vector equation of the plane

Examples. (i) The Cartesian equation of the plane through (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is obviously

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

and this follows also from the vector equation by noting that $x = t_1x_1 + t_2x_2 + t_3x_3$, $y = t_1y_1 + t_2y_2 + t_3y_3$, $z = t_1z_1 + t_2z_2 + t_3z_3$, $1 = t_1 + t_2 + t_3$.

(ii) Prove Ceva's Theorem—that if three lines AD, BE, CF, concurrent in P, are drawn through the vertices of a triangle ABC to meet the opposite sides in D, E, F, then $\dfrac{BD}{DC}\dfrac{CE}{EA}\dfrac{AF}{FB} = 1$

Let **a**, **b**, **c**, be the position vectors of A, B, C; then the position vector of P is $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ where $x + y + z = 1$. The point **d** given by $\dfrac{x\mathbf{a} + y\mathbf{b}}{x + y}$ divides

AB in the ratio y/x, but $\overrightarrow{OP} = (x+y)\mathbf{d} + z\mathbf{c}$, where $(x+y)+z = 1$, and therefore the point given by $\mathbf{d}$ lies on PC, i.e. $\mathbf{d}$ is the position vector of F. Thus $\dfrac{AF}{FB} = \dfrac{y}{x}$; similarly $\dfrac{BD}{DC} = \dfrac{z}{y}$, $\dfrac{CE}{EA} = \dfrac{x}{z}$ and the product of the ratios is unity.

8.1. Products of Vectors. From two vectors $\mathbf{a}$, $\mathbf{b}$ we can form two kinds of products that occur in applications. One of them is a scalar and the other is a vector.

8.11. The Scalar Product. The *scalar product* of $\mathbf{a}$ and $\mathbf{b}$ is defined to be $ab\cos\theta$ where a, b are the moduli of $\mathbf{a}$, $\mathbf{b}$ respectively and θ the angle between $\mathbf{a}$, $\mathbf{b}$. It will be written $\mathbf{a}.\mathbf{b}$ (pronounced *a dot b*), although other notations are used by some writers.

It is the product of the modulus of the one vector and the magnitude (positive or negative) of the vector projection on it of the other; and the *commutative law* $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$ is obviously obeyed.

The *distributive law* $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$ is also obeyed since the projection of the sum of two vectors on any line is equal to the sum of the projections of these vectors on that line.

If $\mathbf{a}$ is *perpendicular* to $\mathbf{b}$, $\mathbf{a}.\mathbf{b} = 0$ and if $\mathbf{a} = \mathbf{b}$, we write $\mathbf{a}.\mathbf{a}$ as $\mathbf{a}^2$

Thus $\quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1, \quad \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = \mathbf{i}.\mathbf{j} = 0.$

Again, if (x_1, y_1, z_1), (x_2, y_2, z_2) are the *components* of $\mathbf{a}$, $\mathbf{b}$ respectively, we have

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}).(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}).\\ &= x_1x_2 + y_1y_2 + z_1z_2\end{aligned}$$

and this may be called the Cartesian form of the scalar product.

Note. In n dimensions, the scalar product of two vectors A_r, B_r (expressed in rectangular components) is defined to be $A_1B_1 + A_2B_2 + \ldots + A_nB_n$.

8.12. The Angle between Two Directions. Let (l_1, m_1, n_1), (l_2, m_2, n_2) be the direction cosines of two lines (drawn in specified directions).

Unit vectors along these directions are

$$l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k} \quad l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k}.$$

Therefore $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$, where θ is the angle between the directions.

Examples. (i) Let $\mathbf{a}$ be the displacement of the point of application of a force $\mathbf{F}$. If F, a are the moduli of $\mathbf{F}$, $\mathbf{a}$ respectively and θ the angle between $\mathbf{F}$ and $\mathbf{a}$, then W, the work done by the force is defined to be $Fa\cos\theta$, i.e. $W = \mathbf{F}.\mathbf{a}$.

It follows that if $\mathbf{F}_r$ ($r = 1$ to n) is a system of forces acting at a point which is moved through the displacement $\mathbf{a}$, then the total work done $= \sum_{r=1}^{n} \mathbf{F}_r.\mathbf{a} = (\sum_{r=1}^{n} \mathbf{F}_r).\mathbf{a}$, i.e. is equal to the work done by the resultant $\sum_{r=1}^{n} \mathbf{F}_r$.

(ii) If two pairs of opposite edges of a tetrahedron are orthogonal, so also are the remaining pair.

Take the tetrahedron to be $OABC$ and let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$. If $\overrightarrow{OC} \perp \overrightarrow{AB}$, $\mathbf{c}.(\mathbf{a}-\mathbf{b}) = 0$; and if $\overrightarrow{OB} \perp \overrightarrow{CA}$, $\mathbf{b}.(\mathbf{a}-\mathbf{c}) = 0$. Therefore $\mathbf{a}.\mathbf{c} = \mathbf{b}.\mathbf{c} = \mathbf{a}.\mathbf{b}$, i.e. $\mathbf{a}.(\mathbf{b}-\mathbf{c})$ or $\overrightarrow{OA} \perp \overrightarrow{BC}$

(iii) *The Normal to a Plane* Let unit-vector along the normal from O towards a plane be $\mathbf{c}$ and let the normal meet the plane in A where the modulus of $\overrightarrow{OA} = p$. (If O is in the plane, $p = 0$ and $\mathbf{c}$ may be taken along either normal to the plane at O.)

If $\mathbf{r}$ is the position vector of $P(x, y, z)$, any point on the plane, then $\mathbf{r} \cdot \mathbf{c} = p$.

If (l, m, n) are the direction cosines of $\mathbf{c}$ (i.e. of $\overrightarrow{OA}$), the Cartesian equation of the plane is $lx + my + nz = p$. Conversely, the equation $Ax + By + Cz + D = 0$ represents a plane whose normal has direction cosines *proportional* to A, B, C.

8.13. Vector Areas. Let a plane area of absolute magnitude k be determined by a point P that describes a closed path which does not cross itself, by displacements $\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \ldots, \overrightarrow{P_nP_1}$. (*Fig. 6.*) We can construct a vector $\mathbf{A}$ whose modulus is k and whose direction is that normal to the plane from which the boundary $P_1P_2 \ldots P_nP_1$ appears to be described counter-clockwise. The vector $\mathbf{A}$ is called a *Vector Area*. It is usually more convenient, however, to *prescribe* one of the normals to the plane by means of a unit-vector $\mathbf{p}$ along that normal. Thus the vectors determined by areas described in the plane are of the form $A\mathbf{p}$,

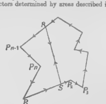


FIG. 6

FIG. 7

where A is the magnitude of the vector and may be positive or negative: positive, if the area is described counter-clockwise from a point on the prescribed normal and negative, if the description is clockwise. The prescribed normal is often called the *positive* normal. It is useful to note, however, that the direction of the vector area is *completely* indicated by drawing arrows in the boundary to show in which direction the boundary is described. Thus if a boundary is described once in each direction, the vector area indicated is zero. Also if a line be drawn from a point R of the boundary to a second point S (*Fig. 6*) and RS is described once in each direction, the vector area is unaltered; and the sum of the two vector areas corresponding to the two subdivisions is correctly given as the total vector area. More generally, the given area may be divided up into a network (*Fig. 7*), and the total effect of the circuits round the various meshes is equivalent to a circuit round the boundary.

8.14. Vector Projection of a Vector Area. Let **A** be the vector area described by a point P in the plane α for which the prescribed normal is determined by the unit vector **p**. Then $\mathbf{A} = A\mathbf{p}$ where A is the magnitude of **A**. Let β be a second plane for which the prescribed normal

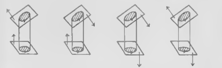


FIG. 8

is determined by the unit vector **q**. Let Q be the projection of P on β. Then as P describes **A** in α, Q describes an area **B** in β which is called the *vector projection* of **A** and $\mathbf{B} = B\mathbf{q}$ where B is the magnitude of **B**. If θ is the angle between **p** and **q**, then $|B| = |A \cos\theta|$ and the aspect of the description of **B** from a point on the prescribed normal of β is the *same* as the aspect of the description of **A** from a point on the prescribed normal of α if $\cos\theta$ is *positive*, whilst the aspect is the *opposite* if $\cos\theta$ is *negative*. (*Fig. 8.*) Thus $\mathbf{B} = A \cos\theta\mathbf{q}$, so that our definition of the vector projection of a vector area is consistent with our previous definition of vector projection.

8.15. The Vector Surface of a Closed Polyhedron is Zero. The *vector surface* of a polyhedron is defined to be the sum of the vector areas of its faces, the prescribed normal to a face being the outward normal, and the boundary of each face being described counter-clockwise to that normal. If the areas of the faces are projected on any plane, the sum of the vector projections is indicated by a network of lines each part of which is described once in each direction. The sum of the vector projections is therefore zero, from which it follows that the projection of the vector surface is zero. Since the plane of projection is arbitrary, we deduce that the vector surface is zero. (*Fig. 9.*)

FIG. 9

8.16. The Vector Product. The *vector product* of **a** and **b** is defined to be the *vector area* of the parallelogram formed by **a**, **b**, the boundary

of the parallelogram being described so that the vector $\mathbf{b}$ follows the vector $\mathbf{a}$. (*Fig. 10*) This product will be written $\mathbf{a} \times \mathbf{b}$ (pronounced *a cross b*), although other notations are used by some writers.

It follows from the definition that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$; $\mathbf{a} \times \mathbf{a} = 0$. The area of the parallelogram is equal, in absolute value, to $ab \sin \theta$, where $a = \text{mod } \mathbf{a}$, $b = \text{mod } \mathbf{b}$ and θ (between 0 and π) is the angle between $\mathbf{a}$ and $\mathbf{b}$. Therefore $\mathbf{a} \times \mathbf{b} = (ab \sin \theta)\mathbf{n}$, where $\mathbf{n}$ is the *correctly chosen* unit vector perpendicular to the plane of the parallelogram.

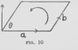


FIG. 10

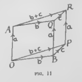


FIG. 11

8.17. The Distributive Law for the Vector Product. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{BP}$ in *Fig. 11* be represented by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively; then $\overrightarrow{OP} = \mathbf{b} + \mathbf{c}$. A triangular prism may be obtained by completing the parallelograms $OBQA$, $OPRA$, $BPRQ$ (the figure, however, being *plane* if the vectors are parallel to the same plane). The vector surface of the prism is zero,

$$\text{i.e.} \quad \overrightarrow{OBQAO} + \overrightarrow{BPRQB} - \overrightarrow{OPRAO} + \overrightarrow{AQRA} - \overrightarrow{OBPO} = 0$$

But $\overrightarrow{OBQAO} = \mathbf{b} \times \mathbf{a}$; $\overrightarrow{BPRQB} = \mathbf{c} \times \mathbf{a}$; $\overrightarrow{OPRAO} = (\mathbf{b} + \mathbf{c}) \times \mathbf{a}$;

$$\overrightarrow{AQRA} = \overrightarrow{OBPO} = \tfrac{1}{2}(\mathbf{b} \times \mathbf{c}).$$

Thus $\mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} = (\mathbf{b} + \mathbf{c}) \times \mathbf{a}$ or

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

so that the distributive law is obeyed.

The above proof is still applicable when the figure is *plane* or more simply from the fact that

$$\overrightarrow{OBPRAO} = \overrightarrow{OBPO} + \overrightarrow{OPRAO} = \overrightarrow{OBQAO} + \overrightarrow{BPRQB} + \overrightarrow{AQRA}.$$

Note. In *Fig. 11*, the arrows are used to denote the vectors, and should not be confused with the use of arrows for indicating the direction in which a boundary is described.

By repeated applications of the distributive law, we find

$$(\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_m) \times (\mathbf{b}_1 + \mathbf{b}_2 + \ldots + \mathbf{b}_n) = \sum_{r=1}^{m} \sum_{s=1}^{n} \mathbf{a}_r \times \mathbf{b}_s.$$

8.18. The Cartesian Form of the Vector Product. Let $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ be chosen as the *positive* normals of the planes YOZ, ZOX, XOY respec-

tively, and consider the vector area of the triangle $\overrightarrow{ABC}$, where $\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{OB} = \mathbf{j}$, $\overrightarrow{OC} = \mathbf{k}$. The direction cosines of one normal to ABC are all positive and the direction cosines of the other normal are all negative. Therefore the magnitudes of the projections of $\overrightarrow{ABC}$ on the co-ordinate planes have the *same* sign. These vector projections are obviously $\overrightarrow{OBC}$, $\overrightarrow{OCA}$, $\overrightarrow{OAB}$, i.e. $\frac{1}{2}(\mathbf{j} \times \mathbf{k})$, $\frac{1}{2}(\mathbf{k} \times \mathbf{i})$, $\frac{1}{2}(\mathbf{i} \times \mathbf{j})$. Thus the magnitudes of $\mathbf{j} \times \mathbf{k}$, $\mathbf{k} \times \mathbf{i}$, $\mathbf{i} \times \mathbf{j}$ have the same sign. But $\mathbf{j} \times \mathbf{k} = \pm \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \pm \mathbf{j}$, $\mathbf{i} \times \mathbf{j} = \pm \mathbf{k}$ (from the definition of vector product), and the signs are all + or all −. If they are all +, the direction of rotation from any of the axes $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ to either of the other two as viewed from a point on the third is counter-clockwise and the system of axes is called a *positive system*. Otherwise the system is *negative*.

For a *positive* system, therefore, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and for both systems $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$.

If $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, then for a positive system

$$\mathbf{a} \times \mathbf{b} = (y_1z_2 - y_2z_1)\mathbf{i} + (z_1x_2 - z_2x_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}$$

Notes. (i) The vector product of (x_1, y_1, z_1), (x_2, y_2, z_2) could therefore be defined simply as the vector $(y_1z_2 - y_2z_1, z_1x_2 - z_2x_1, x_1y_2 - x_2y_1)$.

(ii) The absolute magnitudes of the projections of the triangle formed by $(0, 0, 0)$, (x_1, y_1, z_1), (x_2, y_2, z_2) are

$$\tfrac{1}{2}|y_1z_2 - y_2z_1|, \; \tfrac{1}{2}|z_1x_2 - z_2x_1|, \; \tfrac{1}{2}|x_1y_2 - x_2y_1|.$$

(iii) This representation of vector product as a vector area is peculiar to three dimensions. In n dimensions, the analogue of the vector product is a skew-symmetric tensor of the second order.

Example. The angular velocity of a rigid body rotating about a fixed axis ON (*Fig. 12*) may be regarded as a vector $\boldsymbol{\omega}$ whose modulus is the measure of the ordinary angular velocity and whose direction is $\overrightarrow{ON}$, the rotation being counter-clockwise to an observer on ON looking towards O. Let P be any point of the body and let θ (between 0 and π) be the angle between $\overrightarrow{ON}$ and $\overrightarrow{OP}$. The speed of P is $\omega . OP \sin\theta$ where $\omega = |\boldsymbol{\omega}|$ and its direction is obviously that of $\boldsymbol{\omega} \times \mathbf{r}$ where $\mathbf{r} = \overrightarrow{OP}$, i.e. $\mathbf{v}$ the velocity of P is actually $\boldsymbol{\omega} \times \mathbf{r}$.

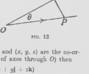


FIG. 12

If $(\omega_1, \omega_2, \omega_3)$ are the components of $\boldsymbol{\omega}$ and (x, y, z) are the co-ordinates of P (referred to a positive system of axes through O) then

$$\mathbf{v} = (\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

so that the components of $\mathbf{v}$ are

$$(\omega_2 z - \omega_3 y), \; (\omega_3 x - \omega_1 z), \; (\omega_1 y - \omega_2 x).$$

8.19. Scalar Triple Products. From the vectors $(\mathbf{a} \times \mathbf{b})$ and $\mathbf{c}$, we can form the product $(\mathbf{a} \times \mathbf{b}).\mathbf{c}$ and this is called a *scalar triple* product.

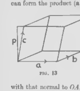


FIG. 13

The modulus of $(\mathbf{a} \times \mathbf{b}).\mathbf{c}$ is Ep where E is the area of the parallelogram formed by $\overrightarrow{OA}\ (=\mathbf{a})$, $\overrightarrow{OB}\ (=\mathbf{b})$, and p is the perpendicular distance from C to the plane OAB. Thus the modulus of $(\mathbf{a} \times \mathbf{b}).\mathbf{c}$ is the volume of the parallelepiped whose coterminous edges are $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ (*Fig. 13*.) The scalar product is positive if $\mathbf{c}$ makes an acute angle with that normal to OAB from which the change of direction from $\overrightarrow{OA}$ to $\overrightarrow{OB}$ appears counter-clockwise.

If $\mathbf{a} = (x_1, y_1, z_1)$, $\mathbf{b} = (x_2, y_2, z_2)$, $\mathbf{c} = (x_3, y_3, z_3)$, then

$$(\mathbf{a} \times \mathbf{b}).\mathbf{c} = \{(y_1z_2 - y_2z_1)\mathbf{i} + (z_1x_2 - z_2x_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}\}.\{x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}\}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

from which it appears that $(\mathbf{a} \times \mathbf{b}).\mathbf{c} = (\mathbf{b} \times \mathbf{c}).\mathbf{a} = (\mathbf{c} \times \mathbf{a}).\mathbf{b}$ as is obvious from the above geometrical interpretation of the triple product. The product is of course equal to

$$-(\mathbf{b} \times \mathbf{a}).\mathbf{c} = -(\mathbf{a} \times \mathbf{c}).\mathbf{b} = -(\mathbf{c} \times \mathbf{b}).\mathbf{a}$$

and therefore without ambiguity may be denoted by the symbol $[\mathbf{abc}]$, if it is understood that the interchange of any two of the letters changes its sign.

Thus $[\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}] = -[\mathbf{acb}] = -[\mathbf{cba}] = -[\mathbf{bac}]$.

Note. The volume of the tetrahedron formed by (x_r, y_r, z_r) $(r = 1 \text{ to } 4)$ is

$$\pm \tfrac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix} = \pm \tfrac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 & 0 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 & 0 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 & 0 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \pm \tfrac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

8.191. Vector Triple Products. Vector Triple Products may be formed in various ways. Consider, for example, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Let $\mathbf{a} = (x_1, y_1, z_1)$, $\mathbf{b} = (x_2, y_2, z_2)$, $\mathbf{c} = (x_3, y_3, z_3)$.

Then by using the Cartesian form of the product $(\mathbf{b} \times \mathbf{c})$, we may easily verify that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$$

where $A = x_2(x_1x_3 + y_1y_3 + z_1z_3) - x_3(x_1x_2 + y_1y_2 + z_1z_2)$, with two similar expressions for B, C.

Thus $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}.$$

Similarly

$$\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{b}.\mathbf{a})\mathbf{c} - (\mathbf{b}.\mathbf{c})\mathbf{a}\ ;\ \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c}.\mathbf{b})\mathbf{a} - (\mathbf{c}.\mathbf{a})\mathbf{b}$$

8.2. Curves in Space. If the co-ordinates x, y, z of a point P are functions of a parameter t, the locus of P is called a *twisted* (or *tortuous*) curve. Under certain conditions the curve may be plane.

FIG. 14

8.21. The Arc. Let x, y, z be continuous functions of t and let A, B be two points of the curve given respectively by the finite parameters, t_0, T ($T > t_0$). Let $(n - 1)$ parameters t_s be chosen so that

$$t_0 < t_1 < t_2 < \ldots < t_{n-1} < T$$

and let t_s correspond to the point P_s where P_0 is A and P_n is B ($t_n = T$). Then $P_sP_{s+1} = \{(x_{s+1} - x_s)^2 + (y_{s+1} - y_s)^2 + (z_{s+1} - z_s)^2\}^{\frac{1}{2}}$. Let us assume also that $\dot{x}\left(= \dfrac{dx}{dt}\right)$, $\dot{y}\left(= \dfrac{dy}{dt}\right)$, $\dot{z}\left(= \dfrac{dz}{dt}\right)$ are continuous and that they do not all vanish for the same value of t. Then, given ε, the length of each sub-interval can be taken sufficiently small to ensure that the oscillations of $\dot{x}^2$, $\dot{y}^2$, $\dot{z}^2$ are all $< \varepsilon$ in each sub-interval.

By the mean value theorem, $x_{s+1} - x_s = (\dot{x})_{t_s'}(t_{s+1} - t_s)$,

$$y_{s+1} - y_s = (\dot{y})_{t_s''}(t_{s+1} - t_s),\quad z_{s+1} - z_s = (\dot{z})_{t_s'''}(t_{s+1} - t_s)$$

where t_s', t_s'', t_s''' are values of t in the interval (t_s, t_{s+1}).

i.e. $$P_sP_{s+1} = (t_{s+1} - t_s)\{(\dot{x})_{t_s}{}^2 + (\dot{y})_{t_s}{}^2 + (\dot{z})_{t_s}{}^2 + \rho\}^{\frac{1}{2}}$$

where $|\rho| < 3\varepsilon$.

Thus $P_sP_{s+1} = \{(\dot{x})_{t_s}{}^2 + (\dot{y})_{t_s}{}^2 + (\dot{z})_{t_s}{}^2\}^{\frac{1}{2}}(t_{s+1} - t_s) + \kappa(t_{s+1} - t_s)$ where $|k| < M\varepsilon$, M being the maximum value of $\frac{3}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{-\frac{1}{2}}$. When each sub-interval tends to zero $\Sigma\kappa(t_{s+1} - t_s) < M\varepsilon(T - t_0)$ and therefore tends to zero.

$$\text{Also } \Sigma\{(\dot{x})_{t_s}{}^2 + (\dot{y})_{t_s}{}^2 + (\dot{z})_{t_s}{}^2\}^{\frac{1}{2}}(t_{s+1} - t_s) \to \int_{t_0}^{T} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}\,dt.$$

Therefore the sum of lengths of the chords tends to the value of the integral $\int_{t_0}^{T} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}\,dt$, so that the integral provides a natural definition of the length of the arc from A to B. If we take the upper limit to be the variable t (corresponding to the variable point P), we can form the function $s = \int_{t_0}^{t} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}\,dt$ which measures the arc AP.

Differentiating, we find $\dfrac{ds}{dt} = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}$. This result is often written $ds^2 = dx^2 + dy^2 + dz^2$, where dx, dy, dz are the differentials corresponding to the same increment dt. Since $(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}$ is the length of the chord joining (x, y, z) to $(x + dx, y + dy, z + dz)$, it follows from the above that lim (chord AP ÷ arc AP) = 1 when $P \to A$.

Notes. (i) The formula for the arc AB is obviously applicable to the case when $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 0$ at either or both of the points A, B.

(ii) The integral for s will exist under less restrictive conditions than those given above. For example $\dot{x}$, $\dot{y}$, $\dot{z}$ may possess a finite number of finite discontinuities (i.e. the curve may have a finite number of *corners*).

(iii) The function s increases steadily with t and is finite, therefore t may be expressed as a function of s in an interval within which $\dot{s}$ is not zero (although $\dot{s}$ may tend to zero at the ends of the interval). It is therefore convenient in theoretical work to regard x, y, z as functions of the arc s.

8.22. Derivatives of Vectors. A vector is said to be a function of *scalar* variables, u, v, . . . if its components are functions of these variables; and is said to be continuous and possess derivatives if its components possess these properties.

Thus if $\mathbf{a} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ and X, Y, Z are functions of the variable u, $\delta\mathbf{a}$ is defined to be $\delta X\mathbf{i} + \delta Y\mathbf{j} + \delta Z\mathbf{k}$ and

$$\frac{\partial \mathbf{a}}{\partial u} = \frac{\partial X}{\partial u}\mathbf{i} + \frac{\partial Y}{\partial u}\mathbf{j} + \frac{\partial Z}{\partial u}\mathbf{k}.$$

Since the distributive law is obeyed for products, we easily deduce the results

$$\frac{\partial}{\partial u}(\mathbf{a}.\mathbf{b}) = \mathbf{a}.\frac{\partial \mathbf{b}}{\partial u} + \frac{\partial \mathbf{a}}{\partial u}.\mathbf{b}, \quad \frac{\partial}{\partial u}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times \frac{\partial \mathbf{b}}{\partial u} + \frac{\partial \mathbf{a}}{\partial u} \times \mathbf{b}$$

$$\frac{\partial}{\partial u}[\mathbf{pqr}] = [\mathbf{p}_u\mathbf{qr}] + [\mathbf{pq}_u\mathbf{r}] + [\mathbf{pqr}_u];$$

$$\frac{\partial}{\partial u}\{(\mathbf{p} \times \mathbf{q}) \times \mathbf{r}\} = (\mathbf{p}_u \times \mathbf{q}) \times \mathbf{r} + (\mathbf{p} \times \mathbf{q}_u) \times \mathbf{r} + (\mathbf{p} \times \mathbf{q}) \times \mathbf{r}_u.$$

8.23. The Tangent to a Curve. Let P, Q be two points of the curve $x = x(t)$, $y = y(t)$, $z = z(t)$, determined by the values t, $t + \delta t$. Let $\overrightarrow{OP}$ be denoted by $\mathbf{r}$ and $\overrightarrow{OQ}$ by $\mathbf{r} + \delta\mathbf{r}$. Then $\overrightarrow{PQ} = \delta\mathbf{r}$.

FIG. 15

When $Q \to P$, the limiting position of the chord PQ is defined to be the tangent at P.

Thus the *direction* of the vector $\frac{d\mathbf{r}}{ds}$ is that of the tangent and since $\lim\,(\text{chord} \div \text{arc}) = 1$ when $Q \to P$, the *modulus* of the vector $\frac{d\mathbf{r}}{ds}$ is unity. If $\mathbf{T}$ is *unit* vector along the tangent at P (in the direction of *increasing* s) it follows that $\mathbf{T} = \frac{d\mathbf{r}}{ds}$.

If (l_1, m_1, n_1) are the direction cosines of the tangent

$$l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k} = \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}$$

i.e. $\quad l_1, m_1, n_1 = \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ where $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$

8.24. *The Spherical Indicatrix.* Take a unit sphere with its centre at some fixed point (O, for example) and draw the vector $\mathbf{T}$ from O meeting the sphere in the point T. As the point P describes the given curve, the point T describes a curve on the sphere (the sphere being called the *spherical indicatrix*). (*Fig. 16.*)

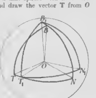


FIG. 16

8.25. *The Principal Normal.* Let OT, OT_1 be parallel to the tangents at P, Q respectively to the given curve. Then since $OT = OT_1 = 1$, the arc TT_1 is equal to the angle $\delta\psi$ between the tangents at P, Q. When $Q \to P$, the limiting position of TT_1 is the tangent at T to the curve described by T on the sphere. Since this tangent is perpendicular to OT, it is parallel to a certain *normal* to the given curve. This normal is called the *principal normal*. Since $\overrightarrow{TT_1} = \delta\mathbf{T}$, it follows that $\dfrac{d\mathbf{T}}{ds}$ is in the direction of the principal normal and its modulus is $\dfrac{d\psi}{ds}$, where ψ is the angle through which the tangent has turned from some initial position.

If $\mathbf{N}$ is *unit* vector along the principal normal

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \text{ where } \kappa = \frac{d\psi}{ds} \text{ (a *positive* scalar).}$$

The number κ is called the *circular curvature* of the given curve and $\rho = 1/\kappa$ is called the *radius of circular curvature.*

If l_2, m_2, n_2 are the direction cosines of the principal normal

$$l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k} = \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{1}{\kappa}\frac{d^2\mathbf{r}}{ds^2} = \frac{1}{\kappa}\left(\frac{d^2x}{ds^2}\mathbf{i} + \frac{d^2y}{ds^2}\mathbf{j} + \frac{d^2z}{ds^2}\mathbf{k}\right)$$

i.e. $(l_2, m_2, n_2) = \dfrac{1}{\kappa}\left(\dfrac{d^2x}{ds^2}, \dfrac{d^2y}{ds^2}, \dfrac{d^2z}{ds^2}\right)$ where

$$\kappa^2 = \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

8.26. *The Osculating Plane.* The plane containing the tangent and the principal normal is called the *Osculating Plane.*

If L, M, N are the direction cosines of either normal to this plane we must have $Ll_1 + Mm_1 + Nn_1 = 0 = Ll_2 + Mm_2 + Nn_2$ and

therefore the equation of the osculating plane is

$$\begin{vmatrix} \xi - x & \eta - y & \zeta - z \\ \dfrac{dx}{ds} & \dfrac{dy}{ds} & \dfrac{dz}{ds} \\ \dfrac{d^2x}{ds^2} & \dfrac{d^2y}{ds^2} & \dfrac{d^2z}{ds^2} \end{vmatrix} = 0$$

where ξ, η, ζ are used for the current co-ordinates on the plane.

Notes. (i) A plane can be drawn through the tangent at P parallel to the tangent at Q. (This plane is parallel to OTT_1, in *Fig. 16*). The osculating plane is the limiting position of the above plane when $Q \to P$, and may therefore be described as the plane containing the directions of two consecutive tangents.

(ii) If $\xi = x(s)$, $\eta = y(s)$, $\zeta = z(s)$ is substituted in the equation of the osculating plane at $s = s_1$, the resulting equation in s has a *triple* root at $s = s_1$, since

$$x(s) - x(s_1) = (s - s_1)\left(\frac{dx}{ds}\right)_1 + \frac{(s - s_1)^2}{2}\left(\frac{d^2x}{ds^2}\right)_1 + O[(s - s_1)^3]$$

with similar equations for $y(s) - y(s_1)$ and $z(s) - z(s_1)$.

The osculating plane may therefore be described as the plane through three consecutive points; it is the limiting position of the plane through P, Q, R when $Q, R \to P$.

(iii) Let $c_1 (>0)$ be the limiting value of the radius of the circumcircle of PQR when $Q, R \to P$, and C_1 the centre. Then C_1 by (ii) above is in the osculating plane at $P(s_1)$. The equation $(\mathbf{r} - \mathbf{a}_1)^2 = c_1^2$ where $\mathbf{a}_1 = \overrightarrow{OC_1}$ must have a triple root $s = s_1$.

Therefore $(\mathbf{r}_1 - \mathbf{a}_1)^2 = c_1^2$, $(\mathbf{r}_1 - \mathbf{a}_1).\mathbf{T}_1 = 0$, $\mathbf{T}_1^2 + (\mathbf{r}_1 - \mathbf{a}_1)\kappa\mathbf{N} = 0$.

From the second it follows that $\mathbf{r}_1 - \mathbf{a}_1 = -c_1\mathbf{N}$ since C_1 is in the osculating plane and by substitution in the third we find that $c_1 = 1/\kappa = \rho$ (since it is positive). The radius of curvature is therefore identified as the limiting value of the radius of the circumcircle of P, Q, R when $Q, R \to P$.

8.27 *The Binormal.* The *Binormal* is that normal to the osculating plane (and therefore to the curve) which is such that the tangent, the principal normal and the binormal (in this order) form a positive system.

If $\mathbf{B}$ is *unit* vector along the binormal, we have

$$[\mathbf{TNB}] = 1,\ \mathbf{T}.\mathbf{B} = 0,\ \mathbf{T}.\mathbf{N} = 0,\ \mathbf{B}.\mathbf{N} = 0,\ \mathbf{T} \times \mathbf{N} = \mathbf{B},\ \text{etc.}$$

Since $\mathbf{B}^2 = 1$, $\mathbf{B}.\dfrac{d\mathbf{B}}{ds} = 0$, and since $\mathbf{B}.\mathbf{T} = 0$, $\dfrac{d\mathbf{B}}{ds}.\mathbf{T} = -\mathbf{B}.(\kappa\mathbf{N}) = 0$.

But $\mathbf{B}.\mathbf{N} = 0$; therefore $\dfrac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{B}$ and $\mathbf{T}$ and must be of the form $-\lambda\mathbf{N}$ (where λ is a scalar, positive or negative). Since $|d\mathbf{B}|$ measures the angle ds through which the binormal turns when P is displaced through the arc ds, the magnitude $(-\lambda)$ of $\dfrac{d\mathbf{B}}{ds}$ (in the direction of $\mathbf{N}$) measures the *twisting* of the curve from the osculating plane. The scalar λ is therefore called the *torsion* and $1/\lambda$ the *radius of torsion*. In *Fig. 16*, the extremities of the unit vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ are shown and also the extremities $\mathbf{T}_1$, $\mathbf{N}_1$, $\mathbf{B}_1$ corresponding to $\mathbf{T} + d\mathbf{T}$, $\mathbf{N} + d\mathbf{N}$, $\mathbf{B} + d\mathbf{B}$. Since $\dfrac{d\mathbf{T}}{ds} = k\mathbf{N}$, $\dfrac{d\mathbf{B}}{ds} = -\lambda N$, the arcs TT_1, BB_1

lie respectively along the great circles TN, NB. If l_3, m_3, n_3 are the direction cosines of the binormal

$$l_3\mathbf{i}+m_3\mathbf{j}+n_3\mathbf{k} = \mathbf{B} = \mathbf{T}\times\mathbf{N}$$
$$= \frac{1}{\kappa}\left\{\left(\frac{dy}{ds}\frac{d^2z}{ds^2}-\frac{dz}{ds}\frac{d^2y}{ds^2}\right)\mathbf{i}+\left(\frac{dz}{ds}\frac{d^2x}{ds^2}-\frac{dx}{ds}\cdot\frac{d^2z}{ds^2}\right)\mathbf{j}+\left(\frac{dx}{ds}\cdot\frac{d^2y}{ds^2}-\frac{dy}{ds}\cdot\frac{d^2x}{ds^2}\right)\mathbf{k}\right\}$$

8·28. *The Frenet-Serret Formulae.* Since $\mathbf{N} = \mathbf{B}\times\mathbf{T}$,

$$\frac{d\mathbf{N}}{ds} = (-\lambda\mathbf{N})\times\mathbf{T} + \mathbf{B}\times(\kappa\mathbf{N}) = \lambda\mathbf{B} - \kappa\mathbf{T}$$

The three equations:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N};\quad \frac{d\mathbf{N}}{ds} = \lambda\mathbf{B} - \kappa\mathbf{T};\quad \frac{d\mathbf{B}}{ds} = -\lambda\mathbf{N}$$

comprise what are known as the *Frenet-Serret Formulae.* They are equivalent to the equations

$$\frac{dl_1}{ds} = \kappa l_2,\quad \frac{dl_2}{ds} = \lambda l_3 - \kappa l_1,\quad \frac{dl_3}{ds} = -\lambda l_2$$

with similar equations involving m_1, m_2, m_3 and n_1, n_2, n_3.

8·29. *The Torsion Formula.* The scalar triple product

$$\left[\frac{d\mathbf{r}}{ds},\ \frac{d^2\mathbf{r}}{ds^2},\ \frac{d^3\mathbf{r}}{ds^3}\right]$$

is equal to

$$\left[\mathbf{T},\ \kappa\mathbf{N},\ \kappa(\lambda\mathbf{B} - \kappa\mathbf{T}) + \frac{d\kappa}{ds}\mathbf{N}\right] = \kappa^2\lambda.$$

i.e.
$$\kappa^2\lambda = \begin{vmatrix} dx/ds & dy/ds & dz/ds \\ d^2x/ds^2 & d^2y/ds^2 & d^2z/ds^2 \\ d^3x/ds^3 & d^3y/ds^3 & d^3z/ds^3 \end{vmatrix}$$

Notes and Examples. (i) *The Condition for a Plane Curve.* If the curve lies in the plane $Ax + By + Cz + D = 0$ (A, B, C not all zero), then for all values of s,

$$0 = Ax + By + Cz + D = A\frac{dx}{ds} + B\frac{dy}{ds} + C\frac{dz}{ds}$$
$$= A\frac{d^2x}{ds^2} + B\frac{d^2y}{ds^2} + C\frac{d^2z}{ds^2} = A\frac{d^3x}{ds^3} + B\frac{d^3y}{ds^3} + C\frac{d^3z}{ds^3} = \&c.$$

A necessary condition is therefore $\begin{vmatrix} dx/ds & d^2x/ds^2 & d^3x/ds^3 \\ dy/ds & d^2y/ds^2 & d^3y/ds^3 \\ dz/ds & d^2z/ds^2 & d^3z/ds^3 \end{vmatrix} = 0.$

This condition is also *sufficient.* For let

$$\alpha = \frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2},\ \beta = \frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2},\ \gamma = \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}.$$

If any one of these expressions α, β, γ vanish, the curve is plane. For example, if $\alpha = 0$, $\frac{dy}{ds} = k\frac{dz}{ds}$, where k is independent of s. (In the trivial case when a first derivative vanishes the curve obviously lies in a plane parallel to a co-ordinate plane). Therefore $y = kz + l$, a *plane.*

If two of them vanish, so must the third (except possibly in the trivial case), and the curve reduces to a *straight line.*

If none of them vanishes, we have

$$\alpha\frac{dx}{ds}+\beta\frac{dy}{ds}+\gamma\frac{dz}{ds}=0=\alpha\frac{d^2x}{ds^2}+\beta\frac{d^2y}{ds^2}+\gamma\frac{d^2z}{ds^2}=\alpha\frac{d^3x}{ds^3}+\beta\frac{d^3y}{ds^3}+\gamma\frac{d^3z}{ds^3}$$

Differentiation gives $\frac{d\alpha}{ds}\frac{dx}{ds}+\frac{d\beta}{ds}\frac{dy}{ds}+\frac{d\gamma}{ds}\frac{dz}{ds}=0=\frac{d\alpha}{ds}\frac{d^2x}{ds^2}+\frac{d\beta}{ds}\frac{d^2y}{ds^2}+\frac{d\gamma}{ds}\frac{d^2z}{ds^2}$, i.e. $\frac{1}{\alpha}\frac{d\alpha}{ds}=\frac{1}{\beta}\frac{d\beta}{ds}=\frac{1}{\gamma}\frac{d\gamma}{ds}$, from which it follows that $\beta=k_1\alpha$, $\gamma=k_2\alpha$ where k_1, k_2 are independent of s; i.e. $\frac{dx}{ds}+k_1\frac{dy}{ds}+k_2\frac{dz}{ds}=0$ or x, y, z must lie on a plane

$$x+k_1y+k_2z=k_3.$$

(ii) *Formulae in Terms of a Parameter t.* In applications x, y, z are usually given in terms of a parameter t, and the formulae we have obtained may be expressed more conveniently in terms of x, y, z and the derivatives with respect to t. Let *dots* be used to denote these derivatives.

(a) *The Arc.* s is obtained from $\dot{s}^2=\dot{x}^2+\dot{y}^2+\dot{z}^2$. It should be noted that $\dot{s}\ddot{s}=\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}$.

(b) *Circular Curvature.* $\frac{dx}{ds}=\frac{\dot{x}}{\dot{s}}$, $\frac{d^2x}{ds^2}=\frac{\ddot{x}\dot{s}-\dot{x}\ddot{s}}{\dot{s}^3}$, with similar results for $\frac{dy}{ds}$, $\frac{d^2y}{ds^2}$, $\frac{dz}{ds}$, $\frac{d^2z}{ds^2}$. We may then easily obtain the formula

$$\kappa^2=\frac{1}{\rho^2}=\frac{\ddot{x}^2+\ddot{y}^2+\ddot{z}^2-\ddot{s}^2}{\dot{s}^4}.$$

(c) *Tangent and Normals.*

$$\dot{s}\,(l_1,\ m_1,\ n_1)=\dot{x},\ \dot{y},\ \dot{z}\,.$$
$$\kappa\dot{s}^3(l_2,\ m_2,\ n_2)=\ddot{x}\dot{s}-\dot{x}\ddot{s},\ \ddot{y}\dot{s}-\dot{y}\ddot{s},\ \ddot{z}\dot{s}-\dot{z}\ddot{s}\,.$$
$$\kappa\dot{s}^3(l_3,\ m_3,\ n_3)=\dot{y}\ddot{z}-\ddot{y}\dot{z},\ \dot{z}\ddot{x}-\ddot{z}\dot{x},\ \dot{x}\ddot{y}-\ddot{x}\dot{y}\,.$$

and it may be verified that $\dot{s}^6\kappa^2=(\dot{y}\ddot{z}-\ddot{y}\dot{z})^2+(\dot{z}\ddot{x}-\ddot{z}\dot{x})^2+(\dot{x}\ddot{y}-\ddot{x}\dot{y})^2$

(d) *Torsion.* Since $\dot{\mathbf{r}}=\dot{s}\mathbf{T}$, $\ddot{\mathbf{r}}=\ddot{s}\mathbf{T}+\kappa\dot{s}^2\mathbf{N}$ and

$$\dddot{\mathbf{r}}=\dddot{s}\mathbf{T}+\{\ddot{s}\kappa\dot{s}+\frac{d}{dt}(\kappa\dot{s}^2)\}\mathbf{N}+\kappa\dot{s}^3(\lambda\mathbf{B}-\kappa\mathbf{T})$$

and therefore $[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]=\kappa^2\lambda\dot{s}^6$

or

$$\kappa^2\lambda\dot{s}^6=\begin{vmatrix}\dot{x} & \dot{y} & \dot{z}\\ \ddot{x} & \ddot{y} & \ddot{z}\\ \dddot{x} & \dddot{y} & \dddot{z}\end{vmatrix}$$

and the vanishing of this determinant is the condition (necessary and sufficient) that the curve should be plane.

(iii) *The Circular Helix.* The curve given by $x=a\cos\theta$, $y=a\sin\theta$, $z=a\theta\tan\alpha$ is called a *circular helix*. It obviously lies on the cylinder $x^2+y^2=a^2$. It cuts the generators at a constant angle $(\frac{1}{2}\pi-\alpha)$, a property that may easily be seen by cutting the cylinder along the generator in XOZ and developing the cylinder into a plane. (*Fig. 17.*) The curve then becomes a set of straight lines inclined to AA at angle α.

$\dot{x}=(-a\sin\theta)$, $\dot{y}=(a\cos\theta)$, $\dot{z}=a\tan\alpha$.

$\dot{s}^2=a^2\sec^2\alpha$ and therefore $s=(a\sec\alpha)\theta$ if s is measured from $\theta=0$ in the direction of increasing θ. This is obviously verified by inspection of the developed curve in *Fig. 17.*

$\mathbf{T}=(-\sin\theta\cos\alpha)\mathbf{i}+(\cos\theta\cos\alpha)\mathbf{j}+(\sin\alpha)\mathbf{k}$

$\kappa\mathbf{N}=\{(-\cos\theta\cos\alpha)\mathbf{i}-(\sin\theta\cos\alpha)\mathbf{j}\}\frac{\cos\alpha}{a}$, i.e. $\kappa=\frac{\cos^2\alpha}{a}$, and the principal normal is along the radius of the circular section through P.

$\mathbf{B} = \mathbf{T} \times \mathbf{N} = (\sin\theta \sin\alpha)\mathbf{i} - (\cos\theta \sin\alpha)\mathbf{j} + (\cos\alpha)\mathbf{k}$, thus showing that the binormal makes a constant angle with the generators.

$$\frac{d^2\mathbf{r}}{ds^2} = \{-(\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j}\}\frac{\cos^2\alpha}{a}, \quad \frac{d^3\mathbf{r}}{ds^3} = \{(\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j}\}\frac{\cos^3\alpha}{a^2}.$$

Therefore $\left(\frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3}\right)\cdot\frac{d\mathbf{r}}{ds} = \frac{\cos^4\alpha}{a^3}\sin\alpha = \kappa^2\lambda$, i.e. $\lambda = \frac{\sin\alpha\cos\alpha}{a}$. In particular, $\alpha = 0$, the helix is a circle, $\kappa = 1/a$, $\lambda = 0$.

$\alpha = \pi/2$, the helix is a generator, $\kappa = 0$, $\lambda = 0$.

$\alpha = \pi/4$, $\kappa = 1/2a = \lambda$, and this is the helix of maximum torsion on this cylinder.

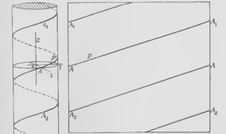


FIG. 17

If the spherical indicatrix (*Fig. 16*) has $x^2 + y^2 + z^2 = 1$ for its equation, the paths of T, N, B are the circular sections $z = \sin\alpha$, $z = 0$, $z = \cos\alpha$ respectively.

Note. The general helix is defined to be a curve that always makes the same angle with a fixed direction.

(iv) *The Intrinsic Equations of a Curve.* Let the axes of reference be the tangent (OX), the principal normal (OY), the binormal (OZ), where O is a point on the curve. The co-ordinates of any other point P (where arc $OP = s$) are then functions of s and can be expressed in terms of κ, λ the curvature and torsion at O. Since any point O may be chosen, these axes may be regarded as *moving* with the point O on the curve; and since κ, λ depend only on s, the co-ordinates are described intrinsic. The Frenet-Serret formulae show that the derivatives can be expressed by equations of the type

$$\frac{d^p x}{ds^p} = l_1 a_p + l_2 b_p + l_3 c_p, \quad \frac{d^p y}{ds^p} = m_1 a_p + m_2 b_p + m_3 c_p; \quad \frac{d^p z}{ds^p} = n_1 a_p + n_2 b_p + n_3 c_p$$

where a_p, b_p, c_p are certain functions of s.

By differentiating these equations and using the formulae, we obtain

$$a_{p+1} = \frac{da_p}{ds} - \kappa b_p; \quad b_{p+1} = \frac{db_p}{ds} + \kappa a_p - \lambda c_p; \quad c_{p+1} = \frac{dc_p}{ds} + \lambda b_p.$$

Also, since O is on the curve,

$$(x_0, y_0, z_0) = (0, 0, 0); \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)_0 = (l_1, m_1, n_1)_0 = (1, 0, 0),$$

$$(l_2, m_2, n_2)_0 = (0, 1, 0), \ (l_3, m_3, n_3)_0 = (0, 0, 1).$$

Thus taking all the values for $(0, 0, 0)$, we have $(a_1, b_1, c_1) = (1, 0, 0)$;

$$(a_2, b_2, c_2) = (0, \kappa, 0); \ (a_3, b_3, c_3) = (-\kappa^2, \kappa', \lambda\kappa);$$
$$(a_4, b_4, c_4) = (-3\kappa\kappa', \kappa'' - \kappa^3 - \kappa\lambda^2, \kappa\lambda' + 2\kappa'\lambda), \ \&c.$$

where accents denote differentiations with respect to s.

$$\text{Thus } x = s - \frac{\kappa^2}{6}s^3 - \frac{\kappa\kappa'}{8}s^4 \ldots; \ y = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + (\kappa'' - \kappa^3 - \kappa\lambda^2)\frac{s^4}{24} \ldots$$

$$z = \frac{\kappa\lambda}{6}s^3 + (\kappa\lambda' + 2\kappa'\lambda)\frac{s^4}{24} \ldots$$

(v) *Example.* Show that the shortest distance between the tangents at O, P is approximately $\frac{1}{12}\kappa\lambda s^3$, when $s\,(= OP)$ is small.

The direction cosines l, m, n of the tangent at P are

$$1 - \frac{\kappa^2}{2}s^2 - \frac{\kappa\kappa'}{2}s^3 \ldots; \ \kappa s + \frac{\kappa' s^2}{2} + (\kappa'' - \kappa^3 - \kappa\lambda^2)\frac{s^3}{6} \ldots;$$
$$\frac{\kappa\lambda s^2}{2} + (\kappa\lambda' + 2\kappa'\lambda)\frac{s^3}{6} \ldots$$

The direction cosines of the line perpendicular to this tangent and the x axis (tangent at O) are $(0, n, -m)/(m^2 + n^2)^{1/2}$, and the shortest distance D is $\dfrac{|ny - mz|}{(m^2 + n^2)^{\frac{1}{2}}}$.

The numerator of D is $\frac{1}{12}\kappa^2\lambda s^4 + O(s^5)$ and the denominator is $\kappa s + O(s^2)$ and therefore $D = |\frac{1}{12}\kappa\lambda s^3 + O(s^4)|$.

(vi) *Curvature and Torsion of an Involute.* A thin string has one end A at the point O of a given curve and lies along the curve in the direction of s increasing. The string is unwound from O, the unwound part being kept taut. The path of A is called an *involute* of the given curve. If the length s is unwound, where the arc $OP = s$, the point A is on the tangent at P at a distance s from P in the direction *opposite* to $\mathbf{T}$.

Thus if $\overrightarrow{OP} = \mathbf{r}$, then $\overrightarrow{OA} = \mathbf{r}_1 = \mathbf{r} - s\mathbf{T}$, where the suffix 1 denotes that the quantity refers to the involute.

$$\mathbf{T}_1 = \frac{d\mathbf{r}_1}{ds_1} = (-s\kappa\mathbf{N})\frac{ds}{ds_1}, \text{ i.e. } \mathbf{T}_1 = -\mathbf{N} \text{ and } \frac{ds}{ds_1} = \frac{1}{\kappa s}$$

$$\kappa_1\mathbf{N}_1 = \frac{(\lambda\mathbf{B} - \kappa\mathbf{T})}{\kappa s}, \text{ i.e. } \kappa^2\kappa_1{}^2s^2 = \lambda^2 + \kappa^2 \text{ so that } \rho_1 \text{, the radius of circular}$$

curvature of the involute, is $\dfrac{\sigma s}{\sqrt{(\rho^2 + \sigma^2)}}$, where ρ, σ are the radii of circular curvature and of torsion of the given curve.

Also $\mathbf{N}_1 = \mathbf{T}\cos\alpha - \mathbf{B}\sin\alpha$ where $\tan\alpha = \lambda/\kappa = \rho/\sigma$.

Therefore $\mathbf{B}_1 = \mathbf{T}_1 \times \mathbf{N}_1 = \mathbf{B}\cos\alpha + \mathbf{T}\sin\alpha$, differentiation gives

$$-\lambda_1\mathbf{N}_1 = \frac{1}{\kappa s}\left\{-\lambda\mathbf{N}\cos\alpha + \kappa\mathbf{N}\sin\alpha + (\mathbf{T}\cos\alpha - \mathbf{B}\sin\alpha)\frac{d\alpha}{ds}\right\} = \frac{1}{\kappa s}\frac{d\alpha}{ds}\mathbf{N}_1$$

$$\text{i.e.} \qquad \lambda_1 = -\frac{1}{\kappa s}\frac{d\alpha}{ds} = \frac{1}{\kappa s}\frac{(\sigma'\rho - \sigma\rho')}{\rho^2 + \sigma^2}, \text{ i.e. } \sigma_1\left(= \frac{1}{\lambda_1}\right) = \frac{s(\rho^2 + \sigma^2)}{\rho(\sigma'\rho - \sigma\rho')}.$$

(vii) *Rate of Change of a Vector* $\mathbf{F}$ *referred to Moving Axes.* Let a system of rectangular axes be rotating with angular velocity $\boldsymbol{\omega}\,(= \omega_1, \omega_2, \omega_3$, referred to these

axes). The unit vectors **i**, **j**, **k** that determine the directions of these axes are functions of the time t. Let $\mathbf{F} = (F_1, F_2, F_3)$.

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega} \times \mathbf{i} = \omega_3\mathbf{j} - \omega_2\mathbf{k}, \text{ similarly } \frac{d\mathbf{j}}{dt} = \omega_1\mathbf{k} - \omega_3\mathbf{i}, \text{ and}$$

$$\frac{d\mathbf{k}}{dt} = \omega_2\mathbf{i} - \omega_1\mathbf{j}.$$

Therefore
$$\frac{d\mathbf{F}}{dt} = \frac{d}{dt}(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})$$
$$= (\dot{F}_1 - \omega_3F_2 + \omega_2F_3)\mathbf{i} + (\dot{F}_2 - \omega_1F_3 + \omega_3F_1)\mathbf{j} + (\dot{F}_3 - \omega_2F_1 + \omega_1F_2)\mathbf{k}$$

giving the components of $\dot{\mathbf{F}}$.

In *two* dimensions, if ω is the angular velocity of the axes, and $\mathbf{F} = (F_1, F_2)$ referred to these axes, then the components of $\dot{\mathbf{F}}$ are $\dot{F}_1 - \omega F_2$, $\dot{F}_2 + \omega F_1$.

(viii) *Example.* A particle moves on a certain curve. Find the components of acceleration referred to the moving axes **T**, **N**, **B**.

Here $\dfrac{d\mathbf{T}}{dt} = \kappa v\mathbf{N}$, $\dfrac{d\mathbf{N}}{dt} = -\kappa v\mathbf{T} + \lambda v\mathbf{B}$, $\dfrac{d\mathbf{B}}{dt} = -\lambda v\mathbf{N}$, from the Frenet-Serret formulae if v is ds/dt; so that the components of angular velocity of the axes are $\omega_1 = \lambda v$, $\omega_2 = 0$, $\omega_3 = \kappa v$.

Thus the components of $\dfrac{d\mathbf{v}}{dt}$ are $\dfrac{dv}{dt}$, κv^2, 0, since the components of **v** are $(v, 0, 0)$.

8.3. Scalar and Vector Functions. A function $V(x_1, x_2, \ldots, x_n)$ that is completely specified by the values of $(x_1, x_2, \ldots, x_n)$ is called a *scalar function* or *invariant*. A *vector function* $\mathbf{V}(x_1, x_2, \ldots, x_n)$ is one that requires (in m dimensions) m components for its specification, i.e. is a vector whose components are scalar functions. For simplicity in exposition, we shall often take $m = 3$, and consider functions of three (or fewer) variables.

If $V(x, y, z)$ is a single valued function defined for all x, y, z, then through any point (x_0, y_0, z_0) there is a single surface given by
$$V(x, y, z) = V(x_0, y_0, z_0).$$
It is often convenient, however, to drop the suffixes in x_0, y_0, z_0 and to refer merely to the V-surface through the point (x, y, z).

8.31. The Normal to a Surface. The straight line through the point $P(x_0, y_0, z_0)$ of direction-cosines l, m, n is given by
$$x = x_0 + lr,\ y = y_0 + mr,\ z = z_0 + nr$$
where r is the distance of the point (x, y, z) from P. It meets the surface $V(x, y, z) = V(x_0, y_0, z_0)$ in points determined by
$$V(x_0 + lr, y_0 + mr, z_0 + nr) - V(x_0, y_0, z_0) = 0$$
i.e. $r\left(l\dfrac{\partial V}{\partial x_0} + m\dfrac{\partial V}{\partial y_0} + n\dfrac{\partial V}{\partial z_0}\right) + O(r^2) = 0$, if V possesses bounded second derivatives at P.

One root is obviously zero, and one (at least) of the other roots tends to zero when the line approaches a direction that satisfies the equation $l\dfrac{\partial V}{\partial x_0} + m\dfrac{\partial V}{\partial y_0} + n\dfrac{\partial V}{\partial z_0} = 0$. If therefore the first derivatives do not *all* vanish at (x_0, y_0, z_0), the lines whose direction cosines satisfy the above

equation are *tangents* to the surface. Thus all the tangent lines to the surface through (x_0, y_0, z_0) lie on the plane

$$(x - x_0)\frac{\partial V}{\partial x_0} + (y - y_0)\frac{\partial V}{\partial y_0} + (z - z_0)\frac{\partial V}{\partial z_0} = 0$$

which is therefore called the *Tangent Plane* at (x_0, y_0, z_0). The normal to this plane is called the *normal to the surface* and its direction cosines are

$$\lambda\frac{\partial V}{\partial x_0},\ \lambda\frac{\partial V}{\partial y_0},\ \lambda\frac{\partial V}{\partial z_0},\ \text{where } \frac{1}{\lambda^2} = \left(\frac{\partial V}{\partial x_0}\right)^2 + \left(\frac{\partial V}{\partial y_0}\right)^2 + \left(\frac{\partial V}{\partial z_0}\right)^2$$

Notes. (i) The ambiguity in sign corresponds to the two directions in which the normal may be drawn. In applications the normal chosen is either stated or implied, and in the case of a simple closed surface is usually taken as the *outward* normal. It is often the case also that this corresponds to increasing λ.

(ii) If the first derivatives of V *all* vanish at (x_0, y_0, z_0), this point is said to be *singular* for the surface.

8.32. *The Directional Derivative for* $V(x, y, z)$. Let P be a given point, through which a straight line is drawn, and let Q be a variable point on this line distant s from P. Then $\lim\{(V_Q - V_P)/PQ\}$ when $Q \to P$ may be called the rate of change of V at P with respect to s.

This limit is $\lim_{s\to 0}\left\{\dfrac{V_x\,\delta x + V_y\,\delta y + V_z\,\delta z}{\delta s}\right\} = l_1V_x + m_1V_y + n_1V_z$

where l_1, m_1, n_1 are the direction-cosines of $\vec{PQ}$ and x, y, z are the co-ordinates of P. This may be called the *derivative of* V *in the direction* (l_1, m_1, n_1).

8.33. *The Gradient of a Function* $V(x, y, z)$. Since $l^2 + m^2 + n^2 = 1$, where (l, m, n) are direction-cosines, the directional derivative

$$lV_x + mV_y + nV_z$$

has obviously a maximum (and a minimum) determined by the equations

$$V_x + \mu l = V_y + \mu m = V_z + \mu n = 0$$

i.e.
$$\frac{l}{V_x} = \frac{m}{V_y} = \frac{n}{V_z} = -\frac{1}{\mu} = \pm\frac{1}{\sqrt{(V_x^2 + V_y^2 + V_z^2)}}$$

The maximum value is therefore $(V_x^2 + V_y^2 + V_z^2)^{\frac{1}{2}}$, and the corresponding direction is along the *normal* to the V-surface through (x, y, z) in the direction of increasing V. The *gradient* of $V(x, y, z)$ is defined to be the vector function whose direction is along the above normal and whose magnitude is the maximum derivative. In terms of the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, it is equal to $V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$ and is usually written ∇V or *grad* V.

Note. In n dimensions, grad $V(x_1, x_2, \ldots, x_n)$ is defined to be the vector function of components $\partial V/\partial x_r$.

8.34. *Vector Operators.* The symbol $\nabla \equiv \dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}$ may be regarded as an operator that gives the gradient when applied to a scalar function V. That it is vectorial in character may be verified directly

by making a co-ordinate transformation. If the axes are rectangular the formulae of transformation are of the type

$$x' = l_1x + m_1y + n_1z\,,\ y' = l_2x + m_2y + n_2z\,; \qquad z' = l_3x + m_3y + n_3z.$$
$$x = l_1x' + l_2y' + l_3z'\,;\ y = m_1x' + m_2y' + m_3z'\,;\ z = n_1x' + n_2y' + n_3z'.$$

If $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ denote unit vectors along the new axes, we have

$$\mathbf{i}' = l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k}\,;\ \mathbf{j}' = l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k}\,; \qquad \mathbf{k}' = l_3\mathbf{i} + m_3\mathbf{j} + n_3\mathbf{k}$$
$$\mathbf{i} = l_1\mathbf{i}' + l_2\mathbf{j}' + l_3\mathbf{k}'\,,\ \mathbf{j} = m_1\mathbf{i}' + m_2\mathbf{j}' + m_3\mathbf{k}'\,,\ \mathbf{k} = n_1\mathbf{i}' + n_2\mathbf{j}' + n_3\mathbf{k}'.$$

Now $(\nabla V)' = \dfrac{\partial V}{\partial x'}\mathbf{i}' + \dfrac{\partial V}{\partial y'}\mathbf{j}' + \dfrac{\partial V}{\partial z'}\mathbf{k}' = \Sigma(V_xl_1 + V_ym_1 + V_zn_1)\mathbf{i}'$

$$= \Sigma V_x(l_1\mathbf{i}' + l_2\mathbf{j}' + l_3\mathbf{k}') = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k} = \nabla V.$$

We can give an obvious interpretation to the operator $\mathbf{a}.\nabla$; for $(\mathbf{a}.\nabla)V$ may be defined as

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)V$$
$$= a_1V_x + a_2V_y + a_3V_z \quad (\text{where } \mathbf{a} = (a_1, a_2, a_3)).$$

Thus $\qquad (\mathbf{a}.\nabla)V = \mathbf{a}.\nabla V.$

We may also take $(\mathbf{a}.\nabla)\mathbf{V}$ to mean $a_1\dfrac{\partial\mathbf{V}}{\partial x} + a_2\dfrac{\partial\mathbf{V}}{\partial y} + a_3\dfrac{\partial\mathbf{V}}{\partial z}$ and no difficulty arises if we write $(\mathbf{a}.\nabla)\mathbf{V}$ as $\mathbf{a}.\nabla\mathbf{V}$, since $\nabla\mathbf{V}$ has not been given a meaning.

Notes. (i) If $\mathbf{a}$ is *unit* vector in the direction (l, m, n), then
$$\mathbf{a}.\nabla V = lV_x + mV_y + nV_z$$
the *directional derivative*.

(ii) If $\mathbf{r}$ is the position vector of (x, y, z), then
$$d\mathbf{r}.\nabla V = V_xdx + V_ydy + V_zdz = dV$$ § S ·3 $\dfrac{d\mathbf{r}}{ds}.\nabla V$ $\dfrac{dV}{ds}$
the *differential* of V.

8·35. *The Operators* $\nabla.$, $\nabla\times$. *Divergence and Curl.* If
$$\mathbf{V} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$$
is a vector function, then $\nabla.\mathbf{V}$ is defined to be
$$\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right).(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}).$$

Since $\mathbf{V}$ is given to be a vector and ∇ is *vectorial* in character, $\nabla.\mathbf{V}$ is a scalar (invariant for change of axes). Its value is $\dfrac{\partial V_1}{\partial x} + \dfrac{\partial V_2}{\partial y} + \dfrac{\partial V_3}{\partial z}$ and it is called the *Divergence* of $\mathbf{V}$ and sometimes written *div* $\mathbf{V}$. Again $\nabla\times\mathbf{V}$ is defined to be $\left(\dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}\right)\times(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$ and must be a vector. Thus

$$\nabla\times\mathbf{V} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

The vector $\nabla\times\mathbf{V}$ is called the *curl* (or rotation) of $\mathbf{V}$ and is sometimes written *curl* $\mathbf{V}$ (or *rot* $\mathbf{V}$)

Note. The reader should verify directly that for a transformation of axes, the expressions given for div $\mathbf{V}$ and curl $\mathbf{V}$ are invariantive and vectorial respectively.

8.36. The Symbols $(\mathbf{a} \times \nabla).\mathbf{V}$, $\mathbf{a}.(\nabla \times \mathbf{V})$, $\nabla.(\mathbf{V} \times \mathbf{a})$. Here it is assumed that $\mathbf{a}$ is a constant vector.

It follows from the formal expansion of the *scalar* triple product that these symbols all represent

$$a_1\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + a_2\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + a_3\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

and so this expansion is equivalent to

$$(\mathbf{a} \times \nabla).\mathbf{V} = \mathbf{a}.\operatorname{curl}\mathbf{V} = \operatorname{div}(\mathbf{V} \times \mathbf{a})$$

8.37. The Symbols $(\mathbf{a} \times \nabla) \times \mathbf{V}$, $\mathbf{a} \times (\nabla \times \mathbf{V})$, $\nabla \times (\mathbf{V} \times \mathbf{a})$. As above, we can find expressions for these from the formulae for *vector* triple products, if it is remembered that the symbol ∇ must precede $\mathbf{V}$.

(i) $(\mathbf{B} \times \mathbf{C}) \times \mathbf{A} = \mathbf{C}(\mathbf{A}.\mathbf{B}) - (\mathbf{C}.\mathbf{A})\mathbf{B}$ gives

$$(\mathbf{a} \times \nabla) \times \mathbf{V} = \nabla(\mathbf{V}.\mathbf{a}) - (\nabla.\mathbf{V})\mathbf{a} = \textit{grad}\,(\mathbf{V}.\mathbf{a}) - (\textit{div}\,\mathbf{V})\mathbf{a}$$

(ii) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{C}.\mathbf{A}) - (\mathbf{A}.\mathbf{B})\mathbf{C}$ gives

$$\mathbf{a} \times (\nabla \times \mathbf{V}) = \nabla(\mathbf{V}.\mathbf{a}) - (\mathbf{a}.\nabla)\mathbf{V}$$

or

$$\mathbf{a} \times \textit{curl}\,\mathbf{V} = \textit{grad}\,(\mathbf{V}.\mathbf{a}) - (\mathbf{a}.\nabla)\mathbf{V}.$$

(iii) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{B}.\mathbf{A})\mathbf{C}$ gives

$$\nabla \times (\mathbf{a} \times \mathbf{V}) = (\nabla.\mathbf{V})\mathbf{a} - (\mathbf{a}.\nabla)\mathbf{V}$$

or

$$\textit{curl}\,(\mathbf{a} \times \mathbf{V}) = (\textit{div}\,\mathbf{V})\mathbf{a} - (\mathbf{a}.\nabla)\mathbf{V}.$$

It may be noted from the above that

$$(\mathbf{a} \times \nabla) \times \mathbf{V} = \mathbf{a} \times \textit{curl}\,\mathbf{V} + \textit{curl}\,(\mathbf{V} \times \mathbf{a})$$

or

$$\nabla \times (\mathbf{V} \times \mathbf{a}) + \mathbf{a} \times (\nabla \times \mathbf{V}) + (\nabla \times \mathbf{a}) \times \mathbf{V} = 0.$$

8.38. Operations on Products of Functions. We can also determine expressions for

(i) ∇UV; (ii) $\nabla.(U\mathbf{V})$; (iii) $\nabla \times (U\mathbf{V})$; (iv) $\nabla.(\mathbf{U} \times \mathbf{V})$;
(v) $\nabla \times (\mathbf{U} \times \mathbf{V})$, (vi) $\nabla(\mathbf{U}.\mathbf{V})$.

The obvious method of determining these is to add the expression obtained when U or $\mathbf{U}$ is constant to that obtained when V or $\mathbf{V}$ is constant. Thus (i) ∇UV $(= \operatorname{grad} UV) = U\nabla V + V\nabla U$ (*vector*)

(ii) $\nabla.(U\mathbf{V})$ $(= \operatorname{div} U\mathbf{V}) = U\nabla.\mathbf{V} + \mathbf{V}.\nabla U$ (*scalar*).

(iii) $\nabla \times (U\mathbf{V})$ $(= \operatorname{curl} U\mathbf{V}) = U(\nabla \times \mathbf{V}) + (\nabla U) \times \mathbf{V}$ (*vector*).

(iv) From § 8.36, $\nabla.(\mathbf{a} \times \mathbf{V}) = -\mathbf{a}.(\nabla \times \mathbf{V})$ when $\mathbf{a}$ is constant. Therefore

$$\nabla.(\mathbf{U} \times \mathbf{V})\ (= \operatorname{div}\mathbf{U} \times \mathbf{V}) = \mathbf{V}.(\nabla \times \mathbf{U}) - \mathbf{U}.(\nabla \times \mathbf{V}) \quad (\textit{scalar}).$$

(v) From § 8.37, $\nabla \times (\mathbf{a} \times \mathbf{V}) = (\nabla.\mathbf{V})\mathbf{a} - (\mathbf{a}.\nabla)\mathbf{V}$. Therefore

$$\nabla \times (\mathbf{U} \times \mathbf{V})\ (= \operatorname{curl}\mathbf{U} \times \mathbf{V}) = (\nabla.\mathbf{V})\mathbf{U} - (\mathbf{U}.\nabla)\mathbf{V} - (\nabla.\mathbf{U})\mathbf{V} + (\mathbf{V}.\nabla)\mathbf{U} \quad (\textit{vector}).$$

(vi) From § 8.37, $\nabla(\mathbf{a}.\mathbf{V}) = \mathbf{a} \times (\nabla \times \mathbf{V}) + (\mathbf{a}.\nabla)\mathbf{V}$. Therefore

$$\nabla(\mathbf{U}.\mathbf{V}) = \mathbf{U} \times (\nabla \times \mathbf{V}) + \mathbf{V} \times (\nabla \times \mathbf{U}) + (\mathbf{U}.\nabla)\mathbf{V} + (\mathbf{V}.\nabla)\mathbf{U} (\textit{vector}).$$

i.e. (i) $\operatorname{grad}(UV) = U \operatorname{grad} V + V \operatorname{grad} U$.
(ii) $\operatorname{div}(U\mathbf{V}) = U \operatorname{div}\mathbf{V} + \mathbf{V}.\operatorname{grad} U$.
(iii) $\operatorname{curl}(U\mathbf{V}) = U \operatorname{curl}\mathbf{V} + (\operatorname{grad} U) \times \mathbf{V}$.
(iv) $\operatorname{div}(\mathbf{U} \times \mathbf{V}) = \mathbf{V}.\operatorname{curl}\mathbf{U} - \mathbf{U} \operatorname{curl}\mathbf{V}$.

(v) $\text{curl}\,(\mathbf{U} \times \mathbf{V}) = (\text{div}\,\mathbf{V})\mathbf{U} - (\text{div}\,\mathbf{U})\mathbf{V} + (\mathbf{V}\,.\,\nabla)\mathbf{U} - (\mathbf{U}\,.\,\nabla)\mathbf{V}$.

(vi) $\text{grad}\,(\mathbf{U}\,.\,\mathbf{V}) = \mathbf{V} \times \text{curl}\,\mathbf{U} + \mathbf{U} \times \text{curl}\,\mathbf{V} + (\mathbf{V}\,.\,\nabla)\mathbf{U} + (\mathbf{U}\,.\,\nabla)\mathbf{V}$.

8.39. Second Order Operators. By applying the operators ∇, $\nabla.$, $\nabla\times$ twice on suitable functions we may obtain the following expressions:
(i) $\nabla.(\nabla V)$ ($=$ div grad V); (ii) $\nabla \times (\nabla V)$ ($=$ curl grad V); (iii) $\nabla.(\nabla \times \mathbf{V})$ ($=$ div curl $\mathbf{V}$); (iv) $\nabla \times (\nabla \times \mathbf{V})$ ($=$ curl curl $\mathbf{V}$); (v) $\nabla(\nabla.\mathbf{V})$ ($=$ grad div $\mathbf{V}$).

Thus

(i) $\nabla.(\nabla V) = \nabla.(V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}) = V_{xx} + V_{yy} + V_{zz}$.

This is written $\nabla^2 V$ (and called the *Laplacian* of V).

The symbol $(\nabla.\nabla)\mathbf{V}$ may also be written $\nabla^2\mathbf{V}$ where this is taken to mean $(\nabla^2 V_1)\mathbf{i} + (\nabla^2 V_2)\mathbf{j} + (\nabla^2 V_3)\mathbf{k}$, ($V_1$, V_2, V_3 being the components of $\mathbf{V}$).

(ii) Since $\mathbf{a} \times \mathbf{a} = 0$, we deduce that $\nabla \times (\nabla V) = 0$.

(iii) Since $\mathbf{a}.(\mathbf{a} \times \mathbf{b}) = 0$, we deduce that $\nabla.(\nabla \times \mathbf{V}) = 0$.

(iv), (v) Since $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\mathbf{a}.\mathbf{b}) - (\mathbf{a}.\mathbf{a})\mathbf{b}$, we deduce that

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla.\mathbf{V}) - \nabla^2\mathbf{V}.$$

Thus
(i) div grad $V = \nabla^2 V = V_{xx} + V_{yy} + V_{zz}$;
(ii) curl grad $V = 0$; (iii) div curl $\mathbf{V} = 0$;
(iv), (v) curl curl $\mathbf{V} =$ grad div $\mathbf{V} - \nabla^2\mathbf{V}$.

Examples. (i) Find ∇r^m, $\nabla.(r^m\mathbf{r})$, $\nabla\times(r^m\mathbf{r})$, where $\mathbf{r}$ is the vector $\overrightarrow{OP}$ and OP is r.

$$\nabla r^m = mr^{m-1}\left(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}\right) = mr^{m-2}\mathbf{r}.$$

$$\nabla.(r^m\mathbf{r}) = (\nabla r^m)\,.\,\mathbf{r} + r^m(\nabla\,.\,\mathbf{r}) = mr^{m-2}r^2 + 3r^m = (m+3)r^m.$$

$$\nabla \times (r^m\mathbf{r}) = (\nabla r^m) \times \mathbf{r} + r^m(\nabla \times \mathbf{r}) = 0 \text{ since}$$
$$(\nabla r^m) \times \mathbf{r}) = mr^{m-2}(\mathbf{r} \times \mathbf{r}) = 0 \text{ and } \nabla \times \mathbf{r} = 0.$$

(ii) If $w\mathbf{v} = \nabla w$ prove that $\mathbf{v}.\text{curl}\,\mathbf{v} = 0$.

$$\text{curl}\,\mathbf{v} = \nabla\left(\frac{1}{w}\right) \times \nabla w + \frac{1}{w}\nabla \times (\nabla w) = \nabla\left(\frac{1}{w}\right) \times \nabla w$$

Therefore $\mathbf{v}.\text{curl}\,\mathbf{v} = \left(\frac{1}{w}\nabla w\right).\left(\nabla\left(\frac{1}{w}\right) \times \nabla w\right) = 0$

8.4. Transformation of Co-ordinates. A vector has a meaning that is independent of any particular co-ordinate system; and therefore a study of the relationship between its components in one system with those in another may be expected to give an indication of the essential characteristics of a vector. In order to deal satisfactorily with the effect on vectors of a transformation of co-ordinates, it is necessary to introduce the special notations and conventions that are used in this analysis.

8.41. The Summation Convention. Kronecker Deltas. In the Tensor (or Absolute) Calculus, quantities occur with various *affixes*, some of which may appear *above* the main symbol (or symbols) as in A^m, a^rb^s or *below* the symbols as in A_m, A_{rs}. In the former case the affix is called a *superscript* and in the latter a *subscript*. Symbols occur having both types of affixes, as for example A^{pq}_{mn} or $a_rb^sc^m_n$. If a symbol contains an affix that is *not* repeated, this affix is called *free* and is supposed to

take all values from 1 to N (where N is the number of dimensions under consideration).

Thus A_r is a symbol denoting the N values $A_1, A_2, \ldots, A_N$, whilst A^r_s denotes the N^2 values $A^1_1, A^2_1, A^1_2, \ldots, A^N_N$.

A *repeated* affix implies a *summation*. Thus $a_r b^r_s$ means the N expressions $\sum_{r=1}^{N} a_r b^r_s$ ($s = 1$ to N).

A repeated affix is sometimes called *umbral* since it may be changed to any other affix not appearing in the symbol. For example:

$$\lambda_{rs}X^s = \lambda_{rn}X^n;\ \lambda_{rs}X^s + \mu_{rt}X^t = \lambda_{rt}X^t + \mu_{rs}X^s.$$

An illustration of the type of quantity that occurs is provided by the Kronecker Delta δ^s_r which is defined to be 1 when $r = s$ and 0 when $r \neq s$. Thus N of these quantities are equal to unity, viz. $\delta^1_1, \delta^2_2, \ldots, \delta^N_N$ and the remaining $N^2 - N$ are zero.

Also $\delta^r_r = N$; and $\delta^r_s A^s = A^r$.

Note. Kronecker Deltas of higher order may also be constructed. Thus δ^{mn}_{pq} is defined to be zero except when (i) $m = p$, $n = q$ ($m \neq n$) its value then being $+1$ and (ii) $m = q$, $n = p$ ($m \neq n$), its value then being -1. It follows that $\delta^{rm}_{sm} = (\delta^{r1}_{s1} + \ldots + \delta^{rN}_{sN}) = 0$ if $r \neq s$ and $= N - 1$ if $r = s$ (since $\delta^{11}_{11} = 0$, for example)

i.e. $$\delta^{rm}_{sm} = (N-1)\delta^r_s;\ \text{also } \delta^{mn}_{mn} = (N-1)\delta^m_m = N(N-1).$$

8.42. *Linear Transformations.* A displacement vector may be indicated by the symbol x^r, where in anticipation of a result to be proved later, the affix is written as a superscript.

A linear transformation is given by the equation

$$\bar{x}^r = a^r_s x^s\ (= a^r_1 x^1 + \ldots + a^r_N x^N,\ r = 1 \text{ to } N)$$

where the determinant of the coefficients, sometimes written $|a^r_s|$, is finite and not zero.

By solving these equations, we express x^r in terms of $\bar{x}^s$ in the form $x^r = \alpha^r_s \bar{x}^s$ where $\Delta\alpha^s_r$ is the cofactor of a^r_s in the determinant $\Delta = |a^r_s|$.

By the substitution of $x^s = \alpha^s_m \bar{x}^m$ in the equation for $\bar{x}^r$ (or by using the properties of a determinant), we see that $a^r_s \alpha^s_m = 1$ when $r = m$ and zero when $r \neq m$,

i.e. $$a^r_s \alpha^s_m = \delta^r_m \text{ and similarly } \alpha^s_r a^m_s = \delta^m_r.$$

In view of more general transformations, we can write these results as

$$\bar{x}^r = \frac{\partial \bar{x}^r}{\partial x^s}\, x^s;\ x^r = \frac{\partial x^r}{\partial \bar{x}^s}\cdot \bar{x}^s;$$

where $$\frac{\partial \bar{x}^r}{\partial x^s}\cdot\frac{\partial x^s}{\partial \bar{x}^t} = \frac{\partial x^r}{\partial \bar{x}^s}\cdot\frac{\partial \bar{x}^s}{\partial x^t} = \delta^r_t.$$

Note. A special case arises when the two sets of axes are orthogonal. Suppose for example, $N = 3$ and (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are the direction cosines of $O\bar{X}$, $O\bar{Y}$, $O\bar{Z}$ with respect to OX, OY, OZ, and that the two systems are of the

same species. Then (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3) are the direction cosines of OX, OY, OZ with respect to $O\bar{X}$, $O\bar{Y}$, $O\bar{Z}$.

Thus $\bar{x} = l_1x + m_1y + n_1z$, $\bar{y} = l_2x + m_2y + n_2z$; $\bar{z} = l_3x + m_3y + n_3z$, and $x = l_1\bar{x} + l_2\bar{y} + l_3\bar{z}$; $y = m_1\bar{x} + m_2\bar{y} + m_3\bar{z}$; $z = n_1\bar{x} + n_2\bar{y} + n_3\bar{z}$.

Typical relations satisfied by these coefficients are $l_1^2 + m_1^2 + n_1^2 = 1$; $l_1^2 + l_2^2 + l_3^2 = 1$; $l_1 = m_2n_3 - m_3n_2$; and the determinants of the coefficients are both equal to 1.

This is a special case when $a^r_s = \alpha^r_s$ and $|a^s_r| = |\alpha^s_r| = 1$

8.43. Functional Transformations. When $\bar{x}^r$ is a function of x^r (not necessarily linear), the differential displacements $d\bar{x}^r$ are connected *linearly* with dx^r at x^r, provided $\partial(x^1, x^2, \ldots, x^N)/\partial(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ is finite and not zero. The new framework of reference consists of the loci $\bar{x}^r =$ constant (which in 3 dimensions are surfaces) and the tangents to the various loci obtained when all but one of the variables $\bar{x}^r$ are constant form the axes of reference at the point x^r. The equations connecting the two displacement vectors $d\bar{x}^r$, dx^r are obviously given by $d\bar{x}^r = \dfrac{\partial\bar{x}^r}{\partial x^s}dx^s$ or $dx^r = \dfrac{\partial x^r}{\partial\bar{x}^s}d\bar{x}^s$, where, as in the last paragraph,

$$\frac{\partial\bar{x}^r}{\partial x^s}\frac{\partial x^s}{\partial\bar{x}^t} = \frac{\partial x^r}{\partial\bar{x}^s}\frac{\partial\bar{x}^s}{\partial x^t} = \delta^r_t.$$

8.44. Covariant and Contravariant Vectors. The *gradient* of $V(x^1, x^2, \ldots, x^N)$ (an invariant) is defined to be the vector $\dfrac{\partial V}{\partial x^r}$, and when the variables are changed we have

$$\frac{\partial V}{\partial\bar{x}^r} = \frac{\partial V}{\partial x^s}\frac{\partial x^s}{\partial\bar{x}^r}$$

Also we have shown that $d\bar{x}^r = \dfrac{\partial\bar{x}^r}{\partial x^s}.dx^s$.

Thus vectors appear to be of *two* kinds, one which is transformed like the gradient and one like the displacement vector. The former is called *Covariant* and the latter *Contravariant*. Thus if X^r is a vector that obeys the law

$$\bar{X}^r = \frac{\partial\bar{x}^r}{\partial x^s}X^s$$

it is called a *Contravariant Vector*, a *superscript* being used to denote contravariance whilst if X_r is a vector obeying the law

$$\bar{X}_r = \frac{\partial x^s}{\partial\bar{x}^r}X_s$$

it is called a *Covariant Vector*, a *subscript* being used to denote covariance.

8.45. Tensors. If the n^2 quantities X_{rs} obey the law of transformation

$$\bar{X}_{rs} = \frac{\partial x^p}{\partial\bar{x}^r}.\frac{\partial x^q}{\partial\bar{x}^s}X_{pq}$$

X_{rs} is called a *Covariant Tensor* of the *second* order

If the n^2 quantities X^{rs} obey the law

$$\bar X^{rs} = \frac{\partial \bar x^r}{\partial x^p}\frac{\partial \bar x^s}{\partial x^q}X^{pq}$$

X^{rs} is called a *Contravariant Tensor* of the *second* order. It is possible also to have a mixed tensor X^r_s if it obeys the law

$$\bar X^r_s = \frac{\partial \bar x^r}{\partial x^p}\cdot\frac{\partial x^q}{\partial \bar x^s}\cdot X^p_q.$$

Similarly, we may have tensors of any order having contravariant and covariant properties. Thus we should write a tensor of the fifth order as X^{pq}_{rst} if it obeyed the law

$$\bar X^{pq}_{rst} = \frac{\partial \bar x^p}{\partial x^\alpha}\frac{\partial \bar x^q}{\partial x^\beta}\frac{\partial x^\gamma}{\partial \bar x^r}\cdot\frac{\partial x^\delta}{\partial \bar x^s}\frac{\partial x^\epsilon}{\partial \bar x^t}X^{\alpha\beta}_{\gamma\delta\epsilon},$$

and there is of course a similar equation for the inverse transformation :

$$X^{pq}_{rst} = \frac{\partial x^p}{\partial \bar x^\alpha}\cdot\frac{\partial x^q}{\partial \bar x^\beta}\cdot\frac{\partial \bar x^\gamma}{\partial x^r}\cdot\frac{\partial \bar x^\delta}{\partial x^s}\frac{\partial \bar x^\epsilon}{\partial x^t}\bar X^{\alpha\beta}_{\gamma\delta\epsilon}.$$

Notes. (i) A vector is a tensor of the first order.

(ii) It is important to note that any free affix appearing on the left of such an equation as the one above should appear in the same place on the right, and that any umbral affix appearing on one side of the equation only should occur once as a superscript and once as a subscript, this is a characteristic feature of Tensor Equations.

8.46. Addition and Multiplication of Tensors. Two tensors of the same order and species may be *added* to form another of the same order and species.

Thus $X_{rs} + Y^m_r Z_{ms}$ is a covariant tensor of the second order if X_{rs}, $Y^m_r Z_{ms}$ are covariant tensors of the second order.

The *product* of two tensors of orders k_1, k_2 is a tensor of order $k_1 + k_2$.

Thus if $\bar X^p_{qr} = \frac{\partial \bar x^p}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar x^q}\cdot\frac{\partial x^\gamma}{\partial \bar x^r}\cdot X^\alpha_{\beta\gamma}$ and $\bar Y^s_t = \frac{\partial \bar x^s}{\partial x^\lambda}\frac{\partial x^\mu}{\partial \bar x^t}Y^\lambda_\mu$ then $X^p_{qr}Y^s_t$ is obviously a tensor of the fifth order that might be denoted by Z^{ps}_{qrt}.

8.47. The Substitution Operator. Since $\frac{\partial \bar x^r}{\partial x^s}\cdot\frac{\partial x^s}{\partial \bar x^t} = \delta^r_t$, then

$$\frac{\partial \bar x^r}{\partial x^s}\cdot\frac{\partial x^s}{\partial \bar x^t}A(t) = A(r)$$

where $A(t)$ is any expression involving the affix t. The operator

$$\frac{\partial \bar x^r}{\partial x^s}\frac{\partial x^s}{\partial \bar x^t}$$

is therefore called a *substitution operator*. Similarly

$$\frac{\partial \bar x^r}{\partial x^s}\cdot\frac{\partial x^t}{\partial \bar x^r}\ (= \delta^t_s)$$

is a substitution operator.

The operator $\dfrac{\partial \bar{x}^r}{\partial x^s}\cdot\dfrac{\partial x^s}{\partial \bar{x}^t}$ is a mixed tensor of the second order. For

$$\delta^\alpha_\beta \frac{\partial \bar{x}^r}{\partial x^\alpha}\cdot\frac{\partial x^\beta}{\partial \bar{x}^t} = \frac{\partial \bar{x}^r}{\partial x^\alpha}\cdot\frac{\partial x^\alpha}{\partial \bar{x}^t} = \delta^r_t = \bar{\delta}^r_t.$$

8.48. Contraction of Tensors. From such a tensor as A^{rst}_{pq} we can form a tensor of lower order by writing a superscript identical with a subscript and obtaining for example A^{rsq}_{pq}.

$$\text{Now } \bar{A}^{rsq}_{pq} = \frac{\partial \bar{x}^r}{\partial x^\alpha}\cdot\frac{\partial \bar{x}^s}{\partial x^\beta}\cdot\frac{\partial \bar{x}^q}{\partial x^\gamma}\cdot\frac{\partial x^\lambda}{\partial \bar{x}^p}\cdot\frac{\partial x^\nu}{\partial \bar{x}^q}\, A^{\alpha\beta\gamma}_{\lambda\nu}$$

$$= \frac{\partial \bar{x}^r}{\partial x^\alpha}\cdot\frac{\partial \bar{x}^s}{\partial x^\beta}\cdot\frac{\partial x^\lambda}{\partial \bar{x}^p}\cdot A^{\alpha\beta\nu}_{\lambda\nu} \text{ since } \frac{\partial \bar{x}^q}{\partial x^\gamma}\cdot\frac{\partial x^\nu}{\partial \bar{x}^q} A^{\alpha\beta\gamma}_{\lambda\nu} = A^{\alpha\beta\nu}_{\lambda\nu}$$

so that A^{rsq}_{pq} is a tensor of the third order. This operation is called the *contraction of tensors*.

Similarly by further contraction we obtain the vector A^{rpq}_{pq}.

Note. Contraction is obtained by making a contravariant affix identical with a covariant affix and no significance is attached to the expression obtained by making two affixes of the same kind identical.

8.49. The Quotient Law Suppose we are given that

$$A(r, s, t)B^{st} = C^r$$

under transformations, where C^r is a certain vector and B^{st} is an *arbitrary* tensor of the second order.

$$\text{Then } \bar{A}(r, s, t).\bar{B}^{st} = \bar{C}^r = \frac{\partial \bar{x}^r}{\partial x^\alpha} C^\alpha = \frac{\partial \bar{x}^r}{\partial x^\alpha} A(\alpha, p, q)B^{pq}$$

$$= \frac{\partial \bar{x}^r}{\partial x^\alpha}\cdot\frac{\partial x^p}{\partial \bar{x}^s}\cdot\frac{\partial x^q}{\partial \bar{x}^t} A(\alpha, p, q)\bar{B}^{st}.$$

But if $\bar{B}^{st}$ is arbitrary, we must therefore have

$$\bar{A}(r, s, t) = \frac{\partial \bar{x}^r}{\partial x^\alpha}\cdot\frac{\partial x^p}{\partial \bar{x}^s}\cdot\frac{\partial x^q}{\partial \bar{x}^t}\cdot A(\alpha, p, q)$$

so that $A(r, s, t)$ is a tensor of the third order that should be denoted by A^r_{st}.

Similarly if $C^r_\alpha\ .\ B^\lambda_\mu$ are tensors, the latter being arbitrary, and if $A(r\ldots, \alpha\ldots, \lambda\ldots)\, B^\lambda_\mu = C^r_\alpha$, we may show that A is a tensor of the type $A^{r\mu}_{\alpha\lambda}$.

8.5. The Fundamental Double Tensors. When the 'distance' ds between the points

$$(\xi^1, \xi^2, \ldots, \xi^N), (\xi^1 + d\xi^1, \xi^2 + d\xi^2, \ldots, \xi^N + d\xi^N)$$

is (as in rectangular Cartesian co-ordinates) given by

$$ds^2 = d\xi^1 d\xi^1 + d\xi^2 d\xi^2 + \ldots + d\xi^N d\xi^N,$$

the space of N dimensions is said to be *flat*.

Notes. (i) More generally, a space is *flat* if by transformation of co-ordinates it can be reduced to the above form.

(ii) The symbol $d\xi^r d\xi^r$ is used for the *square* of $d\xi^r$ so as to avoid ambiguity of

the index symbols. There is, however, little likelihood of confusion in the use of ds^2 for the square of ds.

In a transformation, the *invariant* ds^2 becomes $ds^2 = g_{mn}\,dx^m\,dx^n$ where $g_{mn} = \dfrac{\partial \xi^\lambda}{\partial x^m}\cdot\dfrac{\partial \xi^\lambda}{\partial x^n}$.

If, however, we define the invariant ds^2 by means of this general quadratic form, it cannot in general be transformed into the sum of n positive squares with constant coefficients. Space in which ds^2 is defined by the general quadratic form is therefore, in general, *curved*. One of the investigations occurring in tensor analysis consists in determining the conditions for which the space is flat.

Since dx^n may be regarded as an arbitrary tensor in the expression for the *invariant* ds^2, it follows by the quotient law that $g_{mn}\,dx^m$ is a *covariant vector*; and, by a further application of the law, that g_{mn} is a *covariant tensor* of the second order. The position of the affixes in g_{mn} is therefore justified.

Let $g = |g_{mn}|$ (the determinant of the g's), where $g_{mn} = g_{nm}$, and let the cofactor of g_{mn} be divided by g and the quotient be denoted by g^{mn} (thus anticipating its contravariant character)

Consider $g_{mr}\,g^{nr}$. By the properties of the determinant this is zero when $m \neq n$ and 1 when $m = n$.

Thus $g_{mr}g^{nr}$ is the mixed tensor $\delta^n_m \left(= \dfrac{\partial x^n}{\partial \bar{x}^r}\cdot\dfrac{\partial \bar{x}^r}{\partial x^m}\right)$ so that

$$g_{mr}g^{nr}A(m) = A(n)$$

and if A^m is *any* contravariant vector $g_{mr}g^{nr}A^m = A^n$.

This mixed tensor $g_{mr}g^{nr}$ is here usually denoted by g^n_m. Again, since $g_{mr}A^m$ may be regarded as an *arbitrary* covariant vector and $g^{nr}(g_{mr}A^m) = A^n$, it follows that g^{nr} is a contravariant tensor of the second order.

The tensors g_{mn}, g^n_m, g^{mn} are called the *Fundamental Double Tensors.*

8.51. Raising and Lowering Affixes. From the vector A_n we obtain another A^n by means of the relation $A^n = g^{mn}A_m$. This is called '*Raising the Affix*'. Similarly from A^n we obtain A_n by the relation $A_n = g_{mn}A^m$ and this is called '*Lowering the Affix*'. Although the *form* of the vector is altered in this way, the two forms should be regarded as equivalent ways of representing the *same* vector. For this reason a vector and that obtained by raising or lowering an affix are sometimes called *Associated Vectors.*

Similarly, we may raise or lower affixes in tensors of any order and obtain associated tensors. For example

$$g^{mn}A^{\cdot q}_{rm} = A^{\cdot qn}_{r\cdot}.$$

It is sometimes necessary to indicate which particular affix has been raised or lowered and this can be done by allotting a certain place for each affix in the lower and upper positions.

Thus if A^{pq}_{rsn} be denoted by $A^{pq}{}_{rsn}$ we may write

$$g^{rt}A^{pq}{}_{tsn} = A^{pqr}{}_{sn}.$$

This is unnecessary in certain cases of symmetry, for if $A_{rs} = A_{sr}$, then $g^{mr}A_{rs} = g^{mr}A_{sr} = g^{ms}A_{rs}$ and both can be written A^m_s.

Notes. (i) If there is an *umbral* affix, it may be raised in one place if it is lowered in the other

Thus $A_{mn}B^m = g^{mr}g_{pr}A_{mn}B^p = (g^{mr}A_{mn})(g_{pr}B^p)$

$$= A^r{}_n B_r = A^m{}_n B_m$$

(ii) In a tensor equation, a *free* affix may be raised (or lowered) throughout the equation.

Thus if $$A_{rs}B^s{}_t C^{tp} = D^p_r,$$

then $$A^r{}_s B^s{}_t C^{tp} = D^{pr} \text{ or } A_{rs}B^s{}_t C^t{}_p = D_{pr}.$$

8.52. The Christoffel Symbols. In the further development of tensor analysis, two expressions (which are not tensors) occur which are of fundamental importance.

They are (i) $[\mu\nu, \lambda] = \dfrac{1}{2}\left(\dfrac{\partial g_{\mu\lambda}}{\partial x^\nu} + \dfrac{\partial g_{\nu\lambda}}{\partial x^\mu} - \dfrac{\partial g_{\mu\nu}}{\partial x^\lambda}\right).$

$$\text{(ii)}\ \{\mu\nu, \lambda\} = \frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right) = g^{\lambda\rho}[\mu\nu, \rho].$$

They are called the *Christoffel Symbols* (or the *three-index symbols*) of the first and second kind respectively.

We have seen that when the space is *flat*, $g_{mn} = \dfrac{\partial \xi^r}{\partial x^m}\dfrac{\partial \xi^r}{\partial x^n}$ and we can find expressions for the symbols, in this special case, in terms of the derivatives of ξ^r with respect to x^m.

For $\dfrac{\partial g_{\mu\lambda}}{\partial x^\nu} = \dfrac{\partial^2 \xi^r}{\partial x^\mu \partial x^\nu}\dfrac{\partial \xi^r}{\partial x^\lambda} + \dfrac{\partial \xi^r}{\partial x^\mu}\dfrac{\partial^2 \xi^r}{\partial x^\lambda \partial x^\nu}$ with two similar expressions for $\dfrac{\partial g_{\nu\lambda}}{\partial x^\mu}, \dfrac{\partial g_{\mu\nu}}{\partial x^\lambda}$.

Thus

$$2\left\{\frac{\partial^2 \xi^r}{\partial x^\mu \partial x^\nu}\frac{\partial \xi^r}{\partial x^\lambda} + \frac{\partial^2 \xi^r}{\partial x^\nu \partial x^\lambda}\cdot\frac{\partial \xi^r}{\partial x^\mu} + \frac{\partial^2 \xi^r}{\partial x^\lambda \partial x^\mu}\cdot\frac{\partial \xi^r}{\partial x^\nu}\right\} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} + \frac{\partial g_{\lambda\mu}}{\partial x^\nu}$$

and $[\mu\nu, \lambda] = \dfrac{\partial^2 \xi^r}{\partial x^\mu \partial x^\nu}\cdot\dfrac{\partial \xi^r}{\partial x^\lambda}.$

Examples (i) $[\mu\nu, \lambda] = \{\mu\nu, \lambda\} = 0$ when the g's are constant.
(ii) $g_{\lambda\rho}\{\mu\nu, \rho\} = g_{\lambda\rho}g^{\sigma\rho}[\mu\nu, \sigma] = [\mu\nu, \lambda]$.
(iii) $[\mu\nu, \lambda] = [\nu\mu, \lambda]$; $\{\mu\nu, \lambda\} = \{\nu\mu, \lambda\}$
(iv) $[\mu\nu, \lambda] + [\nu\lambda, \mu] + [\lambda\mu, \nu] = \frac{1}{2}\left(\dfrac{\partial g_{\mu\nu}}{\partial x^\lambda} + \dfrac{\partial g_{\nu\lambda}}{\partial x^\mu} + \dfrac{\partial g_{\lambda\mu}}{\partial x^\nu}\right)$;

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = [\nu\lambda, \mu] + [\lambda\mu, \nu].$$

(v) Show that $\{mp,\ m\} = \dfrac{\partial \log \sqrt{g}}{\partial x^p}$.

Since $dg = G^{mn}\, dg_{mn}$ where G^{mn} is the cofactor of g_{mn} in g, we have

$$\frac{dg}{g} = g^{mn}\, dg_{mn}; \quad \text{also} \quad \{mp,\ m\} = \tfrac{1}{2} g^{mn} \frac{\partial g_{mn}}{\partial x^p}$$

since $g^{mn}\dfrac{\partial g_{np}}{\partial x^m} = g^{mn}\dfrac{\partial g_{mp}}{\partial x^n}$ by symmetry of $g_{\alpha\beta}$

$$= g^{nm}\frac{\partial g_{mp}}{\partial x^n} \text{ by interchange of umbral affixes}$$

i.e.
$$\{mp,\ m\} = \frac{1}{2g}\frac{\partial g}{\partial x^p} = \frac{\partial}{\partial x^p}(\log \sqrt{g}).$$

Note The results in these examples are of course true for any set g_{mn} of fundamental tensors.

8.6. Covariant Derivatives. If A^r is a constant vector in flat space, the corresponding displacement vector at a point P moves parallel to itself as the point P moves. The vector A^r is therefore said to represent a *uniform field*. If P moves along a curve, the vector A^r represents a uniform (or parallel) field of vectors along the curve. When a transformation is made to the system x^m, A^r is transformed into B^r where $A^r = B^m\dfrac{\partial \xi^r}{\partial x^m}$, ($\xi^r$ being the original Cartesian system).

If A^r is constant along a curve (specified by a parameter t)

$$\frac{dB^m}{dt}\cdot\frac{\partial \xi^r}{\partial x^m} + B^m\frac{\partial^2 \xi^r}{\partial x^m\, \partial x^n}\cdot\frac{dx^n}{dt} = 0$$

i.e.
$$g^{\alpha p}\frac{\partial \xi^r}{\partial x^p}\cdot\frac{\partial \xi^r}{\partial x^m}\frac{dB^m}{dt} + g^{\alpha p}\frac{\partial \xi^r}{\partial x^p}\cdot\frac{\partial^2 \xi^r}{\partial x^m\, \partial x^n}B^m\frac{dx^n}{dt} = 0$$

giving
$$g^{\alpha p}g_{mp}\frac{dB^m}{dt} + \{mn,\ \alpha\}B^m\frac{dx^n}{dt} = 0$$

or
$$\frac{dB^\alpha}{dt} + \{mn,\ \alpha\}B^m\frac{dx^n}{dt} = 0.$$

More generally, if B^α represents a uniform field throughout flat space, since the above equation is true for all curves passing through any point P, we have

$$\frac{\partial B^\alpha}{\partial x^\beta} + \{m\beta,\ \alpha\}B^m = 0.$$

If X_α is a given covariant vector, then $X_\alpha B^\alpha$ is an invariant and its partial derivative a covariant vector.

This partial derivative is

$$\frac{\partial X_\alpha}{\partial x^\beta}B^\alpha + X_\alpha\frac{\partial B^\alpha}{\partial x^\beta} = \frac{\partial X_\alpha}{\partial x^\beta}B^\alpha - X_\alpha\{m\beta,\ \alpha\}B^m$$

$$= \left(\frac{\partial X_\alpha}{\partial x^\beta} - \{\alpha\beta,\ n\}X_n\right)B^\alpha$$

But since B^α may be taken arbitrarily, it follows that

$$\frac{\partial X_\alpha}{\partial x^\beta} - \{\alpha\beta,\ n\}X_n$$

is a tensor. It is called the *Covariant Derivative* of X_α and written $X_{\alpha,\ \beta}$.

The covariant derivative of X^α may be obtained similarly, but it is more instructive to obtain it from the above result. The covariant derivative of X^α is written $X^\alpha{}_{,\ \beta}$ and may be defined simply as $g^{\alpha p}X_{p,\ \beta}$.

Now $X_p = g_{pq}X^q$ and therefore by covariant differentiation

$$X_{p,\ \beta} = \frac{\partial X_p}{\partial x^\beta} - \{p\beta,\ s\}X_s = g_{pq}\frac{\partial X^q}{\partial x^\beta} + X^q\frac{\partial g_{pq}}{\partial x^\beta} - \{p\beta,\ s\}g_{sq}X^q$$

$$= g_{pq}\frac{\partial X^q}{\partial x^\beta} + [q\beta,\ p]X^q$$

since $\dfrac{\partial g_{pq}}{\partial x^\beta} = [q\beta,\ p] + [p\beta,\ q]$ and $g_{sq}\{p\beta,\ s\} = [p\beta,\ q]$

i.e. $$X^\alpha{}_{,\ \beta} = g^{\alpha p}X_{p,\ \beta} = g^{\alpha p}g_{pq}\frac{\partial X^q}{\partial x^\beta} + g^{\alpha p}[q\beta,\ p]X^q$$

$$= \frac{\partial X^\alpha}{\partial x^\beta} + \{q\beta,\ \alpha\}X^q$$

Note. Although we have used the properties of flat space to obtain these derivatives, the expressions are tensors in any space for which $ds^2 = g_{mn}\,dx^m\,dx^n$. It is possible to choose a flat space for which the values of g_{mn} and their first derivatives agree with a given space at a given point. The Tensor law is satisfied, the value of g_{mn} under a transformation being $\bar{g}_{mn} = \dfrac{\partial x^p}{\partial \bar{x}^m}\dfrac{\partial x^q}{\partial \bar{x}^n}g_{pq}$. ?

8.61. Tensor Derivatives. The covariant derivative of a tensor $X^{\alpha}_{\cdot\beta\cdot}$ may now be obtained by writing down the ordinary derivative $\dfrac{\partial X}{\partial x^n}$ and adding (i) $-\{\beta n,\ r\}X_{\cdot r}$ for every covariant affix β, (ii) $\{rn,\ \alpha\}X^{r}_{\cdot\cdot}$ for every contravariant affix α.

It is sufficient to prove this for a particular case such as X^m_p since the method indicated is general.

Let A^α, B_β be two arbitrary uniform fields; then $X^m_p A^p B_m$ is invariant and its ordinary derivative is a covariant vector. This derivative is

$$\frac{\partial X^m_p}{\partial x^n}A^p B_m + X^m_p\frac{\partial A^p}{\partial x^n}B_m + X^m_p A^p\frac{\partial B_m}{\partial x^n}$$

$$= \frac{\partial X^m_p}{\partial x^n}A^p B_m - X^m_p\{rn,\ p\}A^r B_m + X^m_p\{mn,\ r\}A^p B_r$$

$$= \left(\frac{\partial X^m_p}{\partial x^n} - \{pn,\ q\}X^m_q + \{qn,\ m\}X^q_p\right)A^p B_m.$$

i.e. $\dfrac{\partial X^m_p}{\partial x^n} - \{pn,\ q\}X^m_q + \{qn,\ m\}X^q_p$ is a tensor of the third order that may be written $X^m_{p,\ n}$.

8.62. Rules for Covariant Differentiation. The ordinary rules for the differentiation of a sum or a product are conserved in covariant differentiation,
i.e.

$$(X^r + Y^r)_n = (X^r)_n + (Y^r)_n;\quad [(X^r)(Y^s)]_n = (X^r)_n(Y^s) + (X^r)(Y^s)_n.$$

The first result is obvious and the second is obvious for at least a *free* index appearing in the product. If, however, there is a repeated affix α as in the case $X^\alpha Y_\alpha$, than in $(X^\alpha)_n$, there is a term $\{rn, \alpha\}X^r$, and in $(Y_\alpha)_n$ there is a term $-\{\alpha n, r\}Y_r$. These two terms in the expression for the derivative give

$$\{rn, \alpha\}X^r Y_\alpha - \{\alpha n, r\}X^\alpha Y_r$$

which vanishes by the rule for repeated affixes.

8.63. The Covariant Derivatives of g_{mn} are zero.

$$g_{mn,r} = \frac{\partial g_{mn}}{\partial x^r} - \{mr, \alpha\}g_{\alpha n} - \{nr, \alpha\}g_{m\alpha}$$

$$= \frac{\partial g_{mn}}{\partial x^r} - [mr, n] - [nr, m] = 0$$

The fundamental tensors may therefore be regarded as constants in covariant differentiation, and affixes may be raised or lowered before or after differentiation.

Note. This result is, as shown above, true for any space, but is obviously true for flat space, since the variables may be changed to give a value for ds^2 in which g_{mn} is constant, and therefore the ordinary derivatives of g_{mn} and the Christoffel symbols vanish.

8.64. Gradient, Divergence, Laplacian. We have already seen that the gradient of an invariant ϕ is a covariant vector which we may denote by $\nabla\phi$ or $\dfrac{\partial\phi}{\partial x^n}$ or $(\phi)_n$.

The tensor formed from X^r by covariant differentiation is $X^r_{,s}$, and the contracted tensor $X^r_{,r}$ is therefore invariant. This invariant is called the *Divergence* (and agrees with the corresponding definition in Cartesian co-ordinates).

Since $$X^n = g^{mn}X_m,\ \text{div}\ X^n = g^{mn}X_{m,n}$$

The divergence of the gradient $(\phi)_m$ may be written $(\phi)_{m,n}$ or $\nabla^2(\phi)$ and is called the *Laplacian*.

Thus $$\nabla^2\phi = g^{mn}\phi_{m,n}.$$

8.65. Magnitudes of Vectors and Scalar Products. Since the magnitude of ds is $(g_{mn}\,dx^m\,dx^n)^{\frac{1}{2}}$, the magnitude of the contravariant vector A^m is A where

$$A = (g_{mn}A^mA^n)^{\frac{1}{2}} = (g^{mn}A_mA_n)^{\frac{1}{2}}.$$

The *scalar product* of A^m, B^n is defined to be $g_{mn}A^mB^n$ and this is equivalent to $A_nB^n = A^nB_n = g^{mn}A_mB_n$.

Note. The scalar product (which is sometimes called the *inner* product) obviously agrees with that defined in terms of Cartesian co-ordinates; for when the axes are rectangular, the invariant becomes $AB\cos\theta$ where θ is the angle between the vectors.

8.66. Orthogonal Co-ordinates. The magnitude of the single displacement dx^r is $\sqrt{g_{rr}}\,dx^r$ and therefore the angle θ_{rs} between the displacements dx^r, dx^s is given by $\sqrt{g_{rr}}\sqrt{g_{ss}}\,dx^r\,dx^s\cos\theta_{rs} = g_{rs}\,dx^r\,dx^s$, where from now until the end of the paragraph we drop the summation convention for a repeated affix (unless otherwise stated).

i.e. $$\cos\theta_{rs} = g_{rs}/\sqrt{(g_{rr}g_{ss})}$$

The co-ordinates are therefore orthogonal if $g_{rs} = 0$, $(r \neq s)$,

i.e. $$ds^2 = g_{11}dx^1\,dx^1 + \ldots + g_{NN}\,dx^N\,dx^N.$$

If we write $g_{rr} = h_r^2$ (the square of h_r), then

$$g = h_1^2h_2^2\ldots h_N^2,\quad g^{rr} = \frac{1}{h_r^2}.$$

It may easily be verified that

$$[rr, r] = h_r\frac{\partial h_r}{\partial x^r},\quad [rs, t] = 0\ (r, s, t \text{ all different}),$$

$$[rs, s] = h_s\frac{\partial h_s}{\partial x^r} = -[ss, r],\ (r \neq s),$$

$$\{rr, r\} = \frac{1}{h_r}\frac{\partial h_r}{\partial x^r};\quad \{rs, s\} = \frac{1}{h_s}\frac{\partial h_s}{\partial x^r},\quad \{ss, r\} = -\frac{h_s}{h_r^2}\frac{\partial h_s}{\partial x^r};\quad \{rs, t\} = 0$$

Example. Determine the Divergence and Laplacian for orthogonal co-ordinates in three dimensions. This is merely taken as an illustration and is, of course, not the best method of obtaining the formulae in three dimensions.

$$\operatorname{div} X^r = X^r_{,r} = \frac{\partial X^r}{\partial x^r} + \{rp, r\}X^p = \frac{\partial X^r}{\partial x^r} + \frac{\partial}{\partial x^p}(\log\sqrt{g})X^p,$$

$$= \frac{\partial X^r}{\partial x^r} + \frac{\partial(\log\sqrt{g})}{\partial x^r}X^r = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}(\sqrt{g}X^r),$$

a formula applicable to the general case and the summation convention being used.

For orthogonal co-ordinates in three dimensions

$$ds^2 = \sum_1^3 h_r^2(dx^r)^2$$

A *unit* displacement along the x^1 axis corresponds to an increase of $1/h_1$ in x^1. A displacement U_1 along x^1 corresponds therefore to an increase of U_1/h_1 in x^1. A vector X^r of components of actual magnitudes U_1, U_2, U_3 is given therefore by $X^r = U_r/h_r$.

Thus, in three dimensions,

$$\operatorname{div}(U_1, U_2, U_3) = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial x^1}(h_2h_3U_1) + \frac{\partial}{\partial x^2}(h_3h_1U_2) + \frac{\partial}{\partial x^3}(h_1h_2U_3)\right\}.$$

Again $$\nabla^2\phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}\left(\sqrt{g}\,g^{rs}\frac{\partial\phi}{\partial x^s}\right)\text{ since }(\phi)^r = g^{rs}(\phi)_s\text{ (summation),}$$

$$= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial x^1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x^3}\right)\right\}.$$

For example, in spherical polar co-ordinates, where

$$ds^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2,$$

$$\operatorname{div}(U_1, U_2, U_3) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2U_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,U_2) + \frac{1}{r\sin\theta}\frac{\partial U_3}{\partial\phi},$$

where U_1, U_2, U_3 are the physical components of a vector and

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2}.$$

8.67. The Second Covariant Derivatives of X_r. The Riemann-Christoffel Tensor. Since $X_{r,\,s} = \dfrac{\partial X_r}{\partial x^s} - \{rs,\, p\}X_p$, then

$$X_{r,\,st} = \frac{\partial X_{r,\,s}}{\partial x^t} - \{rt,\, m\}X_{m,\,s} - \{st,\, m\}X_{r,\,m}$$

i.e. $$X_{r,\,st} = \frac{\partial^2 X_r}{\partial x^s\,\partial x^t} - \{rs,\, p\}\frac{\partial X_p}{\partial x^t} - \left(\frac{\partial}{\partial x^t}\{rs,\, p\}\right)X_p$$
$$- \{rt,\, m\}\left(\frac{\partial X_m}{\partial x^s} - \{ms,\, p\}X_p\right) - \{st,\, m\}\left(\frac{\partial X_r}{\partial x^m} - \{rm,\, p\}X_p\right)$$
$$= \frac{\partial^2 X_r}{\partial x^s\,\partial x^t} - \{rs,\, p\}\frac{\partial X_p}{\partial x^t} - \{rt,\, m\}\frac{\partial X_m}{dx^s} - \{st,\, m\}\frac{\partial X_r}{\partial x^m}$$
$$- \left(\frac{\partial}{\partial x^t}\{rs,\, p\} - \{rt,\, m\}\{ms,\, p\} - \{st,\, m\}\{rm,\, p\}\right)X_p$$

Similarly

$$X_{r,\,ts} = \frac{\partial^2 X_r}{\partial x^s\,\partial x^t} - \{rt,\, p\}\frac{\partial X_p}{\partial x^s} - \{rs,\, m\}\frac{\partial X_m}{\partial x^t} - \{st,\, m\}\frac{\partial X_r}{\partial x^m}$$
$$- \left(\frac{\partial}{\partial x^s}\{rt,\, p\} - \{rs,\, m\}\{mt,\, p\} - \{st,\, m\}\{rm,\, p\}\right)X_p.$$

But $\{rt,\, m\}\dfrac{\partial X_m}{\partial x^s} = \{rt,\, p\}\dfrac{\partial X_p}{\partial x^s}$ and $\{rs,\, p\}\dfrac{\partial X_p}{\partial x^t} = \{rs,\, m\}\dfrac{\partial X_m}{\partial x^t}$.

Therefore

$$X_{r,\,st} - X_{r,\,ts} = \left[\frac{\partial}{\partial x^s}\{rt,\, p\} - \frac{\partial}{\partial x^t}\{rs,\, p\}\right.$$
$$\left. + \{rt,\, m\}\{ms,\, p\} - \{rs,\, m\}\{mt,\, p\}\right]X_p.$$

But X_p is any vector and $X_{r,\,st} - X_{r,\,ts}$ is a tensor. Therefore the coefficient of X_p is a tensor of the type $R^p{}_{rst}$. It is called the *Riemann-Christoffel Tensor*.

i.e. $$R^p{}_{rst} = \frac{\partial}{\partial x^s}\{rt,\, p\} - \frac{\partial}{\partial x^t}\{rs,\, p\} + \{rt,\, m\}\{ms,\, p\} - \{rs,\, m\}\{mt,\, p\}.$$

The associated tensor R_{prst} is $g_{pq}R^q{}_{rst}$.

Also $g_{pq}\{rt,\, q\} = [rt,\, p]$, and therefore

$$g_{pq}\frac{\partial}{\partial x^s}\{rt,\, q\} = \frac{\partial}{\partial x^s}[rt,\, p] - \{rt,\, q\}([ps,\, q] + [qs,\, p])$$

and similarly $g_{pq}\dfrac{\partial}{\partial x^t}\{rs,\, q\} = \dfrac{\partial}{\partial x^t}[rs,\, p] - \{rs,\, q\}([pt,\, q] + [qt,\, p])$.

Therefore

$$R_{prst} = \frac{\partial}{\partial x^s}[rt,\, p] - \frac{\partial}{\partial x^t}[rs,\, p] + \{rs,\, q\}[pt,\, q] - \{rt,\, q\}[ps,\, q].$$

Since $R_{prst} = 0$ for Cartesian co-ordinates, it follows that $R_{prst} = 0$

for any transformation. Thus the vanishing of R_{prst} is a necessary condition for flat space.

Notes. (i) The condition $R_{prst} = 0$ may also be proved to be *sufficient* for flat space.

(ii) The form of R_{prst} shows that

$$R_{prst} = -R_{rpst}; \quad R_{prst} = -R_{prts}; \quad R_{prst} = R_{stpr};$$
$$R_{prst} + R_{ptrs} + R_{pstr} = 0.$$

The component is zero, when $p = r$ or $s = t$. There are $m\ (= {}^nC_2)$ different pairs of different affixes. The number of ways of selecting 2 pairs (repetition being allowed) is m (when there is a repetition) $+ {}^mC_2$ (when the selections are different). Finally there are ${}^nC_4$ relations involving three components ($n > 3$). The number of independent components is therefore ${}^nC_2 + \frac{1}{2}{}^nC_2({}^nC_2 - 1) - {}^nC_4$, which will be found to be $\frac{1}{12}n^2(n^2 - 1)$. For $n = 4$, there are 20 independent components.

For $n = 3$, the 6 components may be taken with the following arrangements of affixes 1212, 1213, 1223, 1312, 1323, 2323. For $n = 2$, there is one component R_{1212}.

In this case

$$R_{1212} = \frac{\partial}{\partial x^1}[22, 1] - \frac{\partial}{\partial x^2}[12, 1] + \{12, 1\}[12, 1] - \{22, 1\}[11, 1]$$
$$+ \{12, 2\}[12, 2] - \{22, 2\}[11, 2]$$

and the first two terms are $-\dfrac{1}{2}\dfrac{\partial^2 g_{11}}{\partial x^2\,\partial x^2} + \dfrac{\partial^2 g_{12}}{\partial x^1\,\partial x^2} - \dfrac{1}{2}\dfrac{\partial^2 g_{22}}{\partial x^1\,\partial x^1}$.

For example, if $ds^2 = E\,du^2 + G\,dv^2$,

$$R_{1212} = -\frac{1}{2}\frac{\partial^2 E}{\partial v^2} - \frac{1}{2}\frac{\partial^2 G}{\partial u^2} + \frac{1}{4E}\left(\frac{\partial E}{\partial v}\right)^2 + \frac{1}{4G}\left(\frac{\partial G}{\partial u}\right)^2 + \frac{1}{4E}\left(\frac{\partial E}{\partial u}\right)\left(\frac{\partial G}{\partial u}\right) + \frac{1}{4G}\left(\frac{\partial E}{\partial v}\right)\left(\frac{\partial G}{\partial v}\right)$$

since $\{12, 1\} = \dfrac{1}{2E}E_v$; $[12, 1] = \dfrac{1}{2}E_v$; $\{22, 1\} = -\dfrac{1}{2E}G_u$; $[11, 1] = \dfrac{1}{2}E_u$;

$\{12, 2\} = \dfrac{1}{2G}G_u$; $[12, 2] = \dfrac{1}{2}G_u$; $\{22, 2\} = \dfrac{1}{2G}G_v$; $[11, 2] = -\dfrac{1}{2}E_v$.

The tensor vanishes if

$$E_{vv} - \frac{E_v}{2EG}\frac{\partial}{\partial v}(EG) + G_{uu} - \frac{G_u}{2EG}\frac{\partial}{\partial u}(EG) = 0$$

i.e.

$$\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{(EG)}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{(EG)}}\right) = 0$$

or

$$\frac{\partial}{\partial v}\left(\frac{1}{h_2}\frac{\partial h_1}{\partial v}\right) + \frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial h_2}{\partial u}\right) = 0, \quad (E = h_1^2,\ G = h_2^2).$$

In particular, $\left(\dfrac{\partial^2}{\partial u^2} + \dfrac{\partial^2}{\partial v^2}\right)\log h = 0$, if $h_1 = h_2 = h$.

(iii) Since $\dfrac{dx^r}{ds}$ (the tangent vector) is constant for a straight line in flat space, the co-ordinates of a point on a straight line must satisfy the equation

$$\frac{d}{ds}\left(\frac{dx^r}{ds}\right) + \{mn, r\}\frac{dx^m}{ds}\frac{dx^n}{ds} = 0$$

i.e. $\dfrac{d^2x^r}{ds^2} + \{mn, r\}\dfrac{dx^m}{ds}\dfrac{dx^n}{ds} = 0$ is the equation of a straight line in curvilinear co-ordinates.

When the space is not flat, it may be shown that the above equation represents a geodesic, where the geodesic curve through A, B is defined to be that for which $\displaystyle\int_A^B ds$ is a minimum.

In flat space the system X^r that satisfies the equation $\dfrac{dX^r}{dt} + \{mn, r\}X^m\dfrac{dx^n}{dt} = 0$ and has given values at a point A of a curve AB is a system of parallel vectors, and the values at B are independent of the curve joining AB. In curved space, the equations may be used to define parallelism along a curve AB, but the values at B then depend on the curve chosen. Thus whilst the description of a closed path by a point P in flat space does not alter a constant vector drawn through P, the corresponding result is not necessarily true in curved space.

(*Refs.* Eisenhart, *Riemannian Geometry*. Eddington, *The Mathematical Theory of Relativity*. McConnell, *Applications of the Absolute Differential Calculus*.)

Examples VIII

1. Show that the vectors $\mathbf{a} - \mathbf{b} + \mathbf{c}$, $2\mathbf{a} - 3\mathbf{b}$, $\mathbf{a} + 3\mathbf{c}$ are parallel to the same plane.

2. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular vectors of equal magnitude, then $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is equally inclined to $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

3. Interpret geometrically the equation $(\mathbf{r} - \mathbf{a}).(\mathbf{r} + \mathbf{a}) = 0$.

4. If $(\mathbf{c} - \frac{1}{2}\mathbf{a}).\mathbf{a} = (\mathbf{c} - \frac{1}{2}\mathbf{b}).\mathbf{b} = 0$, prove that $(\mathbf{a} - \mathbf{b})$ is perpendicular to $\mathbf{c} - \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and interpret the result geometrically.

Prove the results given in *Examples 5–9*, where the centroid G of a system of particles of weights w_r at P_r is given by the position vector $(\Sigma w_r\mathbf{a}_r)/(\Sigma w_r)$, the position vector of P_r being $\mathbf{a}_r$.

5. G is independent of the origin of reference.

6. The centroid of w_1 at P_1 and w_2 at P_2 divides P_1P_2 in the ratio $w_2 : w_1$.

7. The centroid of equal weights at P_1, P_2, P_3 is the intersection of the medians of the triangle P_1, P_2, P_3.

8. If G is the centroid of w_r at P_r ($r = 1$ to n) and G' the centroid of w_r' at P_r' ($r = 1$ to m), then the centroid of the combined system is the centroid of Σw_r at G and $\Sigma w_r'$ at G'.

9. The centroid of the area of a quadrilateral $ABCD$ is the same as that of particles of weights $1, 1, 1, 1, -1$, at A, B, C, D, E respectively where E is the intersection of AC, BD.

10. The eight vertices of a unit cube, referred to rectangular axes OX, OY, OZ specified by unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are $O, A, B, C, P, A_1, B_1, C_1$, where A, B, C, P are given respectively by $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{i} + \mathbf{j} + \mathbf{k}$ and A_1, B_1, C_1 are the projections of P on YOZ, ZOX, XOY respectively. Find the position vectors of the midpoints of $OB, BC_1, C_1P, PB_1, B_1C, CO$ and show that these points form a regular hexagon in a plane perpendicular to the vector $\mathbf{i} - \mathbf{j} - \mathbf{k}$.

11. The points P, Q, R divide $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BO}$ in the ratios $k_1 : 1$, $k_2 : 1$, $k_3 : 1$, respectively. Find $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ in terms of $\mathbf{a}$ $(= \overrightarrow{OA})$, $\mathbf{b}$ $(= \overrightarrow{OB})$ and show that P, Q, R are collinear if $k_1 k_2 k_3 = -1$. (*Menelaus's Theorem.*)

12. If $\mathbf{a}, \mathbf{b}$ are two vectors of different directions, prove that the points whose position vectors are $p_1\mathbf{a} + q_1\mathbf{b}$, $p_2\mathbf{a} + q_2\mathbf{b}$, $p_3\mathbf{a} + q_3\mathbf{b}$ are collinear if

$$\begin{vmatrix} 1 & p_1 & q_1 \\ 1 & p_2 & q_2 \\ 1 & p_3 & q_3 \end{vmatrix} = 0.$$

13. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three vectors not parallel to the same plane, show that the points given by $p_n\mathbf{a} + q_n\mathbf{b} + r_n\mathbf{c}$ ($n = 1, 2, 3, 4$) are coplanar if

$$\begin{vmatrix} 1 & p_1 & q_1 & r_1 \\ 1 & p_2 & q_2 & r_2 \\ 1 & p_3 & q_3 & r_3 \\ 1 & p_4 & q_4 & r_4 \end{vmatrix} = 0.$$

14. Prove, by vectors, that the midpoints of the diagonals of a complete quadrilateral are collinear

15. If C, D are points that divide $\overrightarrow{AB}$ internally and externally in the ratio k_1, k_2, ($k_1 \neq k_2$), prove that $\overrightarrow{OC}\cdot\overrightarrow{OD} = (k_2^2a^2 - k_1^2b^2)/(k_2^2 - k_1^2)$, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. Deduce that these vectors $\overrightarrow{OC}$, $\overrightarrow{OD}$ are perpendicular if $bk_1 = ak_2$ where $OA = a$, $OB = b$, and interpret the result.

16. Points A_1, B_1, C_1 are taken respectively on OA, OB, OC such that $\overrightarrow{OA_1} = k_1\overrightarrow{OA}$, $\overrightarrow{OB_1} = k_2\overrightarrow{OB}$, $\overrightarrow{OC_1} = k_3\overrightarrow{OC}$. Show that (AB, A_1B_1), (BC, B_1C_1), (CA, C_1A_1) are collinear on a line parallel to the vector

$$k_1(k_2 - k_3)\overrightarrow{OA} + k_2(k_3 - k_1)\overrightarrow{OB} + k_3(k_1 - k_2)\overrightarrow{OC}.$$

17. $ABCD$ is a skew quadrilateral and P, Q, R, S are four coplanar points on $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ respectively dividing these sides in the ratios $k_1:1$, $k_2:1$, $k_3:1$, $k_4:1$. Prove that $k_1k_2k_3k_4 = 1$.

18. If A, B, C, D are four points not in the same plane, show that the six planes obtained by taking two of the points and the midpoint of the join of the other two pass through the centroid of A, B, C, D.

19. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are given by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. From the result $(\mathbf{a}\times(\mathbf{b}-\mathbf{c}))\cdot\mathbf{c} = [\mathbf{abc}]$, prove that the volume of the tetrahedron $OABC$ is $\frac{1}{6}\,OA\cdot BC\,p\sin\theta$ where p is the shortest distance between OA, BC and θ (between 0 and π) is the angle between OA, BC.

20. Deduce from *Example 19*, that if $\mathbf{n}_1$, $\mathbf{n}_2$ are unit vectors along two lines, and P_1, P_2 are two points, one on each line, the shortest distance between the lines is the absolute value of $[\mathbf{n}_1\mathbf{n}_2\,\overrightarrow{P_1P_2}]$ divided by the modulus of $(\mathbf{n}_1\times\mathbf{n}_2)$.

21. If l_1, m_1, n_1 are the direction cosines of a line through (x_1, y_1, z_1) and l_2, m_2, n_2 the direction cosines of a line through (x_2, y_2, z_2) prove that the shortest distance D between the lines is given by

$$D\sin\theta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \end{vmatrix} \quad \text{where } \cos\theta = l_1l_2 + m_1m_2 + n_1n_2.$$

22. Show that the shortest distance between the line joining (x_1, y_1, z_1) to (x_2, y_2, z_2) and the line joining (x_3, y_3, z_3) to (x_4, y_4, z_4) is

$$\{A(x_1 - x_3) + B(y_1 - y_3) + C(z_1 - z_3)\} \div (A^2 + B^2 + C^2)^{\frac{1}{2}}$$

where A, B, C are the cofactors of $(x_1 - x_3)$, $(y_1 - y_3)$, $(z_1 - z_3)$ respectively in the determinant

$$\begin{vmatrix} x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix}.$$

23. Find the shortest distance between the lines given by $x = 3 + 3t$, $y = 3 + 4t$, $z = 4 + 5t$; $x = 3 - 5u$, $y = 4 - 3u$, $z = 2 - 4u$.

24. Find the shortest distance between the intersection of the planes $x - 2y + 3z = 4$, $3x + y + z = 4$ and the intersection of the planes

$$2x - y + 3z = 1,\ 4x + y - 2z = 2.$$

25. Show that the lines joining the midpoints of opposite edges of a tetrahedron are concurrent at the centroid of the tetrahedron.

26. If each edge of a tetrahedron is equal to the edge opposite to it, prove that the lines joining the midpoints of the opposite edges are the shortest distances between these edges; and find the shortest distances in terms of the sides of the tetrahedron.

27. The vector moment about O of a force $\mathbf{F}$ acting at P is *defined* to be $\overrightarrow{OP}\times\mathbf{F}$, prove that its scalar component about any axis is the ordinary moment of $\mathbf{F}$ about that axis. Deduce that the sum of the ordinary moments of a system of forces about an axis is the moment of the resultant about that axis.

28. Show that $(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = \{\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\}.\mathbf{a} = (\mathbf{a}.\mathbf{c})(\mathbf{b}.\mathbf{d}) - (\mathbf{a}.\mathbf{d})(\mathbf{b}.\mathbf{c})$.

29. Show that $[\mathbf{abc}]\mathbf{d} + [\mathbf{cda}]\mathbf{b} = [\mathbf{bcd}]\mathbf{a} + [\mathbf{dab}]\mathbf{c}$, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are four vectors in three dimensions.

30. The equation of motion of a particle of mass m under the action of a force $\mathbf{F}$ is given to be $m\dfrac{d\mathbf{v}}{dt} = \mathbf{F}$ where $\mathbf{v}$ is the velocity. The kinetic energy T is given to be $\frac{1}{2}m\mathbf{v}^2$. The work done (W) by the force in a small displacement $d\mathbf{r}$ is given to be $\mathbf{F}.d\mathbf{r}$. Show that the increase in kinetic energy when the particle moves from P_1 to P_2 along its path is $\displaystyle\int_{P_1}^{P_2} \mathbf{F}.\frac{d\mathbf{r}}{ds}\,ds$ and is therefore equal to the work done by the force.

31. The angular momentum $\mathbf{H}$ of a moving particle of mass m is defined to be $\mathbf{r} \times m\mathbf{v}$ where $\mathbf{r}$ is the position vector of the particle and $\mathbf{v}$ its velocity. Show that $\dot{\mathbf{H}}$ is equal to the vector moment (*see Example* 27) of the force acting on the particle, viz. $\mathbf{r} \times \mathbf{F}$.

32. $OABCD$ is a right pyramid of vertex O and of height h, the base $ABCD$ being a square of side $2a$. Find the shortest distance between OC, AB.

33. The base of a right pyramid of height h is a regular polygon of n sides each of length $2a$. Find the shortest distances between a side of the base and the edges of the pyramid that do not lie in a plane through that side.

34. Find the area of the circular section of the sphere $x^2 + y^2 + z^2 = R^2$ made by the plane $lx + my + nz = p$ (where $l^2 + m^2 + n^2 = 1$) and also the areas of the projections of that section on the co-ordinate planes.

35. Prove that the perpendicular distance of a point P from a line whose direction is specified by a unit vector $\mathbf{a}$ is the modulus of $\overrightarrow{PQ} \times \mathbf{a}$ where Q is any point of the line. Deduce that the equation of the circular cylinder of radius R whose axis passes through (x_0, y_0, z_0) and has direction cosines (l, m, n) is

$$\{m(z - z_0) - n(y - y_0)\}^2 + \{n(x - x_0) - l(z - z_0)\}^2 + \{l(y - y_0) - m(x - x_0)\}^2 = R^2.$$

36. Show that the points $(6, 4, -3)$, $(4, 4, -2)$, $(3, -2, 3)$, $(3, 2, 0)$ are coplanar.

37. Prove that the locus of the midpoint of lines whose extremities are on two given lines and are parallel to a given plane is a straight line.

38. Show that the locus of the midpoint of lines whose extremities lie on two given non intersecting lines is a plane perpendicular to the shortest distance between the given lines.

39. Find the points on the curve $x = t^3$, $y = 3t^2 - 2t$, $z = 3t - 3$ where the osculating planes pass through the origin.

40. For the curve given by $x = 2a(\theta + \sin\theta\cos\theta)$, $y = 2a\sin^2\theta$, $z = 4a\sin\theta$, show that $\rho = \sigma = 8a\cos\theta$.

41. For the curve given by $x = 4at^3$, $y = 3a(1 + 2t^2)$, $z = 6at$ prove that $3a\sigma = y^2$.

42. If for a given curve, ρ/σ is constant, show that the tangent makes a constant angle with a fixed direction (i.e. that the curve is a helix).

43. Show that if ρ, σ are both constant, the curve is a circular helix.

44. If the sphere of centre C and radius R given by $(\mathbf{a} - \mathbf{r})^2 = R^2$, where $\mathbf{a} = \overrightarrow{OC}$, has four-point contact with a given curve at the point whose position vector is $\mathbf{r}$ (a function of s), prove that (i) $(\mathbf{a} - \mathbf{r}).\mathbf{T} = 0$, (ii) $(\mathbf{a} - \mathbf{r}).\mathbf{N} = \rho$, (iii) $(\mathbf{a} - \mathbf{r}).\mathbf{B} = \sigma\rho'$ where $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, ρ, σ refer to the given curve. Deduce that $\overrightarrow{OC} = \mathbf{r} + \rho\mathbf{N} + \sigma\rho'\mathbf{B}$ and $R^2 = \rho^2 + (\sigma\rho')^2$. ($C$ is called the centre and R the radius of *spherical curvature*.)

45. A curve is drawn on a right circular cone so as to cut the generators at a constant angle. Prove that its projection on a plane perpendicular to the axis of the cone is an equiangular spiral.

46. The principal normal at any point P on one curve is given to be also the principal normal at a corresponding point Q on a second curve. Prove that (i) PQ is constant, (ii) the tangent at P makes a constant angle with the tangent at Q, (iii) the curvature and torsion of each curve are linearly connected. (*Bertrand Curves.*)

47. If δ is the shortest distance between the principal normals at two points P, Q of a curve, prove that $\lim_{s\to 0} (\delta/s)$ is equal to $\rho/\sqrt{(\rho^2 + \sigma^2)}$ where s is the arc PQ.

48. A constant length c from each point of a given curve is measured to a point Q along the binormal. Prove that if the torsion of the given curve is constant, the radius of curvature of the locus of Q is $\rho(c^2 + \sigma^2)/\sqrt{\{\sigma^2(c^2 + \sigma^2) + c^2\rho^2\}}$, where ρ, σ refer to the given curve.

49. If $\overrightarrow{PC} = \mathbf{R}$ where P is a point on a given curve and C is the centre of spherical curvature (*Example 44*), show that $\dfrac{d\mathbf{R}}{ds} = \left(\dfrac{RR'}{\rho^2\sigma}\right)\mathbf{B} - \mathbf{T}$.

50. Deduce from *Example 49* that if α is the angle between the radii of curvature at two points P, Q of a given curve

$$\lim_{s\to 0} \frac{\alpha}{s} = \frac{\left\{\sigma^2 + R^2\left(\dfrac{dR}{d\rho}\right)^2\right\}^{\frac{1}{2}}}{R\sigma}, \text{ where } PQ = s.$$

51. Show that $(\mathbf{r}''')^2 = \kappa^4(1 + \lambda^2 R^2)$.

52. Show that the angular velocity of the moving axes determined by the spherical polar co-ordinates r, θ, ϕ is $(\dot\phi \cos\theta, -\dot\phi \sin\theta, \dot\theta)$. Deduce that the velocity $\mathbf{v}$ is $(\dot r, r\dot\theta, r\dot\phi \sin\theta)$ and the acceleration $\dfrac{d\mathbf{v}}{dt}$ is

$$\left\{\left(\ddot r - r\dot\theta^2 - r\sin^2\theta\,\dot\phi^2\right), \left(\frac{1}{r}\frac{d}{dt}(r^2\dot\theta) - r\sin\theta\cos\theta\,\dot\phi^2\right), \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\,\dot\phi)\right\}.$$

53. Prove that the acceleration of a moving particle in cylindrical co-ordinates ρ, ϕ, z is $\left(\ddot\rho - \rho\dot\phi^2, \dfrac{1}{\rho}\dfrac{d}{dt}(\rho^2\dot\phi), \ddot z\right)$.

54. Prove that $\mathbf{u}.\nabla\mathbf{r} = \mathbf{u}$.

55. Show that $\nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$ if $\mathbf{a}$ is constant.

56. Show that $\mathbf{a}.\nabla\left(\mathbf{b}.\nabla\left(\dfrac{1}{r}\right)\right) = \dfrac{3}{r^5}(\mathbf{r}.\mathbf{a})(\mathbf{r}.\mathbf{b}) - \dfrac{1}{r^3}(\mathbf{a}.\mathbf{b})$.

57. If $\operatorname{div}\mathbf{D} = \rho$, $\operatorname{div}\mathbf{H} = 0$, $\operatorname{curl}\mathbf{H} = \dfrac{1}{c}(\dot{\mathbf{D}} + \rho\mathbf{v})$, $\operatorname{curl}\mathbf{D} = -\dfrac{\dot{\mathbf{H}}}{c}$ where c is constant, show that

$$\text{(i) } c^2\nabla^2\mathbf{D} - \ddot{\mathbf{D}} = c^2\nabla\rho + \frac{\partial}{\partial t}(\rho\mathbf{v}), \text{ (ii) } c^2\nabla^2\mathbf{H} - \ddot{\mathbf{H}} = -c\operatorname{curl}(\rho\mathbf{v}).$$

58. Show that in three dimensions

$$g = (1 - \cos^2\theta_{23} - \cos^2\theta_{31} - \cos^2\theta_{12} + 2\cos\theta_{23}\cos\theta_{31}\cos\theta_{12})g_{11}g_{22}g_{33}$$

where $\theta_{23}, \theta_{31}, \theta_{12}$ are the angles between the curves of reference.

59. Show that for

$$x^2(b-a)(c-a) = (\lambda-a)(\mu-a)(\nu-a),\quad y^2(c-b)(a-b) = (\lambda-b)(\mu-b)(\nu-b),$$
$$z^2(a-c)(b-c) = (\lambda-c)(\mu-c)(\nu-c), \text{ (*Elliptic Co-ordinates* } \lambda, \mu, \nu),$$

$$4ds^2 = \frac{(\nu-\lambda)(\mu-\lambda)}{(\lambda-a)(\lambda-b)(\lambda-c)}d\lambda^2 + \frac{(\lambda-\mu)(\nu-\mu)}{(\mu-a)(\mu-b)(\mu-c)}d\mu^2 + \frac{(\mu-\nu)(\lambda-\nu)}{(\nu-a)(\nu-b)(\nu-c)}d\nu^2.$$

60. When $x = uv\cos w$, $y = uv\sin w$, $2z = u^2 - v^2$ (*Parabolic Co-ordinates* u, v, w), show that $ds^2 = (u^2 + v^2)(du^2 + dv^2) + u^2v^2\,dw^2$.

61. Show directly that $V_x^2 + V_y^2 + V_z^2$ and $V_{xx} + V_{yy} + V_{zz}$ are invariant for a change of rectangular axes.

Find values for the expressions given in *Examples 62–4*.

62. $\frac{\partial}{\partial x^a}(g_{mnp}x^m x^n x^p)$, where g_{mnp} is completely symmetrical.

63. $\frac{\partial^2}{\partial x^m \partial x^n}(a_{mr}b_{nr}x^m x^n)$.

64. $\frac{\partial^4}{\partial x^m \partial x^n \partial x^p \partial x^q}(a_m a_n a_p a_q x^m x^n x^p x^q)$.

65. The system δ^{rst}_{mnp} is defined to be (i) zero if two or more of the subscripts (or superscripts) are the same or when r, s, t do not consist of the same three affixes as m, n, p; (ii) $+1$ when rst and mnp differ by an even number of permutations; (iii) -1 when rst and mnp differ by an odd number of permutations. Show that

(i) $\delta^{rsp}_{mnp} = (N-2)\delta^{rs}_{mn}$; $\delta^{rnp}_{mnp} = (N-1)(N-2)\delta^{r}_{m}$.

(ii) $\delta^{123}_{mnp}a^m_1 a^n_2 a^p_3 = |a^r_s|$ when $N = 3$.

66. If X^α, Y_β are vectors, prove that $X^\alpha\frac{\partial Y_\beta}{\partial x^\alpha} + Y_\alpha\frac{\partial X^\alpha}{\partial x^\beta}$ is a vector.

67. Show that $X_{r,s} - X_{s,r} = \frac{\partial X_r}{\partial x^s} - \frac{\partial X_s}{\partial x^r}$.

68. If $ds^2 = f(r)\,(dx^2 + dy^2)$ where $r = \sqrt{(x^2 + y^2)}$ represents a flat space in two dimensions, then $f(r)$ must be of the form ar^b where a, b are constants.

69. If $ds^2 = h_1^2\,dx^2 + h_2^2\,dy^2$, show that the corresponding two-dimensional space is flat when $\frac{\partial^2 h_1}{\partial x\,\partial y}F_x - \frac{\partial h_1}{\partial y}F_{xx} + h_1 F_x^2 F_y = 0$ and $h_2 = \frac{1}{F_x}\frac{\partial h_1}{\partial y}$, where $F(x, y)$ is any function of x, y.

70. Show that the space given by $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ is flat when

$$\frac{\partial}{\partial u}\left\{\frac{FE_v - EG_u}{E\Delta}\right\} + \frac{\partial}{\partial v}\left\{\frac{2EF_u - EE_v - FE_u}{E\Delta}\right\} = 0 \text{ where } \Delta^2 = EG - F^2.$$

71. Prove that for cylindrical co-ordinates ρ, ϕ, z

$$\operatorname{div}(F_1, F_2, F_3) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_1) + \frac{1}{\rho}\frac{\partial F_2}{\partial\phi} + \frac{\partial F_3}{\partial z},$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}.$$

72. Verify that the equations of a straight line in cylindrical co-ordinates are

$$\frac{d^2\rho}{ds^2} - \rho\left(\frac{d\phi}{ds}\right)^2 = 0;\quad \frac{d^2\phi}{ds^2} + \frac{2}{\rho}\frac{d\rho}{ds}\frac{d\phi}{ds} = 0;\quad \frac{d^2 z}{ds^2} = 0.$$

73. Prove that in any space for which $ds^2 = g_{mn}\,dx^m\,dx^n$

$$\overline{[mn, p]} = \frac{\partial^2 x^\mu}{\partial\bar{x}^m\,\partial\bar{x}^n}\cdot\frac{\partial x^\lambda}{\partial\bar{x}^p}g_{\lambda\mu} + \frac{\partial x^\lambda}{\partial\bar{x}^p}\cdot\frac{\partial x^\mu}{\partial\bar{x}^m}\cdot\frac{\partial x^\nu}{\partial\bar{x}^n}[\mu\nu, \lambda].$$

By multiplying this result by $\bar{g}^{pr}\frac{\partial x^\sigma}{\partial\bar{x}^r}$, prove that

$$\frac{\partial^2 x^\rho}{\partial\bar{x}^m\,\partial\bar{x}^n} = \overline{\{mn, r\}}\frac{\partial x^\rho}{\partial\bar{x}^r} - \frac{\partial x^\mu}{\partial\bar{x}^m}\cdot\frac{\partial x^\nu}{\partial\bar{x}^n}\{\mu\nu, \rho\}$$

and deduce that

$$\frac{\partial\bar{X}_m}{\partial\bar{x}^n} - \overline{\{mn, r\}}\bar{X}_r = \left(\frac{\partial X_\mu}{\partial x^\nu} - \{\mu\nu, \rho\}X_\rho\right)\frac{\partial x^\mu}{\partial\bar{x}^m}\cdot\frac{\partial x^\nu}{\partial\bar{x}^n}.$$

74. If $F(x^\alpha, x'^\alpha)$ is invariant, where $x'^\alpha = \frac{dx^\alpha}{ds}$, and if $p_\alpha = \frac{\partial F}{\partial x'^\alpha}$ show that $\frac{dp_\alpha}{ds} - \frac{\partial F}{\partial x^\alpha}$ is a covariant tensor.

Taking $F(x^\alpha, x'^\alpha)$ to be $\frac{1}{2}g_{\alpha\beta}x'^\alpha x'^\beta$ deduce that

$$g^{\alpha\lambda}\left(\frac{dp_\alpha}{ds} - \frac{\partial F}{\partial x^\alpha}\right) = \frac{d^2x^\lambda}{ds^2} + \{\beta\gamma, \lambda\}x'^\beta x'^\gamma$$

is a contravariant tensor A^λ and therefore that $A^\lambda = 0$ determines a set of invariant curves.

Solutions

3. Lines drawn from a point on the surface of a sphere to the extremities of a diameter are orthogonal.

4. The perpendicular bisectors of the sides of a triangle are concurrent.

5. $\dfrac{\Sigma w_r(\mathbf{a}_r + \mathbf{h})}{\Sigma w_r} = \dfrac{\Sigma w_r \mathbf{a}_r}{\Sigma w_r} + \mathbf{h}$, where $\mathbf{h}$ is the displacement of the origin.

9. Let $AE : EC = \lambda : \mu$; the centroid is that of weights $\lambda + \mu, \mu, \lambda + \mu, \lambda$ at B, C, D, A respectively. If λ is added at C and μ at A the centroid is unaltered if $-\lambda - \mu$ is placed at E.

10. $\frac{1}{2}\mathbf{i}$, $\mathbf{j} + \frac{1}{2}\mathbf{i}$, $\mathbf{i} + \mathbf{j} - \frac{1}{2}\mathbf{k}$, $\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$, $\frac{1}{2}\mathbf{i} + \mathbf{k}$, $\frac{1}{2}\mathbf{k}$

11. $\dfrac{k_1\mathbf{a}}{1 - k_1}, \dfrac{\mathbf{a} + k_2\mathbf{b}}{1 + k_2}, \dfrac{\mathbf{b}}{1 + k_3}$

15. If corresponding rays of a harmonic pencil are orthogonal, they are the bisectors of the angles between the other rays. 23. $36/\sqrt{(291)}$

24. Plane through the first line parallel to the second is

$$(x + 2y + 3z - 4) + \lambda(3x + y + z - 4) = 0$$

where $(1 + 3\lambda) + 16(2 - \lambda) + 6(3 + \lambda) = 0$, i.e. is $128x + 11y - 8z = 120$. Plane through the second parallel to the first is similarly $128x + 11y - 8z = 64$. Shortest distance is $56/3\sqrt{(1841)}$, the distance between the planes.

25. Take the position vectors of the vertices as $\mathbf{0}, \mathbf{a}, \mathbf{b}, \mathbf{c}$. One of the lines is $\mathbf{r} = \frac{1}{2}\mathbf{a} + \frac{1}{2}(1 - t)(\mathbf{b} + \mathbf{c})$ which contains the centroid $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ for $t = \frac{1}{3}$.

26. $\mathbf{a}^2 = (\mathbf{b} - \mathbf{c})^2$, $\mathbf{a}.(\mathbf{b} + \mathbf{c} - \mathbf{a}) = 0$, &c. $\delta_1^2 = \frac{1}{2}(b^2 + c^2 - a^2)$, &c.

28. Take $\mathbf{e} = \mathbf{c} \times \mathbf{d}$, so that $(\mathbf{a} \times \mathbf{b}).\mathbf{e} = [\mathbf{abe}] = (\mathbf{b} \times \mathbf{e}).\mathbf{a}$, &c.

29. Take $\mathbf{d} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ then $[\mathbf{dbc}] = x[\mathbf{abc}]$, &c.

31. $\dot{\mathbf{H}} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$.

32. $\dfrac{2ah}{\sqrt{(a^2 + h^2)}}$

33. $\dfrac{2ah\sin\dfrac{r\pi}{n}\sin\dfrac{(r-1)\pi}{n}}{\sqrt{\left(h^2\sin^2\dfrac{\pi}{n} + a^2\cos^2\dfrac{(2r-1)\pi}{n}\right)}}$, $r = 2$ to $n - 1$.

34. $\pi(R^2 - p^2)$, $\pi(R^2 - p^2)^{\frac{1}{2}}$, &c. 39. $t = 0, 1, 2$.

42. $\rho\,d\mathbf{T} + \sigma\,d\mathbf{B} = 0$, i.e. $\mathbf{B} + c\mathbf{T} = \mathbf{a}$, a constant where $c = \rho/\sigma$. Also $\mathbf{T}.\mathbf{a} = c$, so that $\mathbf{T}$ makes a constant angle with $\mathbf{a}$.

45. Prove $\rho\dfrac{d\phi}{d\rho} =$ constant, where ρ, ϕ are cylindrical co-ordinates.

46. Take $\mathbf{r}' = \mathbf{r} + c\mathbf{N}$ where accents refer to the curve Q.

$$\mathbf{T}'\frac{ds'}{ds} = \mathbf{T} + \frac{dc}{ds}\mathbf{N} - c(\lambda\mathbf{B} - \kappa\mathbf{T}),\quad \mathbf{N}' = \mathbf{N},\quad \mathbf{N}.\mathbf{T}' = \mathbf{N}.\mathbf{T} = 0;$$

therefore $\dfrac{dc}{ds} = 0$, also $\left(\dfrac{ds'}{ds}\right)^2 = (1 + \lambda c)^2 + \kappa^2c^2$; take $\dfrac{ds'}{ds} = \mu$, $(1 + \lambda c) = \mu\cos\alpha$, $\kappa c = \mu\sin\alpha$, then $\mathbf{T}' = \mathbf{T}\cos\alpha + \mathbf{B}\sin\alpha$; therefore $\kappa'\mu\mathbf{N} = \kappa\mathbf{N}\cos\alpha - \lambda\mathbf{N}\sin\alpha - \mathbf{T}\sin\alpha\dfrac{d\alpha}{ds} + \mathbf{B}\cos\alpha\dfrac{d\alpha}{ds}$; $\mathbf{N}.\mathbf{T} = 0$ with $\mathbf{N}.\mathbf{B} = 0$ gives $\dfrac{d\alpha}{ds} = 0$. Also $\mathbf{T}'.\mathbf{T} = \cos\alpha$ and finally $(1 - \lambda c)\tan\alpha = \kappa c$.

62. $3g_{mnp}x^mx^p$ 63. $a_{mr}b_{nr} + a_{nr}b_{mr}$ 64. $24a_ma_na_pa_q$

74. $\frac{dx^\alpha}{ds} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\beta}{ds}$; $\delta F = p_\alpha \delta x'^\alpha + \frac{\partial F}{\partial x^\alpha}\delta x^\alpha = p_\alpha \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\delta \bar{x}'^\beta + p_\alpha \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\bar{x}'^\beta \delta \bar{x}^\gamma$ $+ \frac{\partial F}{\partial x^\alpha}\cdot\frac{\partial x^\alpha}{\partial \bar{x}^\gamma}\delta \bar{x}^\gamma$ so that $\bar{p}_\beta = p_\alpha \frac{\partial x^\alpha}{\partial \bar{x}^\beta}$ and $\frac{\partial \bar{F}}{\partial \bar{x}^\beta} = p_\alpha \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\bar{x}'^\gamma + \frac{\partial F}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \bar{x}^\beta}$; and $\frac{d\bar{p}_\beta}{ds} - \frac{\partial \bar{F}}{\partial \bar{x}^\beta} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\left(\frac{dp_\alpha}{ds} - \frac{\partial F}{\partial x^\alpha}\right)$. If $F = \frac{1}{2}g_{\alpha\beta}x'^\alpha x'^\beta$; $p_\alpha = g_{\alpha\beta}x'^\beta$ and

$$\frac{dp_\alpha}{ds} = g_{\alpha\beta}\frac{d^2x^\beta}{ds^2} + x'^\beta \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}x'^\gamma = g_{\alpha\beta}\frac{d^2x^\beta}{ds^2} + \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\right)x'^\beta x'^\gamma$$

and therefore $\frac{dp_\alpha}{ds} - \frac{\partial F}{\partial x^\alpha} = g_{\alpha\beta}\frac{d^2x^\beta}{ds^2} + [\beta\gamma, \alpha]x'^\beta x'^\gamma$.

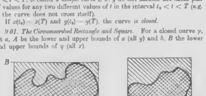
CHAPTER IX

DOUBLE AND MULTIPLE INTEGRALS. LINE, VOLUME AND SURFACE INTEGRALS.

9. Simple Curves (Plane). The locus determined by $x = x(t)$, $y = y(t)$, where $x(t)$, $y(t)$ are continuous functions of t in the interval $t_0 < t < T$, is called a *simple curve* if x, y do not assume the same pair of values for any two different values of t in the interval $t_0 < t < T$ (e.g. if the curve does not cross itself).

If $x(t_0) = x(T)$ and $y(t_0) = y(T)$, the curve is *closed*.

9.01. The Circumscribed Rectangle and Square. For a closed curve γ, let a, A be the lower and upper bounds of x (all y) and b, B the lower and upper bounds of y (all x).



(I) (II)

FIG. 1

The rectangle determined by $x = a$, $x = A$, $y = b$, $y = B$ may be called the *circumscribed rectangle* of γ. (*Fig. 1* (i).) If c is the greater of $A - a$, $B - b$ (or their common value if equal), a square of side c can be drawn with its sides parallel to OX, OY enclosing γ and having one point (at least) in common with γ on at least two opposite sides. (*Fig. 1* (ii).) The area c^2 of such a square may be denoted by $sq.\ (\gamma)$.

Note. The term area or a symbol A which specifies it will often be used to refer to a two-dimensional set of points and also to its measure, when there is no likelihood of ambiguity.

9.02. Elementary Quadratic Closed Curve. Let the circumscribed rectangle of γ be given by $a < x < A$, $b < y < B$. It will be found sufficient for a simple development of the theory to assume that γ is such that every line $x = c$ where $a < c < A$, and every line $y = c'$, where $b < c' < B$ meets the curve in *two* points and two points only. The

closed curve is then of the type illustrated in *Fig. 2*, where γ consists of parts of the sides of the circumscribed rectangle joined by curves

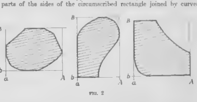


FIG. 2

within the rectangle in each of which y (or x) can be expressed as a single valued function of $x(y)$. Such a curve may be called *quadratic*.

Note. It would be sufficient for most purposes that these single valued functions should be monotonic (in the narrow sense).

9.03. Elementary Closed Curve. It will be assumed as obvious that a simple closed curve divides the points of the plane not on the curve into two categories, the one forming the *interior* of the curve and the other the *exterior* (*Ref. Watson, Cambridge Tract No 15, 1.*) We shall call a simple closed curve *elementary* if the interior can be divided up into a *finite* number of regions by means of lines parallel to the axes such that the boundary of each sub-region is quadratic. (*Fig. 3.*) A curve is elementary, for example, if every line parallel to an axis meets the curve in a finite number of points (except when the line coincides with part of the boundary, the exceptional lines being finite in number).

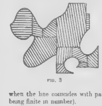


FIG. 3

9.04. The Area determined by a Closed Curve. Let $y = f(x)$ be a bounded function defined for the interval $a \leq x \leq b$. Draw a square of side c whose sides are parallel to OX, OY such that the curve lies entirely within the square. (*Fig. 4.*) Divide the square into n^2 smaller squares of side c/n by lines parallel to the axes.

These smaller squares may be placed in three classes.

(i) Those having some point in common with the curve.

(ii) Those that are interior to the region bounded by $x = a$, $x = b$ the curve and OX, ($f(x)$ for simplicity being assumed > 0).

(iii) The remainder.

Let the total area of the squares in these classes be denoted by K_n, I_n and $c^2 - E_n$ respectively, so that $K_n = E_n - I_n$.

As n increases, it is easy to see that K_n, E_n are decreasing positive functions of n, and I_n an increasing function. Thus when $n \to \infty$, K_n, E_n, I_n tend to limits K, E, I respectively where $K = E - I$.

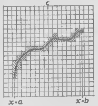


FIG. 4

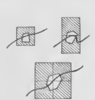


FIG. 5

The limit K may be called the *area covered by the curve* and is not necessarily zero. When $K = 0$, E is equal to I and it is sufficient, but not necessary, in order that K should be zero, that $f(x)$ should be a *continuous* function of x; for E_n, I_n then obviously tend to the common limit $\int_a^b f(x)dx$.

The subdivision of the square of side c may be replaced by a subdivision into *polygons* P_r provided $sq(P_r)$ tends to zero in the continued subdivision, for the polygons that have a point in common with the curve can always be taken sufficiently small as to lie entirely within K_n. (*Fig. 5.*)

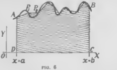


FIG. 6

Again, let P_1, P_2, . . . P_{n-1} be $(n - 1)$ points taken in order on AB. The area of the polygon $AP_1P_2 \ldots BCD$ (*Fig. 6*) tends also to $\int_a^b f(x)dx$ when this exists, if n tends to infinity in such a way that the

upper bound of the lengths of the chords $P_r P_{r+1}$ for a given n tends to zero. (C, D are the points $(b, 0)$, $(a, 0)$ respectively.)

Similarly if we take an elementary closed curve γ (*Fig. 7*) and subdivide a square of side c into smaller squares of side c/n, the limit of the sum of the squares having a point in common with γ is zero when n tends to ∞; and the limit of I_n, the sum of the squares interior to γ, may be called the *area enclosed by* γ.

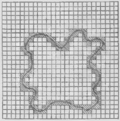


FIG. 7

It follows also that for such a curve, the area enclosed is the limit of the area of a polygon inscribed in the curve provided that the upper bounds (for a given n) of the lengths of the sides tend to zero. (*Fig. 8.*)

Note. A simple curve is not, in general, rectifiable, nor is the area covered by it necessarily zero. It can be proved that if a curve has a length however, the area covered by it is zero, but this is not a necessary condition. For example, the length of the curve $y = f(x)$ from $x = a$ to $x = b$ when $f(x)$ is continuous may not exist, but the area covered by it is zero.

9.1. Double Integrals. Let an area of magnitude Ω, enclosed by an elementary closed curve γ be divided up into N sub-regions of areas $\omega_1, \omega_2, \ldots, \omega_N$. (*Fig. 9.*) Let $f(x, y)$ be a bounded function of x, y determined at all points of Ω including γ. Form the sums:

$$S_1 = \sum_1^N M_r \omega_r, \quad s_1 = \sum_1^N m_r \omega_r$$

where M_r, m_r are the upper and lower bounds of $f(x, y)$ in ω_r (with its boundary γ_r). Also let M, m, be the upper and lower bounds of $f(x, y)$ in Ω (and γ); then

$$M\Omega > S_1 > \sum_1^N f(x_r', y_r')\omega_r > s_1 > m\Omega,$$ lower use s

where (x_r', y_r') is any point in ω_r or on γ_r.

If each sub-region be again subdivided in a similar way and the numbers $M_s\omega_s$, $m_s\omega_s$ be summed over the whole area Ω, where ω_s denotes one of the new sub-regions, and the sums be denoted by S_2, s_2 respectively, we have

$$M\Omega_1 > S_1 > S_2 > \Sigma f(x_s', y_s')\omega_s > s_2 > s_1 > m\Omega.$$

By continuing this process, we form two monotones

$$S_1 > S_2 > S_3 > \ldots; \quad s_1 < s_2 < s_3 < \ldots$$

and these sequences tend to limits as the number of times a subdivision is made tends to infinity. If also this number tends to infinity in such a way that every $sq. (\gamma_r)$ tends to zero, it can be shown that the limits are independent of the mode of subdivision (i.e. of the particular choice

FIG. 8

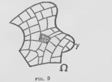


FIG. 9

of the sub-regions ω). Denoting these limits by S, s respectively we have $M\Omega > S > s > m\Omega$. If $S = s$, the common value is called the *Double Integral* of $f(x, y)$ over Ω and is written $\iint_\Omega f(x, y)dx\, dy$.

In particular, if $f(x, y)$ is *continuous* over Ω and γ, it may be proved by a method analogous to that given for functions of one variable that the double integral exists.

Also, the double integral, when it exists, is then obviously equal to the limit of the sum $\Sigma f(x_r', y_r)\omega_r$.

9.11. Mean Value of a Double Integral. Since

$$M\Omega > \iint_\Omega f(x, y)dx\, dy > m\Omega$$

then $\iint_\Omega f(x, y)dx\, dy = k\Omega$ where k is some number for which $M > k > m$. This number k is called the *Mean Value* of $f(x, y)$ over Ω.

Notes. (i) $\iint_\Omega c\, dx\, dy$ is obviously equal to $c\,\Omega$ when c is constant.

(ii) If $f(x, y)$ is continuous over Ω and γ, there is at least one point (x_0, y_0) of the domain for which $f(x_0, y_0) = k$.

Thus $\frac{1}{\Omega}\iint_\Omega f(x, y)dx\, dy = f(x_0, y_0)$ for some point (x_0, y_0) in Ω or on γ.

9.12. Discontinuities in the Integrand. Let γ_m be a curve lying entirely within Ω, which is such that the area covered by it is zero. (*Fig. 10* (i).)

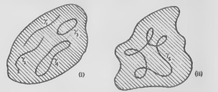


FIG. 10

Let $f(x, y)$ be discontinuous (but bounded) on γ_m. Let K_n be the total area of those subdivisions that have a point in common with γ_m, when the sums that correspond to all the sub-regions are S_n and s_n. Then in $S_n - s_n$, the part belonging to K_n must be less than or equal to $(M - m)K_n$ and therefore tends to zero since $\lim K_n = 0$. Thus these discontinuities have no effect on the value of the double integral. Similarly we may have a *finite* number of curves $\gamma_1, \gamma_2, \ldots, \gamma_m$ if they are of the requisite type.

Notes. (i) It is sufficient that γ_m should consist of a finite number of parts, in each of which either y is expressible as a continuous function of x, or x is a continuous function of y. It is sufficient also, but not necessary, that γ_m should have a finite length.

(ii) It is implied here that the sub-regions ω_r in any sub-division are *positive* (or rather *signless*). When a curve crosses itself, it is necessary in the analytical case to attribute sign to an area bounded by it in order to give a meaning to such an area. Since, however, in this particular case, we are dealing with curves that cover zero area, it is possible for γ_m to be non-simple, as in *Fig. 10* (ii), if it can be broken up into a finite number of simple curves of the right type.

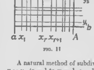


FIG. 11

9.13. Rectangular Boundary. Let Ω be the rectangle bounded by $x = a$, $x = A$, $y = b$, $y = B$ (*Fig. 11*); and let $f(x, y)$ be continuous over Ω and its boundary.

A natural method of subdivision consists in drawing the lines $x = x_r$, $y = y_s$, ($r = 1$ to $m - 1$, $s = 1$ to $n - 1$) where $x_0 = a$, $x_m = A$, $y_0 = b$, $y_n = B$. The sub-regions consist of the rectangles ω_{rs} specified by $x = x_r$, $x = x_{r+1}$, $y = y_s$, $y = y_{s+1}$, the area of this rectangle being $(x_{r+1} - x_r)(y_{s+1} - y_s)$. The double integral is therefore equal to the

limit of $\Sigma\Sigma f(x_r, y_s)(x_{r+1} - x_r)(y_{s+1} - y_s)$ where x_r, y_s are any numbers that satisfy the inequalities $x_r \le x_r' \le x_{r+1}$, $y_s \le y_s' \le y_{s+1}$.

Also the value of the limit is independent of the mode in which m, n tend to infinity provided that $\max(x_{r+1} - x_r)$ and $\max(y_{s+1} - y_s)$ both tend to zero.

The part of the summation belonging to the rectangles of breadth $x_{r+1} - x_r$ is $\sum_0^{n-1} f(x_r', y_s')(y_{s+1} - y_s)(x_{r+1} - x_r)$ and by the definition of a simple integral, the limit of this when n tends to infinity is

$$\left\{\int_b^B f(x_r', y)dy\right\}(x_{r+1} - x_r).$$

The double integral is therefore the limit of

$$\sum_0^{m-1}\left\{\int_b^B f(x_r', y)dy\right\}(x_{r+1} - x_r)$$

which, again from the definition of a simple integral is equal to

$$\int_a^A\left\{\int_b^B f(x, y)dy\right\}dx.$$

By reversing the order of integration we obtain similarly that the double integral is equal to $\int_b^B\left\{\int_a^A f(x, y)dx\right\}dy$. For a rectangle, therefore, we may without ambiguity, write

$$\iint_D f(x, y)dx\,dy = \int_a^A\int_b^B f(x, y)dx\,dy = \int_b^B\int_a^A f(x, y)dy\,dx$$

and regard the two latter integrals as *repeated simple integrals.*

Examples. (i) $\iint(x^2 + y^2)dx\,dy$ over the rectangle bounded by $x = 0$, $x = a$, $y = 0$, $y = b$, is equal to $\int_0^a [x^2y + \frac{1}{3}y^3]_0^b\,dx = \int_0^a (bx^2 + \frac{1}{3}b^3)\,dx = \frac{1}{3}ab(a^2 + b^2)$.

(ii) $\iint f(x)\phi(y)dx\,dy$ over the rectangle given by $a < x < A$, $b < y < B$ is equal to $\left\{\int_a^A f(x)dx\right\}\left\{\int_b^B \phi(y)dy\right\}$

(iii) $\iint e^{-r^2}r\,dr\,d\theta$ over the quadrant of a circle specified by $0 < r < a$, $0 < \theta < \pi/2$ (in polar co-ordinates), is

$$\left\{\int_0^a e^{-r^2}r\,dr\right\}\left\{\int_0^{\pi/2} d\theta\right\} = \frac{\pi}{4}(1 - e^{-a^2}).$$

9.14. Elementary Closed Boundary. Assume first that the boundary is quadratic. (*Fig. 12.*) Then the line $x = c$, $(a < c < A)$ meets the boundary in two points given by $(c, y_1(c))$, $(c, y_2(c))$ where $y_1 > y_2$ and y_1,

y_2 are continuous functions of c. Similarly the line $y = c'$, $(b < c' < B)$ meets the boundary in two points given by $(x_1(c'), c')$, $(x_2(c'), c')$, where $x_1 > x_2$; the circumscribing rectangle being given by $a < x < A$, $b < y < B$. Denote $y_1(x)$, $y_2(x)$, $x_1(y)$, $x_2(y)$ by Y_1, Y_2, X_1, X_2 respectively.

Now define $f(x, y)$ for the *whole* rectangle by taking $f(x, y) = 0$ *outside* the area Ω. The boundary of Ω is therefore a curve of finite discontinuity, and the area covered by it is zero. The double integral over the rectangle is then obviously equal to the double integral over Ω.

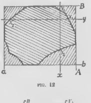


FIG. 12

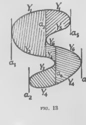


FIG. 13

Now $\int_b^B f(x, y)dy = \int_{Y_2}^{Y_1} f(x, y)dy$ since $f(x, y) = 0$ for $b < y < Y_2$ and $Y_1 < y < B$

i.e. $\iint_\Omega f(x, y)dx\,dy = \int_a^A \left\{\int_{Y_2}^{Y_1} f(x, y)dy\right\}dx$, ($Y_1$, Y_2 being functions of x).

Similarly

$\iint_\Omega f(x, y)dx\,dy = \int_b^B \left\{\int_{X_2}^{X_1} f(x, y)dx\right\}dy$, ($X_1$, X_2 being functions of y).

More generally, we can apply the same method to an area Ω bounded by an elementary curve, since we can divide Ω into a finite number of sub-regions whose boundaries are quadratic. Thus, in *Fig. 13*, the double integral is equal to

$$\int_{a_1}^{a_2}\left\{\int_{Y_2}^{Y_1} f\,dy\right\}dx + \int_{a_2}^{a_3}\left\{\int_{Y_4}^{Y_3} f\,dy\right\}dx + \int_{a_3}^{a_4}\left\{\int_{Y_1}^{Y_5} f\,dy\right\}dx$$
$$+ \int_{a_4}^{a_5}\left\{\int_{Y_3}^{Y_1} f\,dy\right\}dx + \int_{a_3}^{a_5}\left\{\int_{Y_4}^{Y_6} f\,dy\right\}dx$$

and if $f(x, y) = \dfrac{\partial F}{\partial y}(x, y)$, the above result might be written

$$\int_{a_1}^{a_4} F(x, Y_1)dx - \int_{a_2}^{a_1} F(x, Y_2)dx + \int_{a_2}^{a_4} F(x, Y_3)dx$$
$$- \int_{a_2}^{a_5} F(x, Y_4)dx - \int_{a_6}^{a_5} F(x, Y_5)dx - \int_{a_6}^{a_4} F(x, Y_6)dx.$$

Examples. (i) $\iint xy\,dx\,dy$ over the area given by the boundary: $y = 0$, $(0 < x < 3)$; $y = (x-3)^2$, $(2 < x < 3)$; $y = 1$, $(1 < x < 2)$, $y = x$, $(0 < x < 1)$. (*Fig. 14.*) The integral is

$$\int_0^3 [(\tfrac{1}{2}xy^2)_0^x + (\tfrac{1}{2}xy^2)_0^1 + (\tfrac{1}{2}xy^2)_0^{(x-3)^2}]\,dx = \tfrac{131}{120} \text{ (after evaluation)},$$

or, integrating first with respect to x, we verify that the value is

$$\int_0^1 (\tfrac{1}{2}x^2y)_y^{3-\sqrt{y}}\,dy = \int_0^1 (\tfrac{9}{2}y - 3y^{3/2} + \tfrac{1}{2}y^2 - \tfrac{1}{2}y^3)dy = \tfrac{131}{120}.$$

(ii) $\iint x\,dx\,dy$ over the area shown in *Fig. 15.*

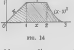


FIG. 14

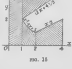


FIG. 15

Here $$\iint x\,dx\,dy = \int_0^1 (xy)_0^4\,dx + \int_1^4 (xy)_{y_1}^4\,dx + \int_1^2 (xy)_0^{y_2}\,dx + \int_2^4 (xy)_0^{y_3}\,dx$$

where $y_1 = \frac{1}{3}(2x + 4)$, $y_2 = x^2 - 4x + 5$, $y_3 = \frac{1}{2}x$.

The value is

$$\int_0^1 4x\,dx + \int_1^4 (\tfrac{8}{3}x - \tfrac{2}{3}x^2)dx + \int_1^2 (x^3 - 4x^2 + 5x)dx + \int_2^4 \tfrac{1}{2}x^2\,dx = \tfrac{77}{4}.$$

9.15. Symmetrical Areas. Let the area Ω be symmetrical about OY as in *Fig. 16* (i); then from the definition of a double integral as a sum, it follows that

$$\iint_\Omega f(x, y)dx\,dy = \iint_{\Omega_1} \{f(x, y) + f(-x, y)\}dx\,dy$$

where Ω_1 is that half of Ω for which $x > 0$. Denote Ω_1 by $\Omega\,(x +, y)$.

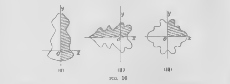
If $f(x, y) = f(-x, y)$, $f(x, y)$ is said to be *even* (x);

and if $f(x, y) = -f(-x, y)$, $f(x, y)$ is said to be *odd* (x).

Therefore for an area $\Omega(x, y)$ symmetrical about OY

$$\iint f(x, y)dx\,dy = 2\iint f(x, y)dx\,dy \text{ over } \Omega(x+, y)$$

if f is even (x) and $\iint_\Omega f(x, y)dx\,dy = 0$ if f is odd (x)



(i) (ii) (iii)

FIG. 16

Similarly for an area symmetrical about OX as in *Fig. 16* (ii)

$$\iint_\Omega f(x, y)dx\,dy = 2\iint f(x, y)dx\,dy \text{ over } \Omega(x, y+)$$

if f is even (y) and $\iint_\Omega f(x, y)dx\,dy = 0$ if f is odd (y).

Finally, if the area is symmetrical about *both* axes as in *Fig. 16* (iii)

$$\iint_\Omega f(x, y)\,dx\,dy = 4\iint f(x, y)dx\,dy \text{ over } \Omega(x+, y+)$$

if f is even (x, y) and is zero if f is odd in either variable.

In particular, if p, q, m, n denote positive integers or zero, for *Fig. 16* (i),

$$\iint_\Omega x^{2p}y^n\,dx\,dy = 2\iint x^{2p}y^n\,dx\,dy \text{ over } \Omega(x+, y)\,, \quad \iint_\Omega x^{2p+1}y^n\,dx\,dy = 0;$$

for *Fig. 16* (ii),

$$\iint_\Omega x^m y^{2n}\,dx\,dy = 2\iint x^m y^{2n}\,dx\,dy \text{ over } \Omega(x, y+)\,; \quad \iint_\Omega x^m y^{2n+1}\,dx\,dy = 0;$$

and for *Fig. 16* (iii),

$$\iint_\Omega x^{2p}y^{2n}\,dx\,dy = 4\iint x^{2p}y^{2n}\,dx\,dy \text{ over } \Omega(x+, y+),$$

whilst

$$\iint_\Omega x^{2p+1}y^{2n}\,dx\,dy = \iint_\Omega x^{2p}y^{2n+1}\,dx\,dy = \iint_\Omega x^{2p+1}y^{2n+1}\,dx\,dy = 0.$$

Note. Similar simplifications may be made when there is any line of symmetry by changing the axes so that the new y axis becomes the line of symmetry. (*See* next §.) In particular if $x = h$ is a line of symmetry, it is obvious from the definition of the double integral that

$$\iint f(x, y)dx\,dy \text{ over } \Omega(x, y) = \iint f(x+h, y)dx\,dy \text{ over } \Omega(x+h, y)$$

Example.

$\iint (ax^2 + 2hxy + by^2 + 2gx + 2fy + c)\,dx\,dy$ over the area bounded by

$|(x - x_0)/p| + |(y - y_0)/q| = 1$, $(p, q > 0)$. (*Fig.* 17.)

Take $x = x_0 + \xi$, $y = y_0 + \eta$.

Then the integrand $\phi(x, y)$ becomes $a\xi^2 + 2h\xi\eta + b\eta^2 + 2g_1\xi + 2f_1\eta + \phi(x_0, y_0)$ and the domain of ξ, η is the parallelogram given by $|\xi/p| + |\eta/q| = 1$, and is symmetrical about the ξ and η axes. Now

$$\iint \xi^2\,d\xi\,d\eta = 4\left(\frac{p^3q}{12}\right), \iint \eta^2\,d\xi\,d\eta = \tfrac{1}{3}pq^3$$

so that the required result is

$$\tfrac{1}{3}pq(ap^2 + bq^2) + 2pq\,\phi(x_0, y_0).$$

FIG. 17

9.16. Change of Variable in a Double Integral. Let u, v be given functions of x, y. Then if these functions are continuous at x_0, y_0 and in

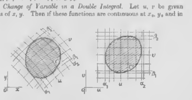


FIG. 18

the neighbourhood and possess partial derivatives of the first order, x, y can be expressed uniquely as functions of u, v when $J \equiv \dfrac{\partial(u, v)}{\partial(x, y)}$ is not zero at x_0, y_0; and these functions tend to $u_0 = u(x_0, y_0)$, $v_0 = v(x_0, y_0)$ when x, y tend respectively to x_0, y_0. We may write these functions as $x(u, v)$, $y(u, v)$. Given a certain region Ω (*Fig.* 18) in the $x - y$ plane, with its boundary γ, there will therefore correspond a set of points Ω' with a boundary γ'. If u, v are single-valued functions of x, y, to each point of Ω and γ there will correspond a single point u, v. But the converse is not necessarily true even when J does not vanish in Ω or on γ. Let us assume, however, that Ω and γ (where γ is a simple closed curve) is transformed into Ω_1 and γ_1 so that the transformation is actually one-one. If this is the case, then γ_1 is also a simple closed curve in one-one correspondence with γ.

Since the case of the simple closed boundary is easily reduced to that of the rectangle, it will be sufficient for us to take the boundary γ_1 as a rectangle $\alpha_1 < u < \alpha_2$, $\beta_1 < v < \beta_2$.

Since we are assuming $J \neq 0$ throughout Ω and γ we may take $J > 0$ (for the variables u, v may be interchanged if necessary).

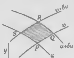


FIG. 19

Corresponding to the division of the rectangle by the lines $u = u_r$ $(r = 1$ to $n-1)$, $v = v_s$ $(s = 1$ to $m - 1)$, we have a subdivision of $\Omega(x, y)$ into curvilinear quadrilaterals $PQRS$. (*Fig. 19.*) It is obviously sufficient, however, to assume that the element of area is the ordinary quadrilateral $PQRS$.

If P is given by $x(u, v)$, $y(u, v)$, then Q, R, S are determined by the parameters $(u + \delta u, v)$, $(u + \delta u, v + \delta v)$, $(u, v + \delta v)$.

Thus $x_Q - x_P = x_u\delta u + O(\delta\rho^2)$, $x_S - x_P = x_v\delta v + O(\delta\rho^2)$,

$$x_R - x_P = x_u\delta u + x_v\delta v + O(\delta\rho^2)$$

with similar expressions for $y_Q - y_P$, $y_S - y_P$, $y_R - y_P$, where

$$\delta\rho = (\delta u^2 + \delta v^2)^{\frac{1}{2}}.$$

If the terms $O(\delta\rho^2)$ were ignored, the points $PQRS$ would form a parallelogram of area

$$(x_Q - x_P)(y_S - y_P) - (x_S - x_P)(y_Q - y_P) = \frac{\partial(x, y)}{\partial(u, v)}\,\delta u\,\delta v$$

i.e. the area of the quadrilateral $PQRS$ is $J_1\,\delta u\,\delta v + O(\delta\rho^3)$ where

$$J_1 = \frac{\partial(x, y)}{\partial(u, v)}.$$

Thus

$$\iint_\Omega f(x, y)dx\,dy$$

$$= \lim \Sigma\Sigma[f\{x(u_r, v_s), y(u_r, v_s)\}]\{J_1(u_r, v_s) + k\}(u_{r+1} - u_r)(v_{s+1} - v_s)$$

where $u_{r+1} - u_r$, $v_{s+1} - v_s$ are written δu_r, δv_s and $k = O(\delta\rho_{rs})$. Given ε, the subdivisions can be taken sufficiently small to ensure that $\delta\rho_{rs} < \varepsilon$ for every r, s. Thus $O(\delta\rho_{rs}) < \lambda\varepsilon$, where λ is bounded.

Thus $\displaystyle\iint_\Omega f(x, y)dx\,dy = \iint_{\Omega_1}[f\{x(u, v), y(u, v)\}]\frac{\partial(x, y)}{\partial(u, v)}du\,dv + \lim K$

where $|K| < \lambda M\varepsilon(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)$ and $M = \max|f(x, y)|$.

i.e. $K \to 0$ and $\displaystyle\iint_\Omega f(x, y)dx\,dy = \iint_{\Omega_1} f(x, y)J_1\,du\,dv = \iint_{\Omega_1} f(x, y)\frac{du\,dv}{J}$

where $J = \dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{1}{J_1}$.

Notes. (i) The Jacobian J_1 may, of course, vanish on the boundary.

(ii) To allow for the case when u, v are interchanged, we may write $|J|$ for J in the formula.

(iii) When the variables are changed, the line element ds is given by

$$ds^2 = (x_u\,du + x_v\,dv)^2 + (y_u\,du + y_v\,dv)^2$$
$$= E\,du^2 + 2G\,du\,dv + F\,dv^2$$

where $E = x_u^2 + y_u^2$, $G = x_u x_v + y_u y_v$, $F = x_v^2 + y_v^2$.

It should be noted that $|J_1| = |(x_u y_v - x_v y_u)| = \sqrt{(EF - G^2)}$.

The curves u = constant, v = constant are *orthogonal* if $G = 0$, since the gradients of these curves are y_v/x_v, y_u/x_u respectively. Thus for orthogonal co-ordinates ds^2 is of the form $h_1^2\,du^2 + h_2^2\,dv^2$ where $|J_1| = h_1h_2$.

For example, $\iint_\Omega f(x, y)dx\,dy = \iint_{\Omega_1} f(r\cos\theta,\ r\sin\theta)r\,dr\,d\theta$, since

$$ds^2 = dr^2 + r^2\,d\theta^2.$$

(iv) In applying the formula for change of variable, care should be taken to ensure that the transformation of Ω (and γ) into Ω_1 (and γ_1) is one-one. More particularly, when the variables are being changed from x, y to u, v, it is essential that to each point of Ω_1 (or γ_1) there should correspond only one point of Ω (or γ). The transformation is not necessarily one-one, when J is of invariable sign in Ω (and γ).

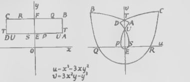


FIG. 20

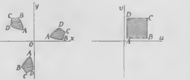


FIG. 21

For example, if $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$, it is easily verified that the rectangle of the $x - y$ plane shown in *Fig. 20* is transformed into an area of the $u - v$ plane whose boundary is not simple. In particular, the points $(\pm\sqrt{3}, 1)$ are both transformed into $(0, 8)$. Also $J = 9(x^2 + y^2)^2$ is never zero in Ω or on γ.

In *Fig. 21*, each of the three areas shown in the $x - y$ plane is transformed into the same rectangle in the $u - v$ plane $(1 < u < 8,\ 1 < v < 8)$; and the transformation is $1 - 1$ for any of these three areas when u, v belong to the rectangle, if the correct functional values (or branches) of x, y are taken.

The following theorem gives a sufficient test for one-one correspondence.

Theorem. If (i) Ω is convex (i.e. such that the straight line joining any two points of Ω lies wholly in Ω)

(ii) $P_1(x_1, y_1)$ $P_2(x_2, y_2)$ are any two points in Ω (distinct or coincident)

(iii) $J_{12} = \dfrac{\partial u}{\partial x_1}\dfrac{\partial v}{\partial y_2} - \dfrac{\partial u}{\partial y_1}\dfrac{\partial v}{\partial x_2}$ is never zero

then the transformation is one-one. (*Demell.*)

If possible, let $u(P) = u(P')$ and $v(P) = v(P')$. Then if the straight line joining PP' has the equation $x\cos\theta + y\sin\theta = p$, we must have $u_x \sin\theta - u_y\cos\theta = 0$ for some point $P_1(x_1, y_1)$ between P, P' and $v_x\sin\theta - v_y\cos\theta = 0$ for some point $P_2(x_2, y_2)$ between P, P' and P_1, P_2 may or may not coincide.

Therefore $\dfrac{\partial u}{\partial x_1}\dfrac{\partial v}{\partial y_2} - \dfrac{\partial v}{\partial x_2}\dfrac{\partial u}{\partial y_1} = 0$, which contradicts one of the conditions of the theorem.

Examples. (i) Find the finite area in the first quadrant bounded by $y^3 = a_1x^3$, $y^3 = a_2x^3$, $xy^2 = b_1^3$, $xy^2 = b_2^3$ $(a_2 > a_1 > 0,\ b_1 > b_2 > 0)$. Take $u = xy^2$, $v = y^3/x^3$. The area in the $u - v$ plane is the rectangle bounded by $a_1 < v < a_2$, $b_2^3 < v < b_1^3$.

Also $\dfrac{\partial(u, v)}{\partial(x, y)} = 7\dfrac{y^4}{x^3}$. Also $x = u^{3/7}v^{-2/7}$, $y = u^{2/7}v^{1/7}$, and the transformation is one-one.

Area $= \frac{1}{7}\iint_{D_1} u^{-2/7}v^{-3/7}\,du\,dv = \frac{7}{20}(b_1^{[illegible]} - b_2^{[illegible]})(a_2^{[illegible]} - a_1^{[illegible]})$.

(ii) Find the area of the loop of the curve $x^3 + y^3 = 3axy$.

Take $v = \dfrac{y^2}{x}$, $u = \dfrac{x^2}{y}$, then $\dfrac{\partial(u, v)}{\partial(x, y)} = 3$, if x, y are not zero. The area of the loop is the limit of $\iint\frac{1}{3}\,du\,dv$ over the triangle specified by $u = \varepsilon_1$, $v = \varepsilon_2$, $u + v = 3a$, when $\varepsilon_1\,(> 0)$, $\varepsilon_2\,(> 0) \to 0$, i.e. is equal to $\iint\frac{1}{3}\,du\,dv$ over the triangle given by $u = 0$, $v = 0$, $u + v = 3a$. Thus the area of the loop is $\frac{3}{2}a^2$.

(iii) Find $\iint\phi(x, y)\,dx\,dy$ over the area of the ellipse $\dfrac{(x - x_0)^2}{a^2} + \dfrac{(y - y_0)^2}{b^2} = 1$ where $\phi(x, y) = Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C$.

Let $x = x_0 + aX$, $y = y_0 + bY$, then $\phi(x, y)$ becomes

$$Aa^2X^2 + 2H\,ab\,XY + Bb^2Y^2 + 2G_1X + 2F_1Y + \phi(x_0, y_0)$$

and the new region is the interior of the circle $X^2 + Y^2 = 1$.

By symmetry $\iint X^2\,dX\,dY = \iint Y^2\,dX\,dY = \frac{1}{2}\iint(X^2 + Y^2)\,dX\,dY = \frac{1}{2}\iint r^3\,dr\,d\theta$, where $X = r\cos\theta$, $Y = r\sin\theta$, $0 < r < 1$, $0 < \theta < 2\pi$

i.e. $$\iint X^2\,dX\,dY = \iint Y^2\,dX\,dY = \tfrac{1}{4}\pi$$

Also $\iint XY\,dX\,dY = \iint X\,dX\,dY = \iint Y\,dX\,dY = 0$; $\iint dX\,dY = \pi$. Thus the required value is $\frac{1}{4}\pi ab(Aa^2 + Bb^2 + 4\phi_0)$.

9.2. Volumes and Surfaces Let $f(x, y) > 0$ over an area Ω. If at each point (x, y) of Ω a distance $z\,(= f(x, y))$ is measured parallel to OZ, the extremity lies on a surface $z = f(x, y)$. (*Fig. 22.*)

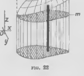


FIG. 22

The lines drawn parallel to OZ from the boundary of ω_r, a sub-region of Ω, determine a cylinder of cross-section ω_r. The volume of that portion of the cylinder cut off between the two planes $z = c$ and $z = c + h$ is $h\omega_r$ and we therefore assume that the volume cut off from the cylinder between $z = 0$ and $z = f(x, y)$ lies

between $m_r\omega_r$ and $M_r\omega_r$. Thus the total volume cut off from the cylinder of section Ω between $z = 0$ and $z = f(x, y)$ may be assumed to lie between $\Sigma m_r\omega_r$ and $\Sigma M_r\omega_r$. Since these sums have a common limit $\iint f(x, y)dx\, dy$, the double integral provides a suitable definition of the volume required.

Note. If $f(x, y)$ has both signs in Ω, the double integral must then give the sum of the volumes for $z > 0$ less the sum of the volumes for $z < 0$.

9.21 *Volume determined by a Closed Surface.* The locus given by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, where u, v are variable parameters belonging to a region Ω_1 in the $u - v$ plane, is called a *simple surface* if x, y, z are continuous functions of u, v not assuming the same set of three values for any pair of values interior to Ω_1. Assuming that x, y, z are differentiable functions, we shall also assume that the Jacobians

$$J_1\left(= \frac{\partial(y, z)}{\partial(u, v)}\right),\ J_2\left(= \frac{\partial(z, x)}{\partial(u, v)}\right),\ J_3\left(= \frac{\partial(x, y)}{\partial(u, v)}\right)$$

do not vanish simultaneously at any point of the surface. A point where these Jacobians all vanish is called *singular*. The loci given by $u =$ constant, $v =$ constant are curves drawn on the surface, and the direction cosines of the tangents to these curves (and therefore to the surface) are proportional to x_v, y_v, z_v and x_u, y_u, z_u respectively. If l, m, n are the direction cosines of the normal to the surface at (u, v) we have

$$lx_u + my_u + nz_u = 0 = lx_v + my_v + nz_v$$

and therefore $l : m : n = J_1 : J_2 : J_3$ so that a unique normal (or tangent plane) does not exist at a singular point; and also it is not in general possible to express any one of the variables x, y, z as a unique function of the other two in the neighbourhood of a singular point. A simple surface may be called *elementary* if it is possible to divide it up into a finite number of regions within each of which one of the variables x, y, z can be expressed as a single valued continuous function of the other two. An elementary surface which is such that all lines parallel to OX, OY and OZ meet the surface in two points at most may be called an elementary *quadratic* surface, and we shall regard it as obvious that the region bounded by an elementary closed surface can be divided up into a finite number of sub-regions, each of which is bounded by an elementary quadratic closed surface. It is sufficient, therefore, in obtaining the ordinary formulae relating to volumes and areas of closed (elementary) surfaces to consider the quadratic type.

By taking a cube of side ϵ whose edges are parallel to the axes and which entirely encloses a portion of a surface $z = f(x, y)$, it may be shown as in the case of areas, that if $f(x, y)$ is continuous, the volume covered by it is zero.

The projection of the points of a quadratic surface on $z = 0$ is obviously an area Ω bounded by a quadratic curve. (*Fig. 23.*) A line drawn through any point $(x, y, 0)$ of Ω parallel to OZ meets the surface

in the two points $(x, y, z_1(x, y))$, $(x, y, z_2(x, y))$ where $z_1 > z_2$, so that since we assume that z_1, z_2 are continuous functions of x, y, the volume enclosed by the surface is $\iint_\Omega (z_1 - z_2)dx\,dy$. We may similarly obtain formulae for the volume as $\iint_{\Omega_1} (x_1 - x_2)dy\,dz, \iint_{\Omega_2} (y_1 - y_2)dz\,dx$, using an obvious notation, Ω_1, Ω_2 being the projections of the surface on $x = 0$, $y = 0$ respectively.

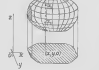


FIG. 23

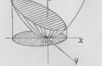


FIG. 24

Example. Find the volume cut from the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$ by the plane $lx + my + nz = p$ (assumed to meet the surface). (*Fig. 24.*)

A line through (x, y) parallel to OZ meeting the volume does so in two points (x, y, z_1) and (x, y, z_2) where

$$z_1 = \frac{p - lx - my}{n}; \quad z_2 = \frac{c}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right).$$

Thus $V = \iint_\Omega (z_1 - z_2)dx\,dy$ where Ω is determined by the equation $z_1 - z_2 = 0$.

Now $\frac{2}{c}(z_2 - z_1) = \frac{(x + x_0)^2}{a^2} + \frac{(y + y_0)^2}{b^2} - \lambda^2$, where $x_0cn = a^2l$, $y_0cn = b^2m$, $\lambda^2 = \frac{2p}{cn} + \frac{a^2l^2}{c^2n^2} + \frac{b^2m^2}{c^2n^2} (\lambda^2 > 0)$.

By taking $x + x_0 = ar\cos\theta$, $y + y_0 = br\sin\theta$ and using the method of *Example (iii)*, § *9.16*, we find that $V = \frac{\pi ab}{4c^2n^4}(a^2l^2 + b^2m^2 + 2pcn)^2$.

9.22. *Area of a Surface*, $z = f(x, y)$. A twisted curve $x = x(t)$, $y = y(t)$, $z = z(t)$ may be called elementary if it can be divided up into a finite number of parts in each of which two of the variables can be expressed as single-valued continuous functions of the third. Its projections on the co-ordinate planes are then elementary plane curves and the projections are closed if the curve is closed. For simplicity in exposition we shall usually assume that these projections are quadratic. In particular, the locus of the point given by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $u = u(t)$, $v = v(t)$, is a curve lying on the surface $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$. If it is elementary and closed, its projection

on the co-ordinate planes are elementary closed curves, which are the boundaries of plane regions divisible into a finite number of closed sub-regions with quadratic boundaries.

Let the projection on $z = 0$ be an elementary curve γ bounding a region Ω (*Fig.* 25), and let the equation of the surface be $z = f(x, y)$ where $f(x, y)$ is single-valued and positive. If $f(x, y)$ is continuous and differentiable, there is a single normal at each point whose direction cosines are proportional to $-f_x, -f_y, 1$ and this normal is never per-

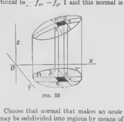


FIG. 25

pendicular to OZ. Choose that normal that makes an *acute* angle with OZ. The area Ω may be subdivided into regions by means of the planes $x = x_r$ ($r = 1$ to $n - 1$), $y = y_s$ ($s = 1$ to $m - 1$), where $x = x_0$, $x = x_n$, $y = y_0$, $y = y_m$ form the rectangle circumscribed to γ.

The sub-regions of Ω consist of *complete* rectangles like $P_1Q_1R_1S_1$ and of *irregular* areas abutting on γ. The subdividing planes divide the surface area also into sub-regions of which the former sub-regions are the corresponding projections. Let the vertices of the representative complete rectangle $P_1Q_1R_1S_1$ be given by $P_1(x, y)$, $Q_1(x + \delta x, y)$, $R_1(x + \delta x, y + \delta y)$, $S_1(x, y + \delta y)$ and let these be the projections of P, Q, R, S of the surface. The direction cosines of the normal to the surface at P are

$$l, m, n = (-z_x, -z_y, 1)/(1 + z_x^2 + z_y^2)^{\frac{1}{2}}.$$

and therefore the direction cosines of the normals to the planes PQR and PSR are of the form $(l + k_1, m + k_2, n + k_3)$, $(l + \lambda_1, m + \lambda_2, n + \lambda_3)$ respectively, where $k_r, \lambda_r = O(\delta\rho)$, $(\delta\rho^2 = \delta x^2 + \delta y^2)$.

But $\Delta P_1Q_1R_1 = (n + k_3)\Delta PQR$ and $\Delta P_1R_1S_1 = (n + \lambda_3)\Delta PRS$,

i.e. $\Delta PQR + \Delta PRS = \dfrac{1 + \sigma}{n}\,\delta x\,\delta y$ where $\sigma = O(\delta\rho)$, since $n \neq 0$.

$$\text{Thus } \Sigma(\Delta PQR + \Delta PRS) = \sum\sum \frac{(x_{r+1} - x_r)(y_{s+1} - y_s)}{n_{rs}}(1 + \sigma_{rs}),$$

where l_{rs}, m_{rs}, n_{rs} are the direction cosines of the normal at (x_r, y_s) and $\sigma_{rs} = O(\delta\rho_{rs})$.

An irregular sub-region of Ω is part of a rectangle E_1 abutting on γ and is the projection of a corresponding part of a skew quadrilateral E, one of whose vertices at least is outside the boundary curve on the surface and one, at least, inside (δx_{rs}, δy_{rs} being small enough). Also E (the sum of one pair of the two triangles of which it is composed) $= E_1/(n + k)$ where l, m, n are the direction cosines of the normal for any point of E_1 and $k = O(\delta\rho)$, $\delta\rho$ being the diagonal of E_1. If therefore γ is a curve that covers zero area, $\Sigma E_1 \to 0$ and therefore $\Sigma E \to 0$ (since $(n + k)^{-1}$ is bounded and $\max \delta\rho \to 0$).

By means of the subdividing planes $x = x_r$, $y = y_s$ we have obtained a number of skew quadrilaterals whose vertices are in the surface, and by joining a diagonal of each quadrilateral we have obtained the triangular faces of a polyhedron inscribed in the surface (which is therefore not the most general type of polyhedron). The projections of certain of these faces are triangles Δ_t lying entirely within γ, and the sum of their areas is $\Sigma\Delta_t(1 + \sigma)/n$ where n refers to a point of Δ_t and $\sigma = O(\epsilon)$, ϵ being $\max(\delta\rho_{rs})$.

The projections of the other faces are triangles abutting on γ and the sum of their areas tends to zero when $\max(\delta\rho_{rs})$ tends to zero. The summation $\sum\frac{\Delta_t}{n}$ tends to $\iint_\Omega \frac{1}{n}\,dx\,dy$ and the summation $\sum\sigma\frac{\Delta_t}{n}$ being $O(\epsilon)\sum\frac{\Delta_t}{n}$ must tend to zero. The double integral $\iint_\Omega \frac{1}{n}\,dx\,dy$ therefore provides a natural definition of the *Surface Area*.

Substituting the value of n, we find for the surface area S under consideration, $S = \iint_\Omega \sqrt{(1 + z_x^2 + z_y^2)}\,dx\,dy$.

9.23. Surface Area in Curvilinear Co-ordinates. Let $u = u(t)$, $v = v(t)$ be an elementary closed curve γ_1 in the $u - v$ plane enclosing an area Ω. Then this curve corresponds to an elementary closed curve γ on the elementary surface defined by the equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$. If we assume that x, y, z are single-valued functions of u, v for all u, v in Ω (and γ_1), then to each such point u, v there corresponds one point within or on γ (although the converse is not, in general, true). We shall assume also that $J_1\left(= \frac{\partial(y, z)}{\partial(u, v)}\right)$, $J_2\left(= \frac{\partial(z, x)}{\partial(u, v)}\right)$, $J_3\left(= \frac{\partial(x, y)}{\partial(u, v)}\right)$ do not all vanish simultaneously at any point of Ω, since the surface is assumed to be simple. The direction cosines of the normal at (u, v) have already been shown to be proportional to J_1, J_2, J_3.

Let Ω be divided up into rectangles by means of the lines $u = u_r$ $(r = 1$ to $n - 1)$, $v = v_s$ $(s = 1$ to $m - 1)$, where $u = u_0$, $u = u_n$, $v = v_0$, $v = v_m$ are fixed lines on the $u - v$ plane forming the circumscribed rectangle of γ_1. The sub-regions are either complete rectangles like $P_1Q_1R_1S_1$ (*Fig. 26*), or irregular areas ω_1 partly bounded by an arc of γ_1. The surface area is correspondingly divided up into curvilinear quadrilaterals $PQRS$ and irregular areas ω partly bounded by an arc of

y. In the light of our work in the last paragraph where the simpler case $x = u$, $y = v$ was considered, it is sufficient, in order to get the required formula, to find an expression for the area of the parallelogram formed by the differential displacements $\overrightarrow{PQ'}$, $\overrightarrow{PS'}$ along the curves v = constant,

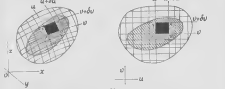


FIG 26

u = constant respectively through P. If x, y, z are the co-ordinates of P

$$PQ' = (x_u\mathbf{i} + y_u\mathbf{j} + z_u\mathbf{k})\delta u\,;\quad PS' = (x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k})\delta v$$

and the vector area of the parallelogram is

$$(J_1\mathbf{i} + J_2\mathbf{j} + J_3\mathbf{k})\delta u\ \delta v$$

its direction being along the appropriate normal to the surface. The absolute magnitude of the surface area is therefore

$$\iint_\Omega (J_1^2 + J_2^2 + J_3^2)^{\frac{1}{2}} du\ dv$$

where $du\ dv$ corresponds to the absolute magnitude of the elementary area of the $u - v$ plane

The symbol $(J_1^2 + J_2^2 + J_3^2)^{\frac{1}{2}} du\ dv$ is often written dS and is called the *Surface Element* and is essentially positive. It is necessary, however, for subsequent development to give greater precision to the notion of surface area by introducing the idea of *Vector Surface Element*. This is defined to be dS **N** where **N** is *unit* normal in such a direction that $[\mathbf{N}\mathbf{a}\mathbf{b}] = +1$, where **a**, **b** are *unit* vectors along the directions u-increasing and v-increasing respectively. It is usually more convenient to *prescribe* this normal to a given surface, and therefore the variables u, v should be interchanged if necessary to secure that $[\mathbf{N}\mathbf{a}\mathbf{b}] = +1$. For a closed surface, for example, it is usual to prescribe the outward-drawn normal.

9.24. The Line-element on a Surface. In rectangular co-ordinates, the line element ds is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

and when x, y, z are functions of u, v this becomes

$$ds^2 = E\,du^2 + 2G\,du\,dv + F\,dv^2$$

where $E = \Sigma x_u^2$, $G = \Sigma x_u x_v$, $F = \Sigma x_v^2$. Thus $dS = \sqrt{(EF - G^2)}du\,dv$.

The curves u = constant, v = constant are *orthogonal* if $G = 0$ and therefore for orthogonal co-ordinates on a surface, ds^2 is of the form $h_1^2\,du^2 + h_2^2\,dv^2$ and $dS = h_1 h_2\,du\,dv$.

Example. The sphere $x^2 + y^2 + z^2 = a^2$ in spherical polar co-ordinates for which $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, is given by $r = a$. The curves θ = constant, ϕ = constant are orthogonal and $ds^2 = a^2(\sin^2\theta\,d\phi^2 + d\theta^2)$ so that $dS = a^2\sin\theta\,d\theta\,d\phi$.

Thus the area between the two parallel planes $z = h_1$, $z = h_2$, where

$$a > h_1 > z > h_2 > -a$$

is given by $\displaystyle\int_{\phi=0}^{\phi=2\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} a^2\sin\theta\,d\theta\,d\phi$. $(h_1 = a\cos\theta_1,\ h_2 = a\cos\theta_2)$

i.e. Area is $2\pi a(h_1 - h_2)$.

9.25. Surfaces of Revolution. Let the part of the curve $y = f(x)$ between $x = x_1$ and $x = x_2$ $(x_2 > x_1)$ be rotated about OX through an angle 2π to form part of a surface of revolution, where $f(x)$ is single-valued, continuous and positive in $x_1 \leq x \leq x_2$. (*Fig. 27.*) Also assume

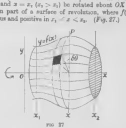


FIG. 27

that $f'(x)$ exists in (x_1, x_2). The $u - v$ curves at a point on the surface may be taken respectively to be, (i) the generating curve at the point, (ii) the circle described by the point. These curves are orthogonal. Let the line element of the generating curve be ds and let θ be the angle through which the ordinate has turned from its initial position (in XOY). Then the line element on the surface is $\sqrt{(ds^2 + y^2\,d\theta^2)}$ and the surface element $dS = y\,ds\,d\theta$.

The surface area required is

$$\int_{x=x_1}^{x=x_2}\int_{\theta=0}^{\theta=2\pi} y\frac{ds}{dx}\,dx\,d\theta = 2\pi\int_{x_1}^{x_2} y(1 + y'^2)^{\frac{1}{2}}\,dx.$$

If the co-ordinates of the generating curve are expressed in terms of s, $S = 2\pi\int_{s_1}^{s_2} y\,ds$ and if in terms of a parameter t, $S = 2\pi\int_{t_1}^{t_2} y(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}\,dt$.

Examples. (i) A square hole of side $2b$ is cut symmetrically through a sphere of radius $a(> b\sqrt{2})$. Find the surface removed. Take OZ as the axis of the hole: then one part of the surface is given by $z = \sqrt{(a^2 - x^2 - y^2)}$ and

$$1 + z_x^2 + z_y^2 = \frac{a^2}{(a^2 - x^2 - y^2)}.$$

Let the sides of the square be parallel to OX, OY. Then $\frac{1}{8}S = 4a\iint \frac{dx\,dy}{\sqrt{(a^2 - x^2 - y^2)}}$ over the square given by $0 < x < b$, $0 < y < b$

i.e. $\quad S = 8a\int_0^b \text{arc sin}\left\{\frac{b}{\sqrt{(a^2 - x^2)}}\right\}dx = 8ab \text{ arc sin}\left\{\frac{b}{\sqrt{(a^2 - b^2)}}\right\} - 8ab\,I$

where $I = \int_0^b \frac{x^2\,dx}{(a^2 - x^2)\sqrt{(a^2 - b^2 - x^2)}} = -\int_0^b \frac{dx}{\sqrt{(a^2 - b^2 - x^2)}} + a^2\int_0^{t_1} \frac{dt}{a^2 + b^2t^2}$

and $\quad x = \sqrt{(a^2 - b^2)}\sin\phi, \quad t = \tan\phi, \; t_1 = \tan\phi_1, \quad b = \sqrt{(a^2 - b^2)}\sin\phi_1.$
On evaluation we find that

$$S = 16ab \text{ arc sin}\left\{\frac{b}{\sqrt{(a^2 - b^2)}}\right\} - 8a^2 \text{ arc tan}\left\{\frac{b^2}{a\sqrt{(a^2 - 2b^2)}}\right\}.$$

(ii) Find the portion intercepted on the surface $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = a\phi\tan\alpha$ by the cylinder $\rho = b$

$x_\rho^2 + y_\rho^2 + z_\rho^2 = 1; \quad x_\phi^2 + y_\phi^2 + z_\phi^2 = \rho^2 + a^2\tan^2\alpha; \quad x_\rho x_\phi + y_\rho y_\phi + z_\rho z_\phi = 0$

Therefore $S = \iint(\rho^2 + a^2\tan^2\alpha)^{\frac{1}{2}}\,d\rho\,d\phi$ over the area $0 < \rho < b$, $0 < \phi < 2\pi$

$$= \pi\left\{b\sqrt{(b^2 + a^2\tan^2\alpha)} + a^2\tan^2\alpha\log\left(\frac{b + \sqrt{(b^2 + a^2\tan^2\alpha)}}{a\tan\alpha}\right)\right\}.$$

If $b = a$, $S = \pi a^2(\sec\alpha + \tan^2\alpha\log\cot\frac{1}{2}\alpha)$.

(iii) The catenary $y = c\cosh x/c$ from $x = 0$ to x_1 is rotated about the axis OX. Find the area of the surface formed.

$$y' = \sinh\frac{x}{c}; \;\; (1 + y'^2)^{\frac{1}{2}} = \cosh\frac{x}{c}\;(= y/c) \text{ and } S = 2\pi\int_0^{x_1} c\cosh^2\frac{x}{c}\,dx$$

i.e. $$S = \pi c\left(x_1 + c\sinh\frac{x_1}{c}\cosh\frac{x_1}{c}\right) = \pi(cx_1 + y_1s_1).$$

9.3. Line Integrals. If x, y, z are the co-ordinates of a point on a given rectifiable curve, they are functions of the arc s (measured from some fixed point) and we can form an integral of the form $\int_A^B f(x, y, z)ds$ where A, B are two points of the curve.

Such an integral is called a *Line Integral.* If P, Q, R are functions of x, y, z, the integral denoted by $\int_A^B (P\,dx + Q\,dy + R\,dz)$ is *defined* to mean the line integral $\int_A^B (lP + mQ + nR)ds$ where

$$l\,(= dx/ds),\; m\,(= dy/ds),\; n\,(= dz/ds)$$

are the direction cosines of the tangent (in the direction of s-increasing).

9.31. Line Integral for a Plane Curve. In two dimensions

$$\int_A^B (P\,dx + Q\,dy) = \int_A^B (lP + mQ)ds.$$

It is usually possible to divide the arc AB (as for example in the case of an elementary curve) into a finite number of parts, within each of which either y is expressible as a single-valued differentiable function of x, or x as a similar function of y. It is then possible to express a given line integral as the sum of a finite number of ordinary integrals. It is essential in such an expression to keep account of the appropriate signs of dx or dy.

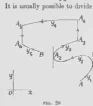


FIG. 28

For example, for the curve shown in *Fig. 28*, if the abscissae of A, A_r, B are denoted by a, a_r, b respectively

$$\int_A^B P(x, y)dx = \int_a^{a_1} P(x, y_1)dx + \int_{a_1}^{a_2} P(x, y_2)dx$$

$$+ \int_{a_3}^{a_4} P(x, y_3)dx + \int_{a_4}^{a_5} P(x, y_4)dx + \int_{a_6}^{b} P(x, y_5)dx$$

where A_2A_3, A_5A_6 in the figure are parallel to OY.

Example. Find $\int_C (x^2 + 2y)dx + (y - x)dy$ where C is the boundary of the closed curve given by $x = 0$, $y = \frac{1}{2}x + 1$, $y = \frac{1}{4}(x - 4)^2$, $y = 0$ and is described counter-clockwise (*Fig. 29*).

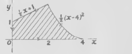


FIG. 29

Along $y = 0$, $I = \int_0^4 x^2\,dx = 21\frac{1}{3}$.

Along $y = \frac{1}{4}(x + 4)^2$, $I = \int_4^2 \{x^2 + (x - 4)^2 + \frac{1}{8}(x - 4)^3 - x(x - 4)\}dx$

$$= -24\frac{1}{3}.$$

Along $y = \frac{1}{2}x + 1$, $I = \int_2^0 (x^2 + x + 2 + \frac{1}{4} - 4x)dx = -9\frac{1}{3}$ and along $x = 0$,

$I \quad \int_1^0 y\,dy = -\frac{1}{2}$.

The value of the line integral is therefore -13.

Note. This may be verified by *Green's Formula*, § 9.32.

9.32. *Green's Formula for an Elementary Plane Closed Curve.* If P, Q are continuous functions of x, y on and within an elementary plane closed curve C and possess continuous partial derivatives, then

$$\oint_C (P\,dx + Q\,dy) = \iint_\Omega (Q_x - P_y)dx\,dy$$

where Ω is the region bounded by C, and where the arrow on the integral sign denotes that the direction in which C is described is the same as the direction from OX to OY as viewed from a point on one side of the plane XOY.

Let C be an elementary quadratic curve as shown in *Fig. 30*, with circumscribing rectangle given by $a < x < A$, $b < y < B$. For a value x for which $a < x < A$ there are two points E, F on C given by $(x, y_1(x))$,

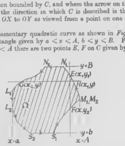


FIG. 30

$(x, y_2(x))$, $(y_1 > y_2)$, and for a value y for which $b < y < B$, there are two points H, G on C given by $(x_1(y), y)$, $(x_2(y), y)$, $(x_1 > x_2)$. On the boundary of the rectangle there may be straight parts L_1L_2, M_1M_2, N_1N_2, S_1S_2, on $x = a$, $x = A$, $y = B$, $y = b$ respectively.

Then, since the derivative is continuous

$$\iint_\Omega Q_x\,dx\,dy = \int_b^B \{Q(x_1, y) - Q(x_2, y)\}dy = \int_{S_1}^{N_1} Q(x, y)dy - \int_{S_2}^{N_2} Q(x, y)dy$$
$$= \oint_C Q(x, y)dy$$

since dy is zero on S_1S_2 and N_1N_2.

Similarly $\iint_\Omega P_y\,dx\,dy = -\oint_C P(x, y)dx$.

Thus $\oint_C (P\,dx + Q\,dy) = \iint_\Omega (Q_x - P_y)\,dx\,dy$ and the direction in which C is described is counter-clockwise as viewed from that side of the plane XOY for which the direction from OX to OY is counter-clockwise.

The theorem is immediately extended to any elementary closed curve C by dividing the region Ω enclosed by C into a finite number of sub-regions $\omega_1, \omega_2, \ldots, \omega_n$ bounded by quadratic curves $\gamma_1, \gamma_2, \ldots, \gamma_n$. (*Fig. 31.*) For then

$$\iint_\Omega (Q_x - P_y)\,dx\,dy = \sum_1^n \iint_{\omega_r} (Q_x - P_y)\,dx\,dy = \sum_1^n \oint_{\gamma_r} (P\,dx + Q\,dy)$$
$$= \oint_C (P\,dx + Q\,dy)$$

since each part of a boundary γ_r not belonging to C is described once in each direction and P, Q are single valued.

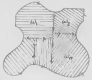


FIG. 31

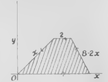


FIG. 32

Example. $\oint_C (x^2 - 3y)dx + (y - 2x)dy$, where C is the quadrilateral specified by $y = x$, $y = 2$, $y = 8 - 2x$, $y = 0$, as shown in *Fig. 32.*

By Green's formula the integral is

$$\iint_\Omega (-2 - 3)dx\,dy = -5\Omega = -25.$$

9.33. Green's Formula when $Q_x = P_y$. If $Q_x = P_y$ and the derivatives are continuous $\oint_C (P\,dx + Q\,dy) = 0$, suppose that $V(x, y)$ is any continuous function for which $V_y = Q(x, y)$. Then $P_y = V_{xy}$ so that P must be of the form $V_x + f(x)$. Take therefore $W = V + X$ where X is any integral of $f(x)$ with respect to x. Then $P = W_x$, $Q = W_y$, i.e. given P, Q and $P_y = Q_x$ we can always find a function W for which $P = W_x$, $Q = W_y$.

By Green's formula $\oint_C (W_x\,dx + W_y\,dy)$, i.e. $\oint_C dW = 0$. This shows that W is a single valued function of x, y for the region Ω, since its final value is the same as its initial value.

Now let AR_1B, AR_2B (*Fig.* 33) be any two elementary curves joining A, B and suppose that in the area between them there is no discontinuity.

Then
$$\int_{AR_1B} dW + \int_{BR_2A} dW = 0$$
i.e.
$$\int_{AR_1B} dW = W_B - W_A = \int_{AR_2B} dW$$
so that if W_A is the initial value of W at A, its value W_B at B is inde-

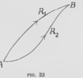


FIG. 33

pendent of the path joining A, B, if the one path can be deformed into the other without encountering a discontinuity.

Examples.

(i) $\displaystyle\int_{0,0}^{x,y} \{2(x+y)e^{2x} + 7e^{2x} + e^{2y}\}dx + \{3(x-y)e^{2y} + e^{2x} + 11e^{2y}\}dy.$

The integrands and their derivatives are continuous for all x, y. Also $Q_x = 2e^{2x} + 3e^{2y} - P_y$. The value of the integral is therefore independent of the path from $(0, 0)$ to (x, y).

Integrate from $(0, 0)$ to $(x, 0)$ along $y = 0$, then
$$I_1 = \int_0^x (2xe^{2x} + 7e^{2x} + 1)dx = xe^{2x} + 3e^{2x} + x - 3.$$
Integrate along $x =$ constant from $y = 0$ to y, then
$$I_2 = \int_0^y (3(x-y)e^{2y} + e^{2x} + 11e^{2y})dy = (x-y)e^{2y} + 4e^{2y} + ye^{2x} - x - 4.$$
Thus $\qquad I = I_1 + I_2 = (x+y+2)e^{2x} + (x-y+4)e^{2y} - 7$

(ii) Find $\oint_C (x^3 + 6x^2y + y^2 - 3y)dx + (2x^3 - 4xy^2 - 2xy + 3y)dy$ where C is the circle given by $(x-2)^2 + (y-2)^2 = 4$.

$I = \iint_C (3 - 4y - 4y^2)dx\,dy = \iint(-21 - 20Y - 4Y^2)dX\,dY$ over $X^2 + Y^2 = 4$.

But $\iint Y^2\,dX\,dY = \frac{1}{2}\iint(X^2 + Y^2)dX\,dY = \frac{1}{2}\iint r^3\,dr\,d\theta$ over $r = 2$ where r, θ are polar co-ordinates in the $X - Y$ plane.

Thus $\iint Y^2\,dX\,dY = 4\pi$; also $\iint dX\,dY = 4\pi$; $\iint Y\,dX\,dY = 0$.

The value of the integral is therefore -100π.

(iii) The area A determined by a closed curve bounded by C is $\displaystyle\oint_C \frac{\lambda y\,dx + \mu x\,dy}{\mu - \lambda}$ where λ, μ are unequal constants, for by Green's formula, this integral is $\iint dx\,dy$.

If $\mu = 0$, $A = -\oint_C y\,dx$, if $\lambda = 0$, $A = \oint_C x\,dy$; if $\lambda = -1$, $\mu = 1$, $A = \frac{1}{2}\oint_C(x\,dy - y\,dx) = \frac{1}{2}\oint_C r^2\,d\theta$ in polar co-ordinates.

Thus $A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C(x\,dy - y\,dx) = \frac{1}{2}\oint_C r^2\,d\theta$

(iv) Let $Q = V_x$, $P = -V_y$ and let the second derivatives of V be continuous. Then $\iint_\Omega \nabla^2 V\,dx\,dy = \oint_C (V_x\,dy - V_y\,dx)$.

Now $\nabla V \times d\mathbf{r} = (V_x\,dy - V_y\,dx)\mathbf{k}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ in the usual notation for vectors.

Thus
$$\oint_C \nabla V \times d\mathbf{r} = (\iint_\Omega \nabla^2 V\,dx\,dy)\mathbf{k}.$$

Denote the rates of change of V along the outward-drawn normal and the tangent to C by $\frac{\partial V}{\partial N}$, $\frac{\partial V}{\partial s}$ (s increasing in the direction of the tangent taken).

Then $\nabla V = \frac{\partial V}{\partial s}\mathbf{T} + \frac{\partial V}{\partial N}\mathbf{N}$; $d\mathbf{r} = ds.\mathbf{T}$ and therefore
$$\nabla V \times d\mathbf{r} = \frac{\partial V}{\partial N}ds(\mathbf{N} \times \mathbf{T}) = \left(\frac{\partial V}{\partial N}ds\right)\mathbf{k}.$$

Thus $(\iint_\Omega \nabla^2 V\,dx\,dy)\mathbf{k} = \left(\oint_C \frac{\partial V}{\partial N}ds\right)\mathbf{k} = \oint_C \nabla V \times d\mathbf{r}$.

(v) If x, y are changed into orthogonal co-ordinates u, v for which
$$ds^2 = h_1^2\,du^2 + h_2^2\,dv^2$$
find the value of $\nabla^2 V$ in terms of u, v.
$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u}\mathbf{i} + \frac{1}{h_2}\frac{\partial V}{\partial v}\mathbf{j};\quad d\mathbf{r} = h_1\,du\,\mathbf{i} + h_2\,dv\,\mathbf{j}$$
where $\mathbf{i}$, $\mathbf{j}$ are the unit vectors at (u, v) along the curves v = constant, u = constant.

Therefore $\iint_{\Omega_1} \nabla^2 V h_1 h_2\,du\,dv\;\mathbf{k} = \int_C \nabla V \times d\mathbf{r} = \int_{C_1}\left(\frac{h_2}{h_1}V_u\,dv - \frac{h_1}{h_2}V_v\,du\right)\mathbf{k}$

i.e.
$$\iint_{\Omega_1} \nabla^2 V h_1 h_2\,du\,dv = \iint_{\Omega_1}\left\{\frac{\partial}{\partial u}\left(\frac{h_2}{h_1}V_u\right) + \frac{\partial}{\partial v}\left(\frac{h_1}{h_2}V_v\right)\right\}du\,dv$$

This result is true when Ω_1 is the interior of a circle of centre (u, v), however small this circle may be taken. The integrands are therefore equal (by an obvious use of the mean-value theorem for double integrals)

i.e.
$$\nabla^2 V = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u}\left(\frac{h_2}{h_1}V_u\right) + \frac{\partial}{\partial v}\left(\frac{h_1}{h_2}V_v\right)\right].$$

For example, in polar co-ordinates, $\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$.

9·34. *Multiply-connected Areas.* Consider the area shown in *Fig. 34*, which has an outer boundary C (an elementary closed curve) and a number (n) of inner boundaries C_r ($r = 1$ to n) (elementary closed curves). By joining a point of C_r to a point of C by means of an elementary curve γ_r, the region becomes one with a single boundary consisting of C, C_r, γ_r (the last being described once in each direction). It is assumed that no curves γ_r intersect. Applying Green's formula to the new region, we note that the integrals along γ_r cancel each other (P, Q being single-

FIG. 34

valued) and that the direction in which C_r is described is opposite to that in which C is described,

i.e. $$\oint_C (P\,dx + Q\,dy) - \sum_1^n \oint_{C_r} (P\,dx + Q\,dy) = \iint_\Omega (Q_x - P_y)\,dx\,dy.$$

An area Ω is said to be *simply-connected* if a line within it joining any two points of its boundary divides Ω into two regions that cannot be connected without crossing the line (called a *cut*). Otherwise Ω is said to be *multiply-connected*. If by means of m cuts (the edges of the cuts, as they are being made, becoming extensions of the boundary), Ω becomes simply-connected, the original region is said to be $(m + 1)$-ply connected. Thus the region shown in *Fig. 34* is $(n + 1)$-ply connected.

9.35. Discontinuities in the Case when $Q_x = P_y$. When discontinuities P, Q or their derivatives occur in Ω, the case of greatest interest is that for which $Q_x = P_y$.

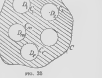


FIG. 35

Let there be n discontinuities at isolated points D_r in Ω ($r = 1$ to m). Draw elementary closed curves C_r surrounding these respectively but not intersecting each other. (*Fig. 35.*) Apply Green's formula to the multiply-connected region between C and C_r; then since $Q_x = P_y$, we have

$$\oint_C (P\,dx + Q\,dy) = \sum_1^m \oint_{C_r} (P\,dx + Q\,dy).$$

The value of the integral is independent of the choice of C_r provided it is of requisite type and does not actually pass through D_r.

In particular, take C_r to be a circle centre D_r and small radius ρ.

Then $$\oint_{C_r} (P\,dx + Q\,dy) = \rho \int_0^{2\pi} (Q\cos\theta - P\sin\theta)\,d\theta$$

where $x = a_r + \rho\cos\theta$, $y = b_r + \rho\sin\theta$ and D_r is the point (a_r, b_r). Although $Q\cos\theta - P\sin\theta$ may not exist when ρ is zero, the integral $\rho\int_0^{2\pi} (Q\cos\theta - P\sin\theta)\,d\theta$ may tend to a definite limit ω_r when $\rho \to 0$. Thus if all the integrals $\oint_{C_r} (P\,dx + Q\,dy)$ tend to limits ω_r, we have

$$\oint_C (P\,dx + Q\,dy) = \sum_1^m \omega_r.$$

Example $\oint_C \frac{(ax - by)dx + (bx + ay)dy}{(x^2 + y^2)}$, ($a$, b constant), where C is an elementary curve enclosing 0.

Here $Q_x = P_y = \frac{b(y^2 - x^2) - 2axy}{(x^2 + y^2)^2}$ and 0 is a point of discontinuity.

Taking $x = \rho\cos\theta$, $y = \rho\sin\theta$, we find that the integral is $b\int_0^{2\pi} d\theta = 2\pi b$.

9.36. Many Valued Integrals. Let $F(x, y) = \int_{x_0, y_0}^{x, y} (P\,dx + Q\,dy)$, where $Q_x = P_y$, and let the path of integration be a particular curve ALB joining A (x_0, y_0) to B (x, y) not passing through a discontinuity of P, Q or their derivatives. (*Fig. 36.*) Let $AL'B$ be any other path joining AB not passing through a discontinuity, the region between the two curves containing within it a number of discontinuities at D_1, D_2, . . ., D_k. Then

$$\int_{AL'B} (P\,dx + Q\,dy) = \int_{ALB} (P\,dx + Q\,dy) + \sum_1^k \oint_{C_r} (P\,dx + Q\,dy)$$

$$= F(x, y) + \sum_1^k \omega_r$$

if these limits ω_r exist.

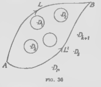


FIG. 36

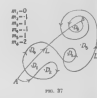


FIG. 37

Therefore the function $\int_{x_0, y_0}^{x, y} P\,dx + Q\,dy$ when the path from A to B is not specified has many values, and its general value (for a simple path) is of the form $F(x, y) + \sum_1^n \lambda_r \omega_r$, where λ_r is 1, -1, or 0 according as the closed path $AL'BLA$ encircles D_r counter-clockwise, encircles D_r clockwise or does not encircle D_r (if the direction from OX to OY be regarded as counter-clockwise).

More generally, if we allow the path $AL'B$ to cross itself, it may be deformed into ALB by taking m_r circuits (m_r being positive, negative or zero) round D_r.

Thus $\int_A^B (P\,dx + Q\,dy) = F(x, y) + \sum_1^n m_r \omega_r$ where m_r is an integer, positive, negative or zero. (*Fig. 37.*)

9.4. Triple and Multiple Integrals. The definition of the double integral and its method of evaluation suggests the corresponding definitions and methods for integrals involving three or more variables.

Thus, if $f(x, y, z)$ is a given function, bounded in a region V enclosed by an elementary closed surface S, we can form, as in the case of double

integrals, the sums $\Sigma M_r v_r$, $\Sigma m_r v_r$, where $v_1, v_2, \ldots$ are sub-regions into which V is divided and M_r, m_r are the upper and lower bounds of $f(x, y, z)$ in v_r (or its boundary). When these sums tend to a common limit as the circumscribing cubes of the sub-regions all tend to zero, this common limit is called the triple integral of $f(x, y, z)$ through V and is written $\iiint_V f(x, y, z)dx\,dy\,dz$.

The integral may be proved to exist when $f(x, y, z)$ is continuous throughout V and S; it also exists when $f(x, y, z)$ is continuous except over a finite number of surfaces, if $f(x, y, z)$ is bounded there and the surfaces cover zero volume.

When the triple integral exists, its value is the limit of the sum $\Sigma f(x_r, y_r, z_r)v_r$ where (x_r, y_r, z_r) is any point of v_r or its boundary.

Again $MV > \iiint_V f(x, y, z)dx\,dy\,dz > mV$ where M, m are the upper and lower bounds of $f(x, y, z)$ throughout V (and S) and

$$\frac{1}{V}\iiint_V f(x, y, z)dx\,dy\,dz,$$

which lies between M and m is called the *Mean Value* of $f(x, y, z)$ throughout V.

9.41. *Evaluation of a Triple Integral.* The method used to evaluate directly a double integral may be extended to the case of a triple integral. Let the boundary S be quadratic. Then a line parallel to OZ through (x, y) that crosses the boundary does so in two points $(x, y, z_1(x, y))$, $(x, y, z_2(x, y))$, $(z_1 > z_2)$ where z_1, z_2 are continuous functions of (x, y). Let the upper and lower bounds on S (i) of x (all y, z) be A, a, (ii) of y (all z, x) be B, b, (iii) of z (all x, y) be C, c respectively. The set of points (x, y) for which z_1, z_2 exist belong to an area Ω in $x-y$ plane whose boundary (an elementary quadratic curve) is determined by the relation $z_1 = z_2$. (*Fig. 38.*)

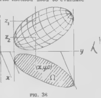


FIG. 38

Thus $\iiint_V f(x, y, z)dx\,dy\,dz$ is equal to $\iint_\Omega\left\{\int_{z_2}^{z_1} f(x, y, z)dz\right\}dx\,dy$, where Ω is the projection of the volume on $z = 0$. Similarly, we may obtain formulae by integrating first with respect to y or x.

If a line parallel to OY in $z = 0$ meets the boundary of Ω in $\{x, y_1(x)\}$, $\{x, y_2(x)\}$, $(y_1 > y_2)$, we may write the triple integral as

$$\int_a^A\left[\int_{y_2}^{y_1}\left\{\int_{z_2}^{z_1} f(x, y, z)dz\right\}dy\right]dx.$$

A convenient notation for this is $\int_a^A dx \int_{y_0}^{y_1} dy \int_{z_0}^{z_1} f\,dz$.

In particular, if V is a rectangular parallelopiped given by $a < x < A$, $b < y < B$, $c < z < C$, the integral may be written

$$\int_{x=a}^{A}\int_{y=b}^{B}\int_{z=c}^{C} f(x,\ y,\ z)dx\,dy\,dz.$$

Example. Evaluate $\iiint xy^3\,dx\,dy\,dz$ throughout the tetrahedron bounded by $z = 0$, $z = x$, $y = x$, $y = a$. (*Fig.* 39.) Integration with respect to z gives $I = \iint_\Omega x^2y^3\,dx\,dy$, where Ω is the triangular area determined by $x = 0$, $y = a$, $y = x$

i.e.
$$I = \int_0^a \tfrac{1}{4}(x^2a^4 - x^6)dx = \tfrac{1}{21}a^7.$$

FIG. 39

The *six* equivalent repeated integrals corresponding to the six ways of effecting the integration are in this example

$$\text{(i) } \int_0^a dx \int_x^a dy \int_0^x xy^3\,dz;\quad \text{(ii) } \int_0^a dy \int_0^y dx \int_0^x xy^3\,dz;\quad \text{(iii) } \int_0^a dy \int_0^y dz \int_z^y xy^3\,dx;$$

$$\text{(iv) } \int_0^a dz \int_z^a dy \int_z^y xy^3\,dx;\quad \text{(v) } \int_0^a dz \int_z^a dx \int_x^a xy^3\,dy;\quad \text{(vi) } \int_0^a dx \int_0^x dz \int_x^a xy^3\,dy.$$

9.42. Multiple Integrals. A *multiple integral* is denoted by

$$\iint \ldots \int f(x_1,\ x_2,\ \ldots,\ x_n)dx_1,\ dx_2\ \ldots\ dx_n$$

and refers to a closed n-dimensional region. The definition is analogous to that of the triple integral, although there is not a correspondingly simple way of illustrating it geometrically. When the region of variation is of a sufficiently simple character, the method of evaluation by repeated integration will not present any difficulty.

Example. Evaluate $\iiiint e^{x+2y+3z+4u}\,dx\,dy\,dz\,du$ over all positive and zero values of x, y, z, u for which $0 < x + y + z + u < a$.

$I = \tfrac{1}{4}\iiint(e^{4a-3x-2y-z} - e^{x+2y+3z})dx\,dy\,dz$ for $0 < x + y + z < a$

$= \iint\{\tfrac{1}{4}e^{4a-3x-2y} - \tfrac{1}{3}e^{3a-2x-y} + \tfrac{1}{12}e^{x+2y}\}dx\,dy$ for $0 < x + y < a$

$$\int_0^a (\tfrac{1}{8}e^{4a-3x} - \tfrac{1}{3}e^{3a-2x} + \tfrac{1}{4}e^{2a-x} - \tfrac{1}{24}e^{x})dx$$

$$\tfrac{1}{24}e^{4a} - \tfrac{1}{6}e^{3a} + \tfrac{1}{4}e^{2a} - \tfrac{1}{6}e^{a} + \tfrac{1}{24} = \tfrac{1}{24}(e^a - 1)^4$$

9.43. Change of Variable in a Multiple Integral. The formula for change of variable in a multiple integral

$$\iint \cdots \int_{A_n} f(x_1, x_2, \ldots x_n)dx_1\,dx_2 \ldots dx_n$$

$$= \iint \cdots \int_{B_n} [f] \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)} du_1\,du_2 \ldots du_n$$

where $x_r = x_r(u_1, u_2, \ldots, u_n)$ $(r = 1 \text{ to } n)$ and B_n is the region in the u_r-space corresponding to A_n in the x_r-space, may be proved by induction.

We shall assume that there is a $1 - 1$ correspondence between A_n with its boundary and B_n with its boundary, and that the Jacobian $J = \dfrac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}$ is continuous and never vanishes in the region.

The region A_n can be divided up into a finite number of sub-regions within each of which one at least of the derivatives $\dfrac{\partial x_1}{\partial u_r}$ is never zero. Otherwise by the process of subdivision and selection it would be possible to find a point in A_n near which all the derivatives $\dfrac{\partial x_1}{\partial u_r}$ (assumed continuous) vanished. This would make J zero, thus contradicting the hypothesis.

Without loss of generality therefore we can assume that $\dfrac{\partial x_1}{\partial u_1} \neq 0$. If we assume that the boundary is an elementary $(n - 1)$-dimensional region (defined in an obvious way), we may integrate first with respect to the variables $x_2, x_3, \ldots x_n$ and obtain $I = \int_{a_1}^{\alpha_1} F(x_1)dx_1$, where $F = \iint \cdots \int_{C_{n-1}} f\,dx_2\,dx_3 \ldots dx_n$, C_{n-1} is the set of points of A_n for which x_1 has a fixed value between a_1, α_1 (the lower and upper bounds of x_1 in A_n, all $x_2, \ldots x_n$).

From the relation $x_1 = x_1(u_1, u_2, \ldots, u_n)$, since $\dfrac{\partial x_1}{\partial u_1} \neq 0$ we can at least, in the neighbourhood of a particular set of values, determine u_1 uniquely as a function of $x_1, u_2, \ldots u_n$. By substituting this value of u_1 in the functions $x_2, \ldots, x_n$, we obtain a transformation from $x_1, x_2, \ldots, x_n$ to $x_1, u_2, \ldots, u_n$ transforming A_n to A_n'; and the transformation must be $1 - 1$ since the *given* transformation is $1 - 1$.

Now F is an integral of multiplicity $(n - 1)$ and x_1 is fixed during the integration. If we *assume* that the formula for change of variable

is true for $(n-1)$ variables, we have

$$F = \iint \cdots \int_{C'_{n-1}} fJ'\, du_2\, du_3 \ldots du_n$$

where
$$J' = \frac{\partial(x_2, \ldots, x_n)}{\partial(u_2, \ldots, u_n)} \ (x_1 \text{ constant}).$$

But $J = \dfrac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x_1, u_2, \ldots, u_n)} \cdot \dfrac{\partial(x_1, u_2, \ldots, u_n)}{\partial(u_1, u_2, \ldots, u_n)} = J' \cdot \dfrac{\partial x_1}{\partial u_1}$, so that J' is not zero (or infinite) and has the value $J \Big/ \dfrac{\partial x_1}{\partial u_1}$.

Thus
$$I = \int_{a_1}^{b_1} \left[\iint \cdots \int_{C'_{n-1}} \left(f.J \Big/ \frac{\partial x_1}{\partial u_1} \right) du_2 \ldots du_n \right] dx_1$$
$$= \iint \cdots \int_{A'_n} \left(f.J \Big/ \frac{\partial x_1}{\partial u_1} \right) dx_1\, du_2\, du_3 \ldots du_n.$$

Let us assume, for simplicity, that the line for which $u_2, u_3, \ldots u_n$ are all constant meets the boundary in two points at most, these points being given by

$$\{X_1(u_2, \ldots u_n), u_2, \ldots, u_n\}, \{X'_1(u_2, \ldots u_n), u_2, \ldots, u_n\}$$
$$(X'_1 > X_1).$$

Then $I = \displaystyle\iint \cdots \int_{D_{n-1}} G(u_2 \ldots u_n) du_2 \ldots du_n$, where

$$G = \int_{X_1}^{X_1'} \left(f.J. \Big/ \frac{\partial x_1}{\partial u_1} \right) dx_1.$$

Now change the variables from $x_1, u_2, \ldots u_n$ to $u_1, u_2, \ldots u_n$ by taking $x_1 = x_1(u_1, u_2, \ldots u_n)$, the transformation being $1-1$ as before. Since $u_2, \ldots, u_n$ are fixed in G, the latter becomes

$$\int_{U_1}^{U_1'} f.J.du_1$$

where U'_1, U_1 are the values of u_1 that correspond to X'_1, X_1 of x_1.

Thus
$$I = \iint \cdots \int_{D_{n-1}} \left\{ \int_{U_1}^{U_1'} f.J\, du_1 \right\} du_2 \ldots du_n$$
$$= \iint \cdots \int_{B_n} f.J.du_1\, du_2 \ldots du_n$$

Example. $\iiiint (x + y + z + u)^n\, xyzu\, dx\, dy\, dz\, du$ over all zero and positive values of x, y, z, u for which $0 < x + y + z + u < 1$ (n being a positive integer or zero).

Take $X = x + y + z + u$, $XY = y + z + u$, $XYZ = z + u$, $XYZU = u$. If we denote $X, XY, XYZ, XYZU$ by ξ, η, ζ, ω respectively, we have

$$\frac{\partial(x, y, z, u)}{\partial(X, Y, Z, U)} = \frac{\partial(\xi, \eta, \zeta, \omega)}{\partial(X, Y, Z, U)} \div \frac{\partial(\xi, \eta, \zeta, \omega)}{\partial(x, y, z, u)} = X^3Y^2Z.$$

The transformed region is determined by the boundaries $u - XYZU = 0$, $z - XYZ(1 - U) = 0$, $y - XY(1 - Z) = 0$, $x - X(1 - Y) = 0$, and

$$x + y + z + u - 1 = X - 1 = 0$$

i.e. consists of the unit 'cube' $0 < X < 1, 0 < Y < 1, 0 < Z < 1, 0 < U < 1$.

The Jacobian vanishes when $u = 0$ but not otherwise. The given integral obviously exists and is therefore the limit when $\varepsilon \longrightarrow 0$ over the region obtained by omitting all values u, for which $0 < u < \varepsilon$. This is equivalent to the integral over the transformed region from which the points given by $0 < XYZU < \varepsilon$ are omitted. This restricted region obviously tends uniformly to the unit 'cube' when $\varepsilon \longrightarrow 0$. The transformed integral over the unit 'cube' gives the required value.

Thus

$$I = \int_0^1 X^{\alpha+7}\,dX \int_0^1 Y^4(1 - Y)\,dY \int_0^1 Z^4(1 - Z)\,dZ \int_0^1 U(1 - U)\,dU = \frac{1}{(\alpha + 8)7!}.$$

9.5. Surface Integrals. An integral of the form $\iint_S \phi(x, y, z)dS$ over a portion S of the surface given by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ is called a *Surface Integral*. Here dS is written for $\sqrt{(EG - F^2)}du\,dv$ and the integral is evaluated over the region Ω in the $u - v$ plane that corresponds to S.

The *vector surface element* is $dS\,\mathbf{N}$ where $\mathbf{N}$ is unit normal in a prescribed direction,

i.e. $$dS\,\mathbf{N} = (ldS)\mathbf{i} + (mdS)\mathbf{j} + (ndS)\mathbf{k}$$

where l, m, n are the direction cosines of the prescribed normal.

The components ldS, mdS, ndS may be positive or negative, and their absolute values are the areas of the projections of the surface element on the co-ordinate planes. If the element $dx\,dy$ that occurs in a double integral over a region in the $x - y$ plane is regarded as positive, ndS may be replaced by $dx\,dy$ if $n > 0$ and may be replaced by $-dx\,dy$ if $n < 0$.

However, the integral $\iint_S F(x, y, z)dx\,dy$ is sometimes used as a *surface* integral and must be taken to mean $\iint_S F(x, y, z)ndS$, so that when $dx\,dy$ occurs in a surface integral, it must be regarded as having *sign* as well as magnitude.

We shall therefore define $\iint_S (P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy)$ to be $\iint_S (lP + mQ + nR)dS$ and for definiteness we shall choose the direction of the normal in such a way that $\mathbf{N}$, $\mathbf{a}$, $\mathbf{b}$ form a positive system (i.e. $[\mathbf{Nab}] = +1$), where $\mathbf{a}$, $\mathbf{b}$ are unit vectors along the tangents to the curves $v =$ constant, $u =$ constant in the directions in which these variables increase. Also in the $u - v$ plane, an area will be regarded as positive if it is described in the same sense as the change in direction from the u-axis to the v-axis.

Let S be an elementary closed surface of quadratic type, so that

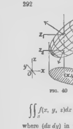


FIG. 40

its projection on $z = 0$ is a quadratic curve γ enclosing an area Ω. (*Fig. 40.*) Let the line through $(x, y, 0)$ of Ω meet S in (x, y, z_1), (x, y, z_2), $(z_1 > z_2)$. The points for which $z > z_1$ (on the line) are *outside* the surface and therefore the normal that makes an *acute* angle with $\overrightarrow{OZ}$ at (x, y, z_1) is the *outward* normal. Similarly the normal that makes an *obtuse* angle with $\overrightarrow{OZ}$ at (x, y, z_2) is also the outward normal. Thus

$$\iint_S f(x, y, z)dx\,dy = \iint_S fndS = \iint_\Omega \{f(x, y, z_1) - f(x, y, z_2)\}dx\,dy$$

where $(dx\,dy)$ in the last integral is *positive*.

Similar results may be obtained for

$$\iint_S \phi(x, y, z)dy\,dz \text{ and } \iint_S \psi(x, y, z)dz\,dx$$

More generally, if a line through $(x, y, 0)$ meets an elementary surface (not necessarily closed) in points (x, y, z_r), ($r = 1$ to m) where

$$z_1 > z_2 > z_3 \ldots > z_m$$

$\iint_S f(x, y, z)dx\,dy$ may be expressed as

$$\iint_{\Omega_1} f(x, y, z_1)dx\,dy - \iint_{\Omega_2} f(x, y, z_2)dx\,dy + \ldots$$
$$+ (-1)^{m-1}\iint_{\Omega_m} f(x, y, z_m)dx\,dy$$

if the normal at (x, y, z_1) makes an *acute* angle with $\overrightarrow{OZ}$, where Ω_r is the region for which z_r exists.

9.51. Green's Formula in Three Dimensions. If an elementary surface S encloses a volume V, then

$$\iiint_V (P_x + Q_y + R_z)dx\,dy\,dz = \iint_S (lP + mQ + nR)dS$$

where l, m, n are the direction cosines of the outward-drawn normal and P, Q, R are continuous functions of x, y, z possessing continuous derivatives.

Let the surface be quadratic. (*Fig. 40.*)

Then

$$\iiint_V R_z\,dx\,dy\,dz = \iint_\Omega \{R(x, y, z_1) - R(x, y, z_2)dx\,dy = \iint_S RndS\ (\S 9.5),$$

Similarly

$$\oiint\!\!\int Q_y\,dx\,dy\,dz = \iint_S Qm\,dS \text{ and } \iiint_V P_x\,dx\,dy\,dz = \iint_S Pl\,dS.$$

Thus $$\iiint_V (P_x + Q_y + R_z)dx\,dy\,dz = \iint_S (lP + mQ + nR)dS.$$

The theorem may be immediately extended to any elementary closed surface by dividing the region enclosed into a finite number of subregions bounded by quadratic surfaces.

Corollary. (i) If P, Q, R are the components of a *vector* function $\mathbf{F}$, then $\mathbf{F.N} = lP + mQ + nR$ and the theorem takes the form

$$\iiint_V \nabla\ \mathbf{F}\,dx\,dy\,dz = \iint_S \mathbf{F}\ \mathbf{dS} \text{ (where } \mathbf{dS} = \mathbf{N}dS).$$

(ii) Let $\mathbf{F} = \nabla E$, then

$$\iiint_V (\nabla^2 E)\,dx\,dy\,dz = \oiint \frac{\partial E}{\partial N}dS.$$

9.52. Harmonic Functions. A three-dimensional *harmonic function* $E(x, y, z)$ may be defined as one that is finite and continuous and possesses first and second derivatives in a given domain and satisfies *Laplace's equation* $\nabla^2 E\left(\equiv \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2}\right) = 0$. Thus if

$$r = \{(x-a)^2 + (y-b)^2 + (z-c)^2\}^{\frac{1}{2}}$$

$\frac{1}{r}$ is harmonic except at (a, b, c).

Example. Let E be harmonic in *Cor.* (ii), § *9.51*, where it is proved that

$$\iiint_V \nabla^2 E\,dx\,dy\,dz = \oiint \frac{\partial E}{\partial N}\,dS.$$

Then $\iint_S \frac{\partial E}{\partial N}dS = 0$ if E is harmonic throughout S and its interior

For example, $\iint_S \frac{\partial}{\partial N}\left(\frac{1}{r}\right)dS = 0$ where $r = \{(x-a)^2 + (y-b)^2 + (z-c)^2\}^{\frac{1}{2}}$ and (a, b, c) is outside S.

9.53. Discontinuities. Let there be m points D_r ($r = 1$ to m) within V at which there are discontinuities. By surrounding each of these points with small closed surfaces S_r, we can, by the method used in the case of Green's formula for two dimensions, deduce that

$$\iint_S \mathbf{F}.\mathbf{dS} = \sum_1^m \iint_{S_r} \mathbf{F}\ \mathbf{dS} + \iiint_{V'} \nabla.\mathbf{F}\,dx\,dy\,dz$$

where V' is the volume between the outer boundary S and the inner boundaries S_r.

In particular if $\nabla\ \mathbf{F} = 0$, we have $\iint_S \mathbf{F}.\mathbf{dS} = \sum_1^m \iint_{S_r} \mathbf{F}\ \mathbf{dS}$. Choos-

ing S_r to be a small sphere of radius ρ and centre D_r we deduce that $\iint_S \mathbf{F}.\mathbf{dS} = \sum_1^m \lim_{\rho \to 0} \iint_{S_r} \mathbf{F}\ \mathbf{dS}$ if these limits exist.

Example. Let $E = \dfrac{1}{r}$ where $r = \{(x-a)^2 + (y-b)^2 + (z-c)^2\}^{\frac{1}{2}}$ and let (a, b, c) be within S.

Then $\iint_S \dfrac{\partial}{\partial N}\left(\dfrac{1}{r}\right) dS = \lim_{\rho \to 0} \iint_{S_1} \left(-\dfrac{1}{\rho^2}\right) dS$ where S_1 is the sphere, centre (a, b, c) and radius ρ.

Thus $\iint_S \dfrac{\partial}{\partial N}\left(\dfrac{1}{r}\right) dS = -4\pi$ if (a, b, c) is within S (its value being zero if (a, b, c) is outside S, § 9.52, *Example*). This is sometimes called *Gauss's Integral*. (*See also Examples IX, No. 122.*)

9.54. Green's Theorems. Let $F = G\nabla E$ where E, G are invariants.

$d\vec{S} = \vec{N} \cdot dS$

Then $\iint_S G\nabla E.\underline{\mathbf{dS}} = \iiint_V (G.\nabla^2 E + \nabla G.\nabla E) dx\, dy\, dz$ (§ 9.51).

Similarly $\iint_S E\nabla G.\underline{\mathbf{dS}} = \iiint_V (E.\nabla^2 G + \nabla E.\nabla G) dx\, dy\, dz$

when there are no discontinuities in V or on S.

Therefore $\iint_S (G\nabla E - E\nabla G)\ \underline{\mathbf{dS}} - \iiint_V (G.\nabla^2 E - E\ \nabla^2 G) dx\, dy\, dz.$

But if there are m discontinuities within V at D_r we have

$$\iint_S (G\nabla E - E\nabla G).\underline{\mathbf{dS}}$$

$$= \Sigma \iint_{S_r} (G\nabla E - E\nabla G)\ \underline{\mathbf{dS}} + \iiint_{V'} (G.\nabla^2 E - E.\nabla^2 G) dx\, dy\, dz$$

(in the notation of § 9.53).

The above result may be called *Green's Theorem* (*General*).

Examples. (i) If E, G are harmonic throughout V and on S we have $\iint_S (G\nabla E - E\nabla G)\underline{\mathbf{dS}} = 0$, i.e. $\iint_S G\dfrac{\partial E}{\partial N} dS = \iint_S E\dfrac{\partial G}{\partial N} dS$.

(ii) Let $G = \dfrac{1}{r}$ where $r = \{(x-a)^2 + (y-b)^2 + (z-c)^2\}^{\frac{1}{2}}$ and E *any* invariant function (possessing bounded second derivatives throughout V and on S).

Then

$$\iint_S \left\{\frac{1}{r}\frac{\partial E}{\partial N} - E\frac{\partial}{\partial N}\left(\frac{1}{r}\right)\right\} dS - \iint_{S_1} \left\{\frac{1}{r}\frac{\partial E}{\partial N} - E\frac{\partial}{\partial N}\left(\frac{1}{r}\right)\right\} dS + \iiint_{V'} \frac{\nabla^2 E}{r} dx\, dy\, dz$$

and we may take the limit of the right-hand side, if it exists, when $\rho \to 0$, where S_1 is the surface of the sphere $r = \rho$.

$$\left|\iint_{S_1} \left(\frac{1}{r}\frac{\partial E}{\partial N}\right) dS\right| = \left|\frac{1}{\rho}\iint \left(\frac{\partial E}{\partial r}\right)_{r=\rho} dS\right| < 4\pi\rho M, \text{ where } M \text{ is max } \frac{\partial E}{\partial r} \text{ on } S_1.$$

Therefore $\iint_{S_1}\left(\frac{1}{r}\frac{\partial E}{\partial N}\right)dS \longrightarrow 0$ when $\rho \longrightarrow 0$.

$\iint_{S_1} E\frac{\partial}{\partial N}\left(\frac{1}{r}\right)dS = -\frac{1}{\rho^2}\iint\{E(a,\ b,\ c)+\rho\lambda\}dS$ where λ is bounded on S_1.

Therefore $\iint_{S_1} E\frac{\partial}{\partial N}\left(\frac{1}{r}\right)dS \longrightarrow -4\pi E(a,\ b,\ c)$ when $\rho \longrightarrow 0$.

$$\left|\iiint_{V-V'}\frac{\nabla^2 E}{r}\,dx\,dy\,dz\right| < \frac{4}{3}M_1\pi\rho^2, \text{ where } M_1 \text{ is max } \nabla^2 E \text{ in } V-V'.$$

Therefore $\iiint_{V'}\frac{\nabla^2 E}{r}\,dx\,dy\,dz \longrightarrow \iiint_V \frac{\nabla^2 E}{r}\,dx\,dy\,dz$ (which is convergent)

i.e. $$\iint_S\left\{\frac{1}{r}\frac{\partial E}{\partial N} - E\frac{\partial}{\partial N}\left(\frac{1}{r}\right)\right\}dS = 4\pi E(a,\ b,\ c) + \iiint_V \frac{\nabla^2 E}{r}\,dx\,dy\,dz.$$

This may be called *Green's Theorem (Special)*.

(iii) In example (ii) let E be harmonic in V and on S, then

$$4\pi E(a,\ b,\ c) = \iint_S\left\{\frac{1}{r}\frac{\partial E}{\partial N} - E\frac{\partial}{\partial N}\left(\frac{1}{r}\right)\right\}dS$$

thus expressing a harmonic function E at a point *inside* S in terms of the values of E, $\frac{\partial E}{\partial N}$ on S.

This may be called *Green's Theorem* (for *Harmonic Functions*).

Note. If (a, b, c) is outside S, $\iint_S\left\{\frac{1}{r}\frac{\partial E}{\partial N} - E\frac{\partial}{\partial N}\left(\frac{1}{r}\right)\right\}dS$ is 0. (*Example* (i).)

(iv) Let $G = \frac{1}{r} - U$, where U is harmonic and let E be harmonic.

Then $\iint_S\left[\left(\frac{1}{r}-G\right)\frac{\partial E}{\partial N} - E\left\{\frac{\partial}{\partial N}\left(\frac{1}{r}-G\right)\right\}\right]dS = 0$. (*Example* (i).)

i.e. $$\iint_S\left\{\frac{1}{r}\frac{\partial E}{\partial N} - E\frac{\partial}{\partial N}\left(\frac{1}{r}\right)\right\}dS = \iint_S\left(G\frac{\partial E}{\partial N} - E\frac{\partial G}{\partial N}\right)dS$$

or $4\pi E(a,\ b,\ c) = \iint_S\left(G\frac{\partial E}{\partial N} - E\frac{\partial G}{\partial N}\right)dS$, if $(a,\ b,\ c)$ is inside E.

If it is given that $G = 0$ on S, we have

$$4\pi E(a,\ b,\ c) = -\iint_S E\frac{\partial G}{\partial N}\,dS.$$

Thus if $E(x,\ y,\ z)$ is a solution of $\nabla^2 E = 0$ with given values E on the boundary, a knowledge of G (which depends on S and $(a,\ b,\ c)$ but *not* on the given values of E) enables us to find the value of E at any point $(a,\ b,\ c)$. The function G has been called *Green's Function* for the surface S, but the term is now used for a class of functions of which the above is a particular case. For example, if it is given that $\frac{\partial G}{\partial N} = 0$ on S, instead of $G = 0$, then

$$4\pi E(a,\ b,\ c) = \iint_S G\frac{\partial E}{\partial N}dS$$

thus giving $E(a,\ b,\ c)$ in terms of the values of $\frac{\partial E}{\partial N}$ on S.

9.55. Stokes's Theorem. Let C be an elementary closed curve (in three dimensions) and S an elementary surface bounded by C. (*Fig. 41.*)

Let the equations of the surface be

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v).$$

Let Ω be the area in the $u - v$ plane corresponding to S and γ (the boundary of Ω) the curve corresponding to C.

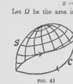


FIG. 41

Let **a**, **b** be unit vectors along the tangents to $v =$ constant, $u =$ constant in the directions in which these variables increase; and let γ be described in the direction of $\mathbf{a} \times \mathbf{b}$. At any point (u, v) of S choose the unit-normal N which is such that $[\mathbf{Nab}] = +1$.

Stokes's Theorem states if P, Q, R are functions of x, y, z possessing continuous derivatives on C and on S, then

$$\int_C (P\,dx + Q\,dy + R\,dz)$$
$$= \iint_S \{l(R_y - Q_z) + m(P_z - R_x) + n(Q_x - P_y)\}dS$$

where l, m, n are the direction cosines of **N**, the direction of description of C having been made definite by the specified description of γ.

$$\int_C (P\,dx + Q\,dy + R\,dz)$$
$$= \int_\gamma (P x_u + Qy_u + Rz_u)du + (Px_v + Qy_v + Rz_v)dv$$
$$= \iint_\Omega (P_u x_v - P_v x_u + Q_u y_v - Q_v y_u + R_u z_v - R_v z_u)du\,dv$$

by Green's formula in two dimensions.

But $P_u = P_x x_u + P_y y_u + P_z z_u$, $P_v = P_x x_v + P_y y_v + P_z z_v$, with similar expressions for Q_u, Q_v, R_u, R_v.

Therefore

$(P_u x_v - P_v x_u) = - J_3 P_y + J_2 P_z$; $(Q_u y_v - Q_v y_u) = - J_1 Q_z + J_3 Q_x$ and $(R_u z_v - R_v z_u) = - J_2 R_x + J_1 R_y$,

i.e. $\displaystyle\int_C (P\,dx + Q\,dy + R\,dz)$
$$= \iint_\Omega \{J_1(R_y - Q_z) + J_2(P_z - R_x) + J_3(Q_x - P_y)\}du\,dv.$$

But $dS\,\mathbf{N} = (J_1\mathbf{i} + J_2\mathbf{j} + J_3\mathbf{k})du\,dv = (ldS\mathbf{i} + mdS\mathbf{j} + ndS\mathbf{k})$

i.e. $J_1\,du\,dv$, $J_2\,du\,dv$, $J_3\,du\,dv$ may be replaced respectively by ldS, mdS, ndS,

or $\displaystyle\int_C (P\,dx + Q\,dy + R\,dz)$
$$= \iint_S \{l(R_y - Q_z) + m(P_z - R_x) + n(Q_x - P_y)\}dS.$$

If $\mathbf{F} = (P, Q, R)$ and $\mathbf{r} = (x, y, z)$, we may write this result:

$$\int_C \mathbf{F}.d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\, d\mathbf{S} = \iint (\mathbf{N}.\text{curl } \mathbf{F})dS.$$

Notes. (i) Green's formula in *two* dimensions is a particular case of *Stokes's Theorem.*

(ii) We deduce that if the normal surface integral of $\mathbf{E}$ over such a surface S is equal to the line integral of $\mathbf{F}$ round every curve C, then $\mathbf{E} = \text{curl } \mathbf{F}$.

9.6. The Simpler Applications of Integration. An account of the commoner applications of integration will naturally involve some recapitulation of work that has already been done. For simplicity in statement we shall assume that closed domains are bounded by elementary quadratic domains, and that the conditions for the existence of the integrals mentioned are satisfied. A line specified by the variation of x (the other variables being fixed) will meet the boundary of a domain in two points which will be denoted by x_1, x_2 $(x_1 > x_2)$.

9.601. The Arc. (i) If the curve is given by $x = x(t)$, $y = y(t)$, $z = z(t)$, the arc s measured from $t = t_0$ to the variable point t is given by

$$s = \int_{t_0}^{t} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}\, dt$$

so that $\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$.

(ii) If the curve is given by $y = f(x)$, and y is single-valued in the interval between x_0 and x, the arc s between x_0 and x is given by

$$s = \int_{x_0}^{x} \{1 + (f'(x))^2\}^{\frac{1}{2}}\, dx.$$

Notes. (i) $f'(x)$ may be discontinuous at a finite number of points if it is bounded.

(ii) In the determination of the absolute magnitude of an arc, care must be exercised in cases where ds/dt vanishes at a point of the curve.

Examples. (i) Find the whole length of the curve given by $x = a\cos^3 t$, $y = a\sin^3 t$, $(0 < t < 2\pi)$.

$\dot{x}^2 + \dot{y}^2 = 9a^2\cos^2 t\sin^2 t$ and $\dot{s} = 3a\cos t\sin t$ when $0 < t < \pi/2$ and $\pi < t < 3\pi/2$; but $\dot{s} = -3a\cos t\sin t$ in the second and fourth quadrants.

By symmetry, however, $s = 4\int_0^{\frac{\pi}{2}} 3a\cos t\sin t\, dt = 6a.$

(ii) Show that the length of the intersection of the paraboloid $x^2 - y^2 = az$ with the cylinder $x^2 + y^2 = a^2$ is the perimeter of the ellipse $x^2 + 5y^2 = 5a^2$. Take $x = a\cos t$, $y = a\sin t$, $z = a\cos 2t$ for the intersection. Then

$$s = a\int_0^{2\pi} (1 + 4\sin^2 2t)^{\frac{1}{2}}\, dt = a\sqrt{5}\int_0^{2\pi} (1 - \tfrac{4}{5}\sin^2 t)^{\frac{1}{2}}\, dt.$$

For the ellipse, take $x = \sqrt{5}a\cos u$, $y = a\sin u$ and find its perimeter

$$s' = a\sqrt{5}\int_0^{2\pi} (1 - \tfrac{4}{5}\sin^2 u)^{\frac{1}{2}}\, du = s.$$

9.602. Line Elements. (i) For the *curve* given by $x = x(t)$, $y = y(t)$, $z = z(t)$

$$ds^2 = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dt^2.$$

(ii) For the *surface* given by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $ds^2 = E\,du^2 + 2G\,du\,dv + F\,dv^2$, where $E = \Sigma x_u{}^2$, $G = \Sigma x_u x_v$, $F = \Sigma x_v{}^2$ and the arc of the curve $u = u(t)$, $v = v(t)$ on the surface is given by

$$\int_{t_0}^{t} (E\dot{u}^2 + 2G\dot{u}\dot{v} + F\dot{v}^2)^{\frac{1}{2}}\,dt.$$

If the co-ordinates are orthogonal, $G = 0$.

(iii) For the change of variables given by $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$,
$ds^2 = g_{11}\,du^2 + g_{22}\,dv^2 + g_{33}\,dw^2 + 2g_{23}\,dv\,dw + 2g_{31}\,dw\,du + 2g_{12}\,du\,dv$
where $g_{11} = \Sigma x_u{}^2$, $g_{22} = \Sigma x_v{}^2$, $g_{33} = \Sigma x_w{}^2$, $g_{23} = \Sigma x_v x_w$, $g_{31} = \Sigma x_u x_w$, $g_{12} = \Sigma x_u x_v$ and the co-ordinates are orthogonal if $g_{12} = g_{23} = g_{31} = 0$.

Example. Find the total length of the plane curve given by $r = a(1 + \cos\theta)$ in polar co-ordinates

$$ds^2 = r^2\,d\theta^2 + dr^2 = 2a^2(1 + \cos\theta)\,d\theta^2, \quad s = 2\int_0^{\pi} 2a\cos\tfrac{1}{2}\theta\,d\theta = 8a.$$

9.603. Plane Areas. (i) The area Ω bounded by a curve $y = f(x)$ (single-valued, > 0), the x-axis $y = 0$, and the ordinates $x = a$, $x = b$ is given by

$$\Omega = \int_a^b f(x)\,dx, \quad (b > a)$$

(ii) The area Ω bounded by a closed curve in the $x - y$ plane is

$$\iint_{\Omega} dx\,dy = \iint_{\Omega_1} \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv$$

when the variables are changed by means of the equations $x = x(u, v)$, $y = y(u, v)$; and Ω_1 is the area in the $u - v$ plane that corresponds to Ω.

If the curves u = constant, v = constant are orthogonal, the line element ds is given by $ds^2 = h_1^2\,du^2 + h_2^2\,dv^2$ and $\Omega = \iint h_1 h_2\,du\,dv$.

In particular, $\Omega = \iint_{\Omega_1} r\,dr\,d\theta$ in polar co-ordinates.

(iii) For a closed curve the area Ω may also be expressed as a line integral in various ways:

$$\Omega = \oint_C x\,dy = -\oint_C y\,dx = \oint_C \frac{px\,dy + qy\,dx}{p - q} \quad (p \neq q)$$

$$= \tfrac{1}{2}\oint_C (x\,dy - y\,dx) = \tfrac{1}{2}\oint_C r^2\,d\theta.$$

(iv) For a closed curve in which y may be given as a function of x, or x a function of y, or r a function of θ.

$$\Omega = \int_a^A (y_1 - y_2)\,dx. \quad (\textit{Fig. 42 (i), quadratic in direction } OY.)$$

$$\Omega = \int_b^B (x_1 - x_2)\,dy. \quad (\textit{Fig. 42 (ii), quadratic in direction } OX.)$$

$$\Omega = \tfrac{1}{2}\int_{\alpha}^{\beta}(r_1^2 - r_2^2)d\theta. \quad (\textit{Fig. 42 (iii)}, \text{ quadratic for a given } \theta, 0 \text{ outside.})$$

$$\Omega = \tfrac{1}{2}\int_{0}^{2\pi} r^2\, d\theta. \quad (\textit{Fig. 42 (iv)}, \text{ quadratic for a given } \theta, 0 \text{ inside.})$$

Examples. (i) Find the area of the three parts into which the circle $x^2 + y^2 = 8ax$ is divided by the parabola $y^2 = 4ax$.

In the first quadrant, the curves meet at $(4a, 4a)$ and therefore the area of each of the equal parts is $\int_0^{4a}(x_1 - x_2)dy$ where $x_1 = 4a - \sqrt{(16a^2 - y^2)}$, and $x_2 = y^2/4a$. This is easily shown to be $4\pi a^2 - \frac{32}{3}a^2$. The third part is $8\pi a^2 + \frac{64}{3}a^2$.

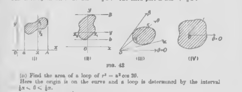


FIG. 42

(ii) Find the area of a loop of $r^2 = a^2 \cos 2\theta$.

Here the origin is on the curve and a loop is determined by the interval $-\frac{1}{4}\pi < \theta < \frac{1}{4}\pi$.

Thus the area $\int_0^{\frac{1}{4}\pi} a^2 \cos 2\theta\, d\theta = \frac{1}{2}a^2$.

9.604. *Areas of Curved Surfaces.* (i) For the surface given by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ where

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

the area S determined by a region Ω in the $u - v$ plane is given by

$$S = \iint_{\Omega} \sqrt{(EG - F^2)}\,du\,dv.$$

Here $E = \Sigma x_u^2$, $F = \Sigma x_u x_v$, $G = \Sigma x_v^2$, $EG - F^2 = \Sigma\left\{\frac{\partial(y, z)}{\partial(u, v)}\right\}^2$.

For *orthogonal* co-ordinates $F = 0$, ds^2 is of the form $h_1^2\,du^2 + h_2^2\,dv^2$ and $S = \iint_{\Omega} h_1 h_2\,du\,dv$.

(ii) For the surface given by $z = z(x, y)$, the area S determined by a region Ω in the $x - y$ plane is given by

$$S = \iint_{\Omega} (1 + p^2 + q^2)^{\frac{1}{2}}\,dx\,dy$$

where $p = z_x$, $q = z_y$, and z is single-valued.

(iii) For a branch of the surface given by $F(x, y, z) = 0$

$$S = \iint_{\Omega} \frac{(F_x^2 + F_y^2 + F_z^2)^{\frac{1}{2}}}{|F_z|}\,dx\,dy \quad (F_z \neq 0)$$

since $F_x + pF_z = 0 = F_y + qF_z$.

(iv) For the surface obtained by rotating an arc PQ of the curve $y=f(x)$ about OX, $S=2\pi\int_P^Q f(x)ds$; and for the surface obtained by rotation about OY, $S=2\pi\int_P^Q x\,ds$ where ds may be replaced by $\sqrt{\{1+(f'(x))^2\}}dx$.

Examples. (i) The area obtained by rotating about OY the arc of the catenary $y=c\cosh\frac{x}{c}$ from $(0, c)$ to (x, y) is

$$2\pi\int_0^s x\,ds = 2\pi\int_0^x x\cosh\frac{x}{c}\,dx = 2\pi c\left(c + x\sinh\frac{x}{c} - c\cosh\frac{x}{c}\right)$$
$$= 2\pi(c^2 + xs - cy).$$

(ii) Find the portion of the surface $az = xy$ intercepted by the cylinder $x^2 + y^2 = b^2$.

Here $ap = y$, $aq = x$, $S = \frac{1}{a}\iint\sqrt{(a^2+x^2+y^2)}\,dx\,dy$ over the area of the circle $x^2+y^2=b^2$. Changing to polar co-ordinates, $x = r\cos\theta$, $y = r\sin\theta$, we find $S=\frac{1}{a}\iint\sqrt{(a^2+r^2)}r\,dr\,d\theta = \frac{2\pi}{3a}\{(a^2+b^2)^{\frac{3}{2}} - a^3\}$.

9.605. Volumes. (i) The volume V cut from the cylinder of cross-section $\Omega(x, y)$ whose generators are parallel to OZ between $z=0$ and $z=f(x, y)$ (single-valued, >0) is given by

$$V=\iint_\Omega f(x, y)dx\,dy.$$

(ii) The volume V determined by a closed surface is given by

$$V=\iiint_\Gamma dx\,dy\,dz = \iiint_{\Gamma_1}\frac{\partial(x, y, z)}{\partial(u, v, w)}du\,dv\,dw$$

when the variables are changed by means of the equations $x=x(u, v, w)$, $y=y(u, v, w)$, $z=z(u, v, w)$, and V_1 is the volume in the u, v, w space that corresponds to V in the x, y, z space.

When the u, v, w co-ordinates are orthogonal, the value of ds^2 is of the form $h_1^2\,du^2 + h_2^2\,dv^2 + h_3^2\,dw^2$ and $V=\iiint_{\Gamma_1}h_1h_2h_3\,du\,dv\,dw$. In particular, for *spherical polar* co-ordinates $V=\iiint_{\Gamma_1}r^2\sin\theta\,dr\,d\theta\,d\phi$ and for *cylindrical* co-ordinates $V=\iiint_{\Gamma_1}\rho\,d\rho\,d\phi\,dz$.

(iii) Let Ω be an area in the x, y plane for which $y>0$, and let this area be rotated about OX through an angle 2π forming a volume of revolution.

The element of length is given by $ds^2 = dx^2 + dy^2 + y^2\,d\phi^2$ where ϕ measures the angle of rotation.

Therefore V_1, the volume of revolution is

$$\iiint y\,dx\,dy\,d\phi = 2\pi\iint_\Omega y\,dx\,dy.$$

This may be expressed as any of the line integrals

$$V_1 = -\pi \oint_C y^2\,dx = 2\pi \oint_C xy\,dy = 2\pi \oint_C \frac{pxy\,dy + qy^2\,dx}{p - 2q} \quad (p \neq 2q)$$

$$= \frac{2\pi}{3} \oint_C y(x\,dy - y\,dx) = \frac{2\pi}{3} \oint_C r^3 \sin\theta\,d\theta. \quad (C \text{ being the boundary.})$$

Similarly if $x > 0$ in Ω, the volume V_2 obtained by a revolution round OY is given by

$$V_2 = 2\pi \iint_\Omega x\,dx\,dy = \pi \oint_C x^2\,dy = -2\pi \oint_C xy\,dx$$

$$= 2\pi \oint_C \frac{px^2\,dy + qxy\,dx}{2p - q} \quad (2p \neq q)$$

$$= \frac{2\pi}{3} \oint_C x(x\,dy - y\,dx) = \frac{2\pi}{3} \oint_C r^3 \cos\theta\,d\theta$$

(iv) From (iii) we deduce when the boundary is quadratic in the appropriate direction that

$$V_1 = \pi \int_a^A (y_1^2 - y_2^2)dx = 2\pi \int_b^B (x_1 - x_2)y\,dy = \tfrac{2}{3}\pi \int_\alpha^\beta (r_1^3 - r_2^3) \sin\theta\,d\theta$$

$$V_2 = \pi \int_b^B (x_1^2 - x_2^2)dy = 2\pi \int_a^A (y_1 - y_2)x\,dx = \tfrac{2}{3}\pi \int_\alpha^\beta (r_1^3 - r_2^3) \cos\theta\,d\theta$$

in the notation of *Fig.* 42 (*i*), (*ii*), (*iii*).

In particular, take the arc of the curve $y = f(x)$ (for which $x > 0$, $y \geq 0$) from P to Q, and suppose for simplicity that $f'(x)$ is of constant sign from P to Q. The volume traced out when this arc makes one revolution

$$\left.\begin{aligned} &(a) \text{ about } OX \text{ is } \pi\int_P^Q \{f(x)\}^2\,dx \\ &(b) \text{ about } OY \text{ is } \pi\int_P^Q x^2 |f'(x)|dx \end{aligned}\right\} \text{where } x_Q > x_P$$

and if the arc is given in polar co-ordinates by the equation $r = f(\theta)$ (single-valued and ≥ 0), the values of these volumes are respectively
(*a*) $\frac{2}{3}\pi \int_P^Q \{f(\theta)\}^3 \sin\theta\,d\theta$. (*b*) $\frac{2}{3}\pi \int_P^Q \{f(\theta)\}^3 \cos\theta\,d\theta$.

(v) If the boundary of the closed surface is quadratic in the direction OZ and Ω is the projection of the volume on $z = 0$, $V = \iint_\Omega (z_1 - z_2)dx\,dy$ in the usual notation, and the boundary of Ω is the curve $z_1 - z_2 = 0$.

(vi) If the area formed by the section of a volume for which z is fixed is $\Omega(z)$, the volume V is given by

$$V = \int_c^C \Omega(z)dz$$

where c, C are the lower and upper bounds of z in V.

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Examples. (i) Find the volume determined by $0 < z < -c \log (x^2/a^2 + y^2/b^2)$ where $z > 0$.

The cross-section for a given z is an ellipse of axes $ae^{-z/c}$, $be^{-z/c}$.

The volume is therefore $\int_0^\infty \pi abe^{-2z/c}\,dz = \frac{1}{2}\pi abc$.

(ii) Find the volume obtained by a revolution of the curve $r = a + b\cos\theta$ $(a > b)$ about OX.

$$V = \frac{2\pi}{3}\int_0^\pi (a + b\cos\theta)^3 \sin\theta\, d\theta = \frac{4}{3}\pi a(a^2 + b^2)$$

(iii) A hole is bored through the solid $x^2 + y^2 = az$ parallel to OY, and the cross-section of the solid is bounded by $z = b$, $x^2 = az\cos^2\alpha$ $(a, b > 0)$. Find the volume removed. The projection on $z = 0$ of the volume removed is bounded by $x = \pm\sqrt{ab}\cos\alpha$, and part of the circle $x^2 + y^2 = ab$ between $\theta = \alpha$, $\theta = \pi - \alpha$ and between $\theta = \pi + \alpha$, $\theta = 2\pi - \alpha$ (r, θ being polar co-ordinates of the x, y plane).

Thus $$V = 4\iint_{A_1}(b - z_1)dx\,dy + 4\iint_{A_2}(b - z_2)dx\,dy$$

where $z_1 = \dfrac{1}{a}(x^2 + y^2)$, $z_2 = \dfrac{x^2}{a\cos^2\alpha}$; A_1 is the area determined by $r = 0$ to $r = \sqrt{ab}$, $\theta = \alpha$ to $\pi/2$. A_2 is the triangle bounded by $x = \sqrt{ab}\cos\alpha$, $y = x\tan\alpha$, $y = 0$.

On evaluation V will be found to be $ab^2\left(\dfrac{\pi}{2} - \alpha + \sin\alpha\cos\alpha\right)$.

(iv) Determine the volume in the octant $(x +, y +, z +)$ bounded by $xyz = c^3$, $x^2y = a_1z$, $x^2y = a_2z$, $xy^2 = b_1z$, $xy^2 = b_2z$, where $a_1 > a_2$, $b_1 > b_2$.

Take $u = xyz$, $v = x^2y/z$, $w = xy^2/z$, then

$$\frac{1}{uvw}\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{5}{xyz}, \text{ i.e. } \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{5vw}.$$

The required volume is $\frac{1}{5}\lim\limits_{\epsilon \to 0}\int_\epsilon^{c^3} du\int_{a_2}^{a_1}\frac{dv}{v}\int_{b_2}^{b_1}\frac{dw}{w} = \frac{1}{5}c^3 \log\left(\frac{a_1}{a_2}\right)\log\left(\frac{b_1}{b_2}\right)$.

9.61. Line, Surface and Volume Integrals. If n points (x_r, y_r, z_r) are given and a function $\phi(x, y, z)$ we can call $\sum\limits_1^n \phi(x_r, y_r, z_r)$ the *sum-function* of ϕ for these n points. The mean value of ϕ for the n points is $\dfrac{1}{n}\sum\limits_1^n \phi_r$. If we suppose that m_r points coincide at (x_r, y_r, z_r) the corresponding sum-function is $\sum\limits_1^s m_r\phi(x_r, y_r, z_r)$ where the number of points n is $\sum\limits_1^s m_r$ and the mean value is $\{\sum\limits_1^s m_r\phi(x_r, y_r, z_r)\}/\sum\limits_1^s m_r$. A real extension may then be made of the meaning of the sum-function by supposing that the numbers m_r may be *any* real numbers positive or negative. The number m_r may then be appropriately called the *weight* associated with the point (x_r, y_r, z_r). A further extension may now be made to continuous distributions of points by using the properties of integrals. For example, let a volume V be divided up into a number of smaller regions of volumes v_r and let the circumscribing cube of v_r be denoted by λ_r. If we make the natural assumption that the mean value of ϕ for the set

of points v_r is $\phi(x_r, y_r, z_r)$ where (x_r', y_r', z_r') is *some* point of λ_r, then the sum-function for V is $\Sigma\phi(x_r', y_r', z_r')v_r$; and since this sum tends to the limit $\iiint_V \phi(x, y, z)dx\,dy\,dz$ (when this exists) when the edge of every λ_r tends to zero and when (x_r', y_r', z_r') is *any* point of λ_r, the triple integral provides a natural definition of the sum-function for a *continuous* distribution V. The mean value of ϕ throughout V is then

$$\left(\frac{1}{V}\right)\iiint_V \phi(x, y, z)dx\,dy\,dz.$$

Similarly the sum-function for a *surface* distribution of area S is given by $\iint_S \phi(x, y, z)dS$ and its mean value is the quotient of this integral by S; finally, the sum-function for a linear distribution of length s is given by $\int_s \phi(x, y, z)ds$ and its mean value is the quotient of the integral by s.

9.62. Mass and Density. For a mass M occupying a volume V, the *mean density* is defined to be M/V. If a small cube of side c and centre $P(x, y, z)$ is taken, the mean density of the mass m occupying this cube is m/c^3, and if this tends to a limit $\rho(x, y, z)$ when c tends to zero, $\rho(x, y, z)$ is called the *density* at P. We deduce therefore that if $\rho(x, y, z)$ is the density at P of a given mass occupying a volume V, then

$$M = \iiint_V \rho(x, y, z)dx\,dy\,dz.$$

Example. The density at P, a point of a solid sphere of radius a and centre O, is given to be

$$\rho_0\{1 + \epsilon\cos\theta + \tfrac{1}{2}\epsilon^2(3\cos^2\theta - 1)\}$$

where θ is the angle OP makes with a fixed radius OC, and ρ_0, ϵ are constants. Find the mean density.

Use spherical polar co-ordinates r, θ, ϕ.

Then total mass $= \iiint\rho_0\{1 + \epsilon\cos\theta + \tfrac{1}{2}\epsilon^2(3\cos^2\theta - 1)\}r^2\sin\theta\,dr\,d\theta\,d\phi$

$$= -\tfrac{2}{3}\pi a^3\rho_0\{\cos\theta + \tfrac{1}{2}\epsilon\cos^2\theta + \tfrac{1}{2}\epsilon^2(\cos^3\theta - \cos\theta)\}_0^\pi$$
$$= \tfrac{4}{3}\pi a^3\rho_0.$$

Therefore ρ_0 is the mean density.

9.63. The nth Powers of Distances. A problem of frequent occurrence is the determination of the sum of the nth powers of the distances of a set of points from (i) a given *point*, (ii) a given *line*, (iii) a given *plane*. For these three cases, the related functions ϕ are respectively

(i) $\{(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2\}^{n/2}$, where (α, β, γ) is the given point.

(ii) $[\{M(z - \gamma) - N(y - \beta)\}^2 + \{N(x - \alpha) - L(z - \gamma)\}^2$
$\qquad + \{L(y - \beta) - M(x - \alpha)\}^2]^{n/2}$

if the given *line* has direction cosines L, M, N and passes through (α, β, γ), i.e. when the line has the equation $(x - \alpha)/L = (y - \beta)/M = (z - \gamma)/N$, $(L^2 + M^2 + N^2 = 1)$.

(iii) $(Lx + My + Nz - P)^n$ if the normal to the given *plane* has direction cosines L, M, N and the perpendicular from the origin to the plane is of absolute magnitude $P\,(> 0)$. When n is odd, the expression has opposite signs on opposite sides of the plane. The *positive* side of the plane is the one that contains (Lr, Mr, Nr) where r is large and positive.

Examples. (i) Find the mean fourth powers of the distances of the points of a solid sphere (radius a) from a point P distant c from the centre of the sphere.

Take the equation of the spherical surface as $x^2 + y^2 + z^2 = a^2$ and P to be the point $(c, 0, 0)$.

Then the sum of the fourth powers is $I = \iiint_V \{(x - c)^2 + y^2 + z^2\}^2\,dx\,dy\,dz$.

By symmetry, the contribution of odd terms is zero, and therefore

$I = \iiint_V \{(x^2 + y^2 + z^2)^2 + 4c^2x^2 + c^4 + 2c^2(x^2 + y^2 + z^2)\}dx\,dy\,dz.$

Also $\iiint_V x^2\,dx\,dy\,dz = \frac{1}{3}\iiint_V(x^2 + y^2 + z^2)dx\,dy\,dz$, by symmetry.

Changing to spherical polar co-ordinates, we find

$I = \iiint_V(r^4 + \frac{10}{3}c^2r^2 + c^4)r^2\sin\theta\,dr\,d\theta\,d\phi = \frac{4}{3}\pi a^3(c^4 + 2a^2c^2 + \frac{3}{7}a^4).$

The mean fourth power is therefore $c^4 + 2a^2c^2 + \frac{3}{7}a^4$.

(ii) Find the mean squared distance of the points of the whole surface of a closed cylinder (height h, radius a) from the centre of one of its ends E.

For the end E, the sum is $\int_0^a 2\pi r^3\,dr = \frac{1}{2}\pi a^4$. For the other end, the sum is

$\int_0^a 2\pi r(r^2 + h^2)dr = \pi(\frac{1}{2}a^4 + a^2h^2)$. For the curved surface, the sum is

$$2\pi a\int_0^h (x^2 + a^2)dx = 2\pi ah(a^2 + \tfrac{1}{3}h^2).$$

The total surface is $2\pi a(a + h)$ and therefore the mean squared distance is

$$\tfrac{1}{2}a(a + h) + \tfrac{1}{3}h^3/(a + h).$$

9.631. Mean Distance from a Plane. Mean Centres. Take n points at (x_s, y_s, z_s) ($s = 1$ to n), with weights m_s. The mean distance of these points from the plane $Lx + My + Nz = P$ is

$$\frac{1}{M}\{\sum_1^n m_s(Lx_s + My_s + Nz_s - P)\}$$

where $M = \sum_1^n m_s$, i.e. is the distance of the point $\bar{x}, \bar{y}, \bar{z}$ from the plane, where

$$\bar{x} = \frac{\Sigma m_s x_s}{\Sigma m_s},\ \bar{y} = \frac{\Sigma m_s y_s}{\Sigma m_s},\ \bar{z} = \frac{\Sigma m_s z_s}{\Sigma m_s}$$

The co-ordinates of this point G are independent of L, M, N, P and therefore G depends only on the relative positions of (x_s, y_s, z_s) and not on the framework of reference. The point G is called the *mean centre* (or *centroid*).

For a continuous distribution V in three dimensions

$$V\bar{x} = \iiint_V x\,dx\,dy\,dz,\ V\bar{y} = \iiint_V y\,dx\,dy\,dz,\ V\bar{z} = \iiint_V z\,dx\,dy\,dz$$

and there are similar formulae for *areal* and *linear* distributions.

Examples. (i) Find the mean centre of the arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, specified by the interval $0 < \theta < 2\pi$. For the cycloid

$$ds = 2a \sin \tfrac{1}{2}\theta \, d\theta.$$

Therefore $\bar{y}\int_0^{2\pi} 2a \sin \tfrac{1}{2}\theta \, d\theta = \int_0^{2\pi} a(1 - \cos\theta)2a \sin \tfrac{1}{2}\theta \, d\theta,$

i.e. $$8a\bar{y}\int_0^{\pi/2} \sin\phi \, d\phi = 16a^2\int_0^{\pi/2} \sin^3\phi \, d\phi, \ (\phi = \tfrac{1}{2}\theta); \ \text{or } \bar{y} = \tfrac{4}{3}a$$

By symmetry $\bar{x} = a\pi$.

(ii) Find the mean centre of the area bounded by a loop of the curve

$$x^4 + y^4 = 4ax^2y \ (a > 0).$$

Take $u = x^3/4ay$, $v = y^3/4ax^2$. Then the area A_1 in the u–v plane is the limit of that determined by $0 < \varepsilon_1 < u$, $0 < \varepsilon_2 < v$, $u + v < 1$, when $\varepsilon_1, \varepsilon_2 \to 0$. Thus if A denotes the given area (when $\varepsilon_1, \varepsilon_2 \to 0$) we have

$$A\bar{x} = \iint_A x \, dx \, dy = 16a^3\iint_{A_1} u \, du \, dv = \tfrac{8}{3}a^3$$

$$A\bar{y} = \iint_A y \, dx \, dy = 16a^3\iint_{A_1} u^{3/4}v^{1/4} \, du \, dv = \tfrac{64}{5}a^3\int_0^1 u^{3/4}(1 - u)^{5/4} \, du.$$

In *Chapter XII*, it is shown that $\int_0^1 u^{p-1}(1 - u)^{q-1} \, du = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p + q)} (p, q > 0)$, where $\Gamma(x)$ is the Gamma Function, and by using the properties of the Gamma Function, we easily find that $A\bar{y} = \dfrac{\pi\sqrt{2}}{2}a^3$. Also

$$A = 4a^2\iint_{A_1} u^{1/4}v^{-1/4} \, du \, dv = \frac{16a^2}{3}\frac{\Gamma(5/4)\,\Gamma(7/4)}{\Gamma(3)} = \frac{\pi\sqrt{2}}{2}a^2$$

i.e. $\bar{x} = \dfrac{8\sqrt{2}a}{3\pi}$ and $\bar{y} = a$.

(iii) The part of the catenary $y = c \cosh \dfrac{x}{c}$ between $x = 0$ and $x = x_1 (> 0)$ is rotated about OY through 2 right angles. Find the mean centre of the *surface* generated.

Here $ds = \cosh \dfrac{x}{c} dx$, $\bar{y}\int_0^{x_1} 2\pi x \, ds = 2\pi\int_0^{x_1} xy \, ds$, $\bar{x} = 0$, on evaluation, it will be found that $4(x_1s_1 - cy_1 + c^2)\bar{y} = cx_1^2 + 2x_1y_1s_1 - cs_1^2$, where $s_1 = c \sinh x_1/c$.

(iv) A solid half-ring is formed by rotating through 180° a circle of radius a about a line in its plane distant c from the centre ($c > a$). Find the distance of its mean centre from the plane passing through the circular ends. Take the line as OZ and the initial position of the circle as $(x - c)^2 + z^2 = a^2$, the direction of rotation being from OX to OY. Using cylindrical co-ordinates, we have

$$\bar{y}\iiint \rho \, d\rho \, d\phi \, dz = \iiint \rho^2 \sin\phi \, d\rho \, d\phi \, dz$$

the surface being $(\rho - c)^2 + z^2 = a^2$ $(0 < \phi < \pi)$.

Thus $\bar{y} = \dfrac{2\iint (c + R\cos\theta)^2 R \, dR \, d\theta}{\pi\iint (c + R\cos\theta) R \, dR \, d\theta}$ where $\rho = c + R\cos\theta$, $z = R\sin\theta$, and $0 < \theta < 2\pi$, $0 < R < a$. Thus $\bar{y} = \dfrac{(a^2 + 4c^2)}{2\pi c}$.

9.632. Pappus's (or Guldin's) Theorems. These theorems determine the relationship of a plane area (or arc) and its mean centre with the

volume (or surface) obtained by rotating that area (or arc) about a line in its plane which does not cross it.

Thus let y be the distance of a point on a plane arc s from a line in its plane (not crossing the arc). The surface S obtained by rotating the arc about the line through an angle θ is given by $S = \int_A \theta y\, ds = s\theta\bar{y}$ where $\bar{y}$ is the distance of the mean centre from the line. $\theta\bar{y}$ is the *path* of the mean centre. Again, if y is the distance of a point of a plane area A from a line in its plane (not crossing the area), then V, the volume generated by rotating the area about the line through an angle θ is given by $V = \iint_A \theta y\, dx\, dy = A\theta\bar{y}$.

Thus: If a plane area (arc), of measure A (s), is rotated through an angle θ about a line in its plane not crossing the area (arc), then V (S), the volume (surface) generated is the product of A (s) and the length of the path of the mean centre.

These theorems are sometimes useful for calculating (*a*) the position of the mean centre, (*b*) surfaces and volumes of revolution.

Examples. (i) On three sides of a square of side a, equilateral triangles are constructed. Find the volume and the surface generated when the figure is rotated about the fourth side of the square. In this case the mean centres of the various parts of the area and perimeter are obvious.

Thus $$V = 2\pi\left\{a^2\frac{a}{2} + 2\frac{a^2\sqrt{3}}{4}\frac{a}{2} + \frac{a^2\sqrt{3}}{4}\left(a + \frac{a\sqrt{3}}{6}\right)\right\} = \tfrac{1}{4}\pi a^3(5 + 4\sqrt{3})$$

and $$S = 2.2\pi\left\{a\frac{a}{4} + a\frac{3a}{4} + a\left(a + \frac{a\sqrt{3}}{4}\right)\right\} = \pi a^2(8 + \sqrt{3})$$

(ii) $BCEF$ is a rectangle in which $BC = b$, $CE = a$. CB is produced to A and BC to D, so that $AB = CD = a$. The points A, F are connected by the quadrant of a circle of centre B, and the points D, E by the quadrant of a circle of centre C. Find the distance of the mean centre of the whole area $ABCDEFA$ from $ABCD$ and also the mean centre of the whole perimeter of this area.

Rotate the figure about $ABCD$ through 2π. In this case, the volume and the surface have obvious values,

$$V = \tfrac{4}{3}\pi a^3 + \pi a^2 b = 2\pi\bar{y}_1(\tfrac{1}{2}\pi a^2 + ab), \quad \bar{y}_1 = \frac{a(4a + 3b)}{3(\pi a + 2b)},$$

$$S = 4\pi a^2 + 2\pi ab = 2\pi\bar{y}_2(2a + 2b + \pi a), \quad \bar{y}_2 = \frac{a(2a + b)}{(\pi + 2)a + 2b}.$$

9.633. *Squared Distances from a Line. Moments of Inertia.* The *moment of inertia* I of a system of masses m_s ($s = 1$ to n) about a given line is defined to be $\sum_1^n m_s D_s^2$ where D_s is the distance of m_s from the line.

We therefore define the moment of inertia of a continuous mass M occupying a volume V by the triple integral $I = \iiint_V \rho D^2\, dx\, dy\, dz$ where ρ is the density of the mass at (x, y, z), and D the distance of (x, y, z) from the line; and M is given by the integral $\iiint_V \rho\, dx\, dy\, dz$.

The mean value of ρD^2 for the distribution is I/M and its value k^2 is called the *radius of gyration* about the line.

If, as is often the case, ρ is constant, then $M = \rho V$ and

$$Vk^2 = \iiint_V D^2\,dx\,dy\,dz$$

so that k^2 is the mean squared distance from the line of the points of V.

If we take $\rho = 1$, then $I = Mk^2 = Vk^2 = \iiint_V D^2\,dx\,dy\,dz$.

Let the origin be taken at a point of the line and let the direction cosines of the line be l, m, n, then D is the modulus of

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$$

in the usual notation or $D^2 = (ny - mz)^2 + (lz - nx)^2 + (mx - ly)^2$, i.e. $I = Al^2 + Bm^2 + Cn^2 - 2Hlm - 2Fmn - 2Gnl$, where

$$A = \iiint_V (y^2 + z^2)\,dx\,dy\,dz\,; \quad B = \iiint_V (z^2 + x^2)\,dx\,dy\,dz\,,$$

$$C = \iiint_V (x^2 + y^2)\,dx\,dy\,dz\,;$$

$$F = \iiint_V yz\,dx\,dy\,dz\,; \quad G = \iiint_V zx\,dx\,dy\,dz\,; \quad H = \iiint_V xy\,dx\,dy\,dz.$$

Thus A, B, C are the moments of inertia about OX, OY, OZ respectively. The quantities F, G, H are called the *products of inertia* about OX, OY, OZ.

9.634. The Theorem of Parallel Axes. Principal Axes. The moment of inertia about a line parallel to OZ through $(x_0, y_0, 0)$ is

$$C_1 = \iiint_V \{(x - x_0)^2 + (y - y_0)^2\}\,dx\,dy\,dz$$

$$= C - 2y_0\iiint_V y\,dx\,dy\,dz - 2x_0\iiint_V x\,dx\,dy\,dz + M(x_0^2 + y_0^2)$$

$$= C + M(x_0^2 + y_0^2) \text{ if the origin is the mean centre } G.$$

Since any given line may be chosen as the z-axis, we deduce that if I is the moment of inertia about any line and I_G is the moment of inertia about a parallel line through G (the mean centre), then $I = I_G + Mh^2$, where h is the perpendicular distance between the lines. This result is called the *Theorem of Parallel Axes.*

Similarly, if A, B, C, F, G, H are the moments and products of inertia for rectangular axes through G, the corresponding quantities for parallel axes through *any* point (x_0, y_0, z_0) are given by

$$A_1 = A + M(y_0^2 + z_0^2)\,; \quad B_1 = B + M(z_0^2 + x_0^2)\,; \quad C_1 = C + M(x_0^2 + y_0^2)\,;$$

$$F_1 = F + My_0z_0\,; \qquad G_1 = G + Mz_0x_0\,. \qquad H_1 = H + Mx_0y_0.$$

It is sufficient therefore to find the values of A, B, C, F, G, H for axes through the mean centre.

A further simplification can be made by choosing the axes of reference in such a way that $F = G = H = 0$ and in such a case, the axes

are called *principal axes* and A, B, C the *principal* moments of inertia. For a discussion of this question, reference may be made to works on Rigid Dynamics, but it is worth while noting that if α, β are two planes of symmetry at right angles intersecting in l, the principal axes at any point P (i.e. such that at P, $F = G = H = 0$) in l are l and the two lines drawn from P perpendicular to l, one in each plane. In this case l obviously contains G.

Example. Find I for a rectangular parallelopiped of edges a, b, c about the line through the centre of a face whose edges are a, b and a corner of the opposite face.

Take the centre of the face as O and the axes OX, OY, OZ parallel to the edges a, b, c respectively. These are obviously principal axes (by symmetry). The values of A, B, C for these axes are

$$A = \iiint(y^2 + z^2)dx\,dy\,dz = M\left(\frac{b^2}{12} + \frac{c^2}{3}\right),\quad B = M\left(\frac{a^2}{12} + \frac{c^2}{3}\right),\quad C = M\left(\frac{a^2}{12} + \frac{b^2}{12}\right).$$

The direction cosines of the line are $(a, b, 2c)/\sqrt{(a^2 + b^2 + 4c^2)}$.

Thus
$$I = M\frac{a^2b^2 + 4a^2c^2 + 4b^2c^2}{6(a^2 + b^2 + 4c^2)}.$$

9·635. *Moments of Inertia for a Plane Lamina.* If the solid is a plane lamina of area A, small thickness h and density ρ, the problem of determining moments of inertia reduces to that of finding the sum of the squares of the distances of the points of an *areal* distribution A, of uniform *surface* density $\sigma = \rho h$. If σ is taken to be unity, the mass and the area are represented by the same number A.

Take O in the plane of A and the z-axis perpendicular to A. Then I, the moment of inertia about an axis through O with direction cosines (l, m, n), is given by

$$I = \iint_A \{n^2y^2 + n^2x^2 + (mx - ly)^2\}dx\,dy$$
$$= \alpha l^2 + \beta m^2 + \gamma n^2 - 2klm$$

where

$$\alpha = \iint_A y^2\,dx\,dy,\quad \beta = \iint_A x^2\,dx\,dy,\quad \gamma = \alpha + \beta,\quad k = \iint_A xy\,dx\,dy.$$

If the axes OX, OY are chosen so that $k = 0$, they are *principal* axes. For example if OX or OY is a line of symmetry, k is obviously zero and the axes are principal. In such a case

$$I = \alpha l^2 + \beta m^2 + (\alpha + \beta)n^2$$

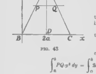


FIG. 43

Examples. (i) Let ABC be an isosceles triangle, in which $AB = AC$ and $BC = 2a$. Find in terms of a and h (the altitude)

(i) the moments of inertia about BC and the bisector of angle A.

(ii) the principal axes at B. (*Fig. 43*)

(i) The moment of inertia about BC is equal to

$$\int_0^h PQ\,y^2\,dy = \int_0^h 2a(h - y)y^2\,dy/h = \tfrac{1}{6}ah^3 = Mh^2/6$$

The moment of inertia about AD, the bisector, is

$$2\int_0^h \tfrac{1}{3}\left(\frac{PQ}{2}\right)^3 dy = \frac{2a^3}{3h^3}\int_0^h (h-y)^3\,dy = M\frac{a^2}{6}.$$

(ii) For the axis through B parallel to AD

$$\beta = \tfrac{1}{6}Ma^2 + Ma^2 = \tfrac{7}{6}Ma^2,$$

also $\alpha = Ma^2/6$, and $k = \iint xy\,dx\,dy$.

In the integral for k, put $\xi = x - a$. Then

$$k = \iint(\xi + a)\,d\xi\,dy = aA\bar{y} = M\frac{ah}{3}.$$

If $x\sin\theta - y\cos\theta = 0$ is a principal axis through B

$$\iint(x\sin\theta - y\cos\theta)(x\cos\theta + y\sin\theta)\,dx\,dy = 0$$

i.e. $\sin\theta\cos\theta(\tfrac{7}{6}a^2 - \tfrac{1}{2}h^2) = (\cos^2\theta - \sin^2\theta)\dfrac{ah}{3}$ or $\tan 2\theta = \dfrac{4ah}{7a^2 - 3h^2}$.

(ii) Find the moment of inertia about OZ of the solid ellipsoid determined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$I = \iiint_{V_1}(x^2 + y^2)\,dx\,dy\,dz = \iiint_{V_1}(a^2\xi^2 + b^2\eta^2)abc\,d\xi\,d\eta\,d\zeta$, where $x = a\xi$, $y = b\eta$, $z = c\zeta$ and V_1 is the sphere given by $\xi^2 + \eta^2 + \zeta^2 = 1$. Using symmetry, we find $I = \frac{1}{3}abc(a^2 + b^2)\iiint_{V_2} r^4\sin\theta\,dr\,d\theta\,d\phi$ where the variables are changed to spherical polars from ξ, η, ζ and V_2 is the unit sphere $r = 1$; i.e.

$$I = \frac{4\pi abc}{15}(a^2 + b^2) = M\frac{a^2 + b^2}{5}.$$

(iii) Find in terms of the moments and products of inertia for the mean centre (1) the sum of the squares of the distances of the points of a volume V from the point (x_0, y_0, z_0), (2) the sum of the squares of the distances of the points from the plane $lx + my + nz = p$ $(l^2 + m^2 + n^2 = 1)$.

(1) Sum $= \iiint_V\{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\}\,dx\,dy\,dz$

$= \frac{1}{2}(A + B + C) + V(x_0^2 + y_0^2 + z_0^2)$, since O is the mean centre.

(2) Sum $= \iiint_V(lx + my + nz - p)^2\,dx\,dy\,dz$

$$= \tfrac{1}{2}(B + C - A)l^2 + \tfrac{1}{2}(C + A - B)m^2 + \tfrac{1}{2}(A + B - C)n^2 + 2Fmn + 2Gnl + 2Hlm + p^2V.$$

(iv) Find the moment of inertia of a uniform thin hollow sphere of mass M and radius a about a tangent. The moment of inertia about a diameter is $\iint(x^2 + y^2)\,dS$ if we take the equation of the sphere as $x^2 + y^2 + z^2 = a^2$. By symmetry this is

$$\tfrac{2}{3}\iint(x^2 + y^2 + z^2)\,dS = \tfrac{2}{3}Ma^2.$$

About a tangent, the moment of inertia is therefore $\frac{5}{3}Ma^2$.

(v) Find the moment of inertia of a uniform solid circular cone of base radius a and height h, about a line that meets the axis of the cone at distance b from the vertex inclined to the axis at an angle θ. (*Fig. 44.*)

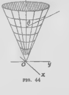


FIG. 44

Let OZ be the axis of the cone and O the vertex. Take the equation of the line to be

$$x\cos\theta - (z - b)\sin\theta = 0.$$

Then $\qquad I = \iiint_V (x\cos\theta - (z-h)\sin\theta)^2 + y^2)dx\,dy\,dz.$

Now
$$\begin{aligned}\iiint x^2\,dx\,dy\,dz &= \iiint y^2\,dx\,dy\,dz\\ &= \tfrac{1}{2}\iiint (x^2+y^2)dx\,dy\,dz\\ &= \tfrac{1}{2}\iiint \rho^3\,d\rho\,d\phi\,dz\end{aligned}$$
in cylindrical co-ordinates.

This gives $\dfrac{\pi}{4}\displaystyle\int_0^h z^4\tan^4\alpha\,dz$, (where α is the semi-vertical-angle of cone)

$$= \frac{\pi}{20}h^5\tan^4\alpha = \frac{3}{20}Ma^2.$$

$$\iiint z^2\,dx\,dy\,dz = \pi\int_0^h z^4\tan^2\alpha\,dz = \tfrac{3}{5}Mh^2.$$

$$\iiint xz\,dx\,dy\,dz = 0 = \iiint x\,dx\,dy\,dz\,. \quad \iiint z\,dx\,dy\,dz = M\bar{z} = \tfrac{3}{4}Mh.$$

Therefore $\qquad \dfrac{I}{M} = \dfrac{3a^2}{20}(1+\cos^2\theta) + \dfrac{3h^2}{5}\sin^2\theta + b^2\sin^2\theta - \dfrac{3bh}{2}\sin^2\theta$

or $$I = M\left\{\frac{3a^2}{10}\cos^2\theta + \left(\frac{3a^2}{20} + \frac{3h^2}{5} + b^2 - \frac{3bh}{2}\right)\sin^2\theta\right\}.$$

In particular, the moment of inertia about

(i) an axis perpendicular to axis of cone passing through the vertex

$$(\theta = 90^\circ,\ b = 0) \text{ is } M\left(\frac{3a^2}{20} + \frac{3h^2}{5}\right)$$

(ii) a diameter of the base $(\theta = 90^\circ,\ b = h)$ is $M\left(\dfrac{3a^2}{20} + \dfrac{h^2}{10}\right)$

(iii) an axis perpendicular to axis of cone passing through the mean centre $(\theta = 90^\circ,\ b = \tfrac{3}{4}h)$ is $M\left(\dfrac{3a^2}{20} + \dfrac{3h^2}{80}\right)$

(iv) the axis of the cone $(\theta = 0)$ is $M\dfrac{3a^2}{10}$

(v) a generator $(\theta = \alpha,\ b = 0)$ is $3M\dfrac{a^2(a^2+6h^2)}{20(a^2+h^2)}$.

9.64. Fluid Pressure. Centre of Pressure. It is shown in the theory of hydrostatics that the normal thrust on one side of a plane area A immersed in a fluid is given by $p(x', y', z')A$ where p (an invariant) is a function of x, y, z, and (x', y', z') is some point of A. We deduce that the thrust $\mathbf{F}$ on one side of a surface S is given by the surface integral $\displaystyle\iint_S p dS\,\mathbf{N}$ where $\mathbf{N}$ is unit normal drawn to that side of S.

The components of the total thrust on S are therefore $\displaystyle\iint_S pl dS$, $\displaystyle\iint_S pm dS$, $\displaystyle\iint pn dS$, where l, m, n are the direction-cosines of the normal.

A simple case arises when the only external force acting on the fluid is gravity, the pressure in that case being proportional to the distance of the point from the free surface. Thus, if we neglect atmospheric pressure, the pressure of a liquid at a depth y is wy, where w is the weight of unit volume of the liquid.

9.641. The Total Thrust of a Liquid under Gravity on a Plane Lamina. Let a plane lamina be wholly or partly immersed in a given liquid. Take axes OX in the free surface and OY in the lamina at right angles to OX and downwards. (*Fig. 45.*) The total thrust $= \iint_{A_1} wy\,dx\,dy \cos\theta$ where A_1 is the area immersed and the lamina is inclined to the vertical at an angle θ ($\neq 90°$), i.e. the total thrust is equal to $A_1 w\bar{h}$ where $\bar{h}$ is the depth of the mean centre of A_1. If $\theta = 90°$, the total thrust is obviously $A_1 wh$ where h is the depth of the lamina (which is horizontal). In all cases, therefore, the total thrust is the product of the area immersed and the pressure at the mean centre of that area.

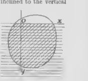


FIG. 45

9.642. The Centre of Pressure of a Plane Lamina. The *centre of pressure* is the point of the lamina through which the resultant thrust acts. It is obviously independent of the inclination of the lamina to the vertical provided this is not 90°. In finding centres of pressure it is sufficient to assume that the lamina is vertical.

(i) A simple case occurs when there is a vertical line of symmetry since the centre of pressure must obviously lie on this line. Also if y_C is the depth of the centre of pressure, we have

$\left\{\iint_A wy\,dx\,dy\right\} y_C = \iint_C wy^2\,dx\,dy$ in the notation of last paragraph, i.e. $y_C = k^2/\bar{y}$, where k is radius of gyration of the part of lamina immersed about the line in the surface and in the plane of the lamina.

Example. A semi-circular lamina, radius a, is completely immersed in a liquid with its bounding diameter horizontal, uppermost and at a depth c. Find its centre of pressure. The radius perpendicular to the bounding diameter is a line of symmetry. The value of k^2 for this diameter is $a^2/4$ and therefore k^2 for the line in the surface is $\frac{1}{4}a^2 - \left(\frac{4a}{3\pi}\right)^2 + \left(\frac{4a}{3\pi} + c\right)^2$. The depth of the centre of pressure is therefore

$$\frac{3\pi a^2 + 32ac + 12\pi c^2}{4(4a + 3\pi c)}.$$

(ii) More generally, take OX, OY in the immersed part of the lamina. (*Fig. 46.*) Then the line in the surface has an equation of the form $x\cos\theta + y\sin\theta + q = 0$. (Take $q > 0$.)

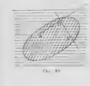


FIG. 46

$$Tx_C = \iint_{A_1} (x\cos\theta + y\sin\theta + q)x\,dx\,dy$$

$$Ty_C = \iint_{A_1} (x\cos\theta + y\sin\theta + q)y\,dx\,dy.$$

where $$T = \iint_{A_1} (x \cos\theta + y \sin\theta + q)\,dx\,dy$$

i.e. $$hx_C = q\bar{x} + k_1^2 \cos\theta + \lambda \sin\theta$$

and $$hy_C = q\bar{y} + \lambda \cos\theta + k_2^2 \sin\theta$$

where h is the depth of the mean centre of A_1, k_1, k_2 are the radii of gyration for OY, OX respectively, and $A_1\lambda$ is $\iint_{A_1} xy\,dx\,dy$, the product of inertia for these axes.

If $\lambda = 0$, i.e. if principal axes are taken at O (for example when OX or OY is a line of symmetry)

$$hx_C = q\bar{x} + k_1^2 \cos\theta; \quad hy_C = q\bar{y} + k_2^2 \sin\theta$$

If, in addition, O is the mean centre (at depth h), then $hx_C = k_1^2 \cos\theta$ and $hy_C = k_2^2 \sin\theta$.

Example. A vertical semi-circular lamina (radius a) is immersed completely in a liquid with the upper end of its bounding diameter in the surface. This diameter is inclined to the surface at an angle α. Taking OX downwards along the bounding diameter and OY along the radius perpendicular to OX, find the co-ordinates of the centre of pressure.

Here $k_1^2 = k_2^2 = \frac{1}{4}a^2$ and $\lambda = 0$ (by symmetry). $h = a \sin\alpha + \dfrac{4a}{3\pi} \cos\alpha$,

$q = a \sin\alpha$; $\bar{x} = 0$; $\bar{y} = \dfrac{4a}{3\pi}$. $a\left(\sin\alpha + \dfrac{4}{3\pi}\cos\alpha\right)x_C = \dfrac{a^2}{4}\sin\alpha$;

$$a\left(\sin\alpha + \frac{4}{3\pi}\cos\alpha\right)y_C = \frac{4a^2}{3\pi}\sin\alpha + \frac{a^2}{4}\cos\alpha$$

i.e. $$x_C = \frac{3\pi a \sin\alpha}{4(3\pi \sin\alpha + 4\cos\alpha)}, \quad y_C = \frac{a(3\pi\cos\alpha + 16\sin\alpha)}{4(3\pi\sin\alpha + 4\cos\alpha)}.$$

In particular $\alpha = 0$, $x_C = 0$, $y_C = \dfrac{3\pi a}{16}$; $\alpha = \dfrac{\pi}{2}$, $x_C = \dfrac{a}{4}$, $y_C = \dfrac{4a}{3\pi}$.

9.65. *Potential and Attractions.* The theory of *potential* and *attractions* provides other illustrations of the use of line, surface, and volume integrals; and whilst some of the problems provide useful exercises in the direct evaluation of these integrals, results may often be obtained more simply by the use of general theorems.

The attraction between two particles of masses m_1, m_2 may be measured by $m_1 m_2/r^2$ where r is the distance between the masses. The intensity at a point P due to a mass m_1 at Q_1 is defined to be the attraction on unit particle at P of the mass m_1 at Q_1 and is therefore given by $-m_1/r_1^2$ in the direction $\overrightarrow{Q_1P}$, i.e. the intensity is given by the vector $\nabla(m_1/r_1)$. The scalar function m_1/r_1 is called the *potential* at P due to the mass m_1 at Q_1.

Note. It is easily shown that m_1/r_1 is the work done by the attraction on unit particle in bringing it from ∞ to P along any path that does not pass through Q_1.

For a system of particles m_r at Q_r, the potential $V = \Sigma \dfrac{m_r}{r_r}$ and clearly the attraction at P is given by ∇V.

For a continuous distribution for which the density is given by $\rho(x', y', z')$, the potential is naturally defined to be $\iiint_D \rho \frac{dx'\,dy'\,dz'}{r'}$ where D is the domain for which ρ exists (ρ being zero elsewhere) and

$$r' = \{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{\frac{1}{2}}.$$

It is, of course, not immediately obvious that this triple integral exists even for a finite domain (or distribution) D, and when ρ is continuous in D.

The attraction (if it is properly defined by such an integral) is given by (V_x, V_y, V_z) where $V_x = \iiint \frac{\rho(x' - x)\,dx'\,dy'\,dz'}{r'^3}$ with similar expressions for V_y, V_z.

Two cases must be distinguished, (i) when $P(x, y, z)$ does not belong to D, and (ii) when P belongs to D.

(i) If P is not a point of D, $1/r'$ may be expanded as an infinite power series in x, y, z, uniformly convergent if ρ is bounded and D finite; so that if in particular ρ is continuous in D, V is finite and continuous and possesses derivatives of all orders; it is not difficult to show also that as (x, y, z) tends to infinity, V becomes infinite like M/R where M is the total mass of the distribution and R is the distance of (x, y, z) from the mean point of D.

Also since $\nabla^2\left(\frac{1}{r'}\right) = 0$, we deduce that $\nabla^2 V = 0$ for all points not belonging to D.

(ii) Let P belong to D and let a small sphere S of radius ε be taken whose centre is P. The contribution to V due to S is (in absolute value)

$$\iiint_S \frac{|\rho|\,dx'\,dy'\,dz'}{\varepsilon} < \mu \tfrac{4}{3}\pi\varepsilon^2, \text{ where } \mu = \max \rho \text{ in } S$$

i.e. this contribution tends to zero as $\varepsilon \to 0$; we therefore define V for an interior point to be $\lim_{\varepsilon \to 0} \iiint_{D-S} \frac{\rho\,dx'\,dy'\,dz'}{r'}$ since this limit exists.

Again, consider $I_1 = \iiint_S \frac{\rho(x' - x)\,dx'\,dy'\,dz'}{r'^3}$

Here $|I_1| < \mu \iiint_S \frac{2\varepsilon}{\varepsilon^3}\,dx'\,dy'\,dz' < \frac{8}{3}\mu\pi\varepsilon$ which tends to 0 with ε.

Thus the integral that defines V_x is given by

$$\lim_{\varepsilon \to 0} \iiint_{D-S} \frac{\rho(x' - x)\,dx'\,dy'\,dz'}{r'^3}$$

although we have not proved that this limit is actually the derivative of the limit that defines V. Assuming this to be true, we see that not only is V finite and continuous throughout D, but it possesses first derivatives that are finite and continuous. The integrals that define the second derivatives may also be proved to exist, although the contribu-

tions due to the small sphere no longer tend to zero in general, and the nature of these second derivatives (so far as continuity and differentiability are concerned) is directly related to the nature of the function ρ.

Notes. (i) For a formal justification of these results, see Goursat, *Cours d'Analyse, III*, § 535 *et foll.*, where it is shown that in the interior of D, V and its first derivatives are differentiable at all points, and V possesses continuous second derivatives. The proof given there of the continuity of the second derivatives within D assumes that the derivatives of ρ are bounded and integrable, although these conditions are not necessary.

(ii) The proof that V and its first derivatives are continuous applies also to a point on the boundary of D. The second derivatives are, however, discontinuous, in general, on the boundary, since ρ is, in general, discontinuous there.

9.651. Poisson's Theorem. We have seen that $\nabla^2 V = 0$ at an exterior point P. Poisson's Theorem gives the value of $\nabla^2 V$ at an interior point. In §§ *9.52*, *9.53*, we proved that (i) $\iint_S \frac{\partial}{\partial N}\left(\frac{1}{r}\right)dS = 0$ if S is a closed surface, $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, (x, y, z) a point on S; $\frac{\partial}{\partial N}\left(\frac{1}{r}\right)$ the rate of change along the (outward) normal, and (a, b, c) exterior to S; and (ii) $\iint_S \frac{\partial}{\partial N}\left(\frac{1}{r}\right)dS = -4\pi$ if (a, b, c) is within S.

Since $V = m/r$ for a single particle, we infer that

$$\iint_S \frac{\partial V}{\partial N}dS = -4\pi M,$$

where M is the total mass *within* S, i.e. the normal surface integral of intensity over S is equal to -4π times the total mass enclosed by the surface. (*Gauss's Theorem.*)

By Green's formula $\iint_S \frac{\partial V}{\partial N}dS = \iiint_v \nabla^2 V\,dv$ where v is the volume enclosed by S,

i.e.
$$\iiint_v (\nabla^2 V + 4\pi\rho)dv = 0.$$

If therefore $\nabla^2 V$ and ρ are continuous within D, we have

$$(\nabla^2 V + 4\pi\rho)_{x_0', y_0', z_0'} = 0,$$

where (x_0', y_0', z_0') is some point of v.

By taking v to be a small sphere of centre (x_0, y_0, z_0) and radius ϵ, and letting $\epsilon \to 0$, we deduce that $\nabla^2 V = -4\pi\rho$ (at any interior point). This result is known as Poisson's Theorem and is sometimes written in the form $V = \iiint_D \frac{\rho\,dv}{r} = -\frac{1}{4\pi}\iiint_D \frac{\nabla^2 V\,dv}{r}$.

Examples. (i) Let S be a surface containing within it or on it no points of the distribution; then V is regular on and within S.

Take $\mathbf{F} = V\nabla V$ in Green's Formula $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_v \nabla\cdot\mathbf{F}\,dv$ (§ *9.51*).

Then $\iint_S V\frac{\partial V}{\partial N}dS = \iiint(\nabla V)^2 dv$.

If therefore V is constant on S, $(\nabla V)^2 = 0$, since $\dfrac{\partial V}{\partial N} = 0$, i.e. $V_x = V_y = V_z = 0$ or V is constant everywhere in the region for which it is regular, if it is constant over a surface drawn in that region. Similarly V is constant in the region if its normal derivative vanishes over S.

(ii) The potential cannot have a maximum or minimum at a point of free space.

From § 9.54 (iii) we have $4\pi V(a, b, c) = \iint_S \left\{\dfrac{1}{r}\dfrac{\partial V}{\partial N} - V\dfrac{\partial}{\partial N}\left(\dfrac{1}{r}\right)\right\} dS$ where S is a surface drawn in free space, (a, b, c) a point within S and r the distance of (a, b, c) from a point of S.

Take S to be the sphere centre (a, b, c) and radius R; then since $\iint \dfrac{\partial V}{\partial N} dS = 0$ and $\dfrac{\partial}{\partial N}\left(\dfrac{1}{r}\right) = -\dfrac{1}{R^2}$ we obtain $V(a, b, c) = \dfrac{1}{4\pi R^2}\iint V dS$. Thus the mean value of V over the sphere is the value at the centre. The function V cannot, therefore, have a maximum or minimum at (a, b, c).

(iii) Find, by direct integration, the potential and attraction at a distance c from the centre of a uniform solid sphere of unit density and radius a; and verify the results by using Gauss's theorem.

(1) $c > a$, $V = \iiint \dfrac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{\sqrt{(c^2 - 2cr\cos\theta + r^2)}}$ (taking spherical polar co-ordinates referred to the centre of the sphere as origin, and the point of distance c on $\theta = 0$).

Integrating, we find $V = \dfrac{2\pi}{c}\displaystyle\int_0^a r\{\sqrt{(c + r)^2} - \sqrt{(c - r)^2}\}dr = \dfrac{2\pi}{c}\int_0^a 2r^2 \, dr = \dfrac{4}{3}\dfrac{\pi a^3}{c}$.

(2) $c < a$: for $0 < r < c - \varepsilon$, $V_1 = \dfrac{4}{3}\dfrac{\pi}{c}(c - \varepsilon)^3$ by (1) and for $c + \varepsilon < r < a$ we have $V_2 = \dfrac{2\pi}{c}\displaystyle\iint r\{\sqrt{(c + r)^2} - \sqrt{(r - c)^2}\}dr = 2\pi\int_{c+\varepsilon}^{a} 2r \, dr = 2\pi\{a^2 - (c + \varepsilon)^2\}$.

Thus $V_1 + V_2 \longrightarrow \frac{4}{3}\pi c^2 + 2\pi(a^2 - c^2) = 2\pi(a^2 - \frac{1}{3}c^2)$ when $\varepsilon \longrightarrow 0$, i.e. $V = \dfrac{4}{3}\dfrac{\pi a^3}{c}$ $(c > a)$, $2\pi(a^2 - \frac{1}{3}c^2)$ $(c < a)$. The functions V, $\dfrac{\partial V}{\partial c}$ are continuous at $c = a$, but $V''(a - 0) - V''(a + 0) = -4\pi$.

Otherwise, take S to be the sphere centre O and radius c; then V, $\dfrac{\partial V}{\partial N}$ are constant over this sphere, by symmetry. Applying Gauss's Theorem

(1) $c > a$: $\dfrac{\partial V}{\partial c} . 4\pi c^2 = -4\pi . \dfrac{4}{3}\pi a^3$; i.e. $\dfrac{\partial V}{\partial c} = -\dfrac{4}{3}\dfrac{\pi a^3}{c^2}$.

(2) $c < a$, $\dfrac{\partial V}{\partial c} . 4\pi c^2 = -4\pi . \dfrac{4}{3}\pi c^3$; i.e. $\dfrac{\partial V}{\partial c} = -\dfrac{4}{3}\pi c$, giving the attraction.

Integrating with respect to c, and using the facts that (a) $V \longrightarrow 0$ when $c \longrightarrow \infty$ (b) V is continuous at $c = a$, we find the same values of V as above.

(iv) The cross section of a right solid circular cylinder of unit density and of height h is a semi-circle of radius a. Find the attraction at the mid point of the bounding diameter of a semi-circular end in the direction perpendicular to the rectangular face of the solid. Use cylindrical co-ordinates, taking the equation of the cylindrical surface to be $\rho = a$, and measuring ϕ from the rectangular face. The solid is given by $0 < \phi < \pi$, $0 < z < h$, $0 < \rho < a$. The attraction is

$$\iiint \frac{\rho^2 \sin\phi \, d\rho \, d\phi \, dz}{(\rho^2 + z^2)^{3/2}} = 2\iint \frac{\rho^2 \, d\rho \, dz}{(\rho^2 + z^2)^{3/2}}$$

$$= 2\int_0^a \frac{h \, d\rho}{(\rho^2 + h^2)^{1/2}} = 2h \log\left\{\frac{a + \sqrt{(a^2 + h^2)}}{h}\right\}$$

9.66. Other Illustrations of the Use of Integrals. (i) Mean Value. Three points are taken at random on a straight line of length a. Find the mean (numerical) distance of the intermediate point from the mid point of the line.

Let the segments measured from one end of the line be

$$x\ (>0),\ y\ (>0),\ z\ (>0),\ a-x-y-z\ (>0)$$

We therefore require the mean value c of $\left|\frac{a}{2}-x-y\right|$ for the tetrahedron given by $0<x+y+z<a$.

The set of points for which $x+y<a/2$ has the same measure as the set for which $x+y>a/2$. Also the mean value for the one set is the same as that for the other.

Thus $\frac{1}{2}c\iiint dx\,dy\,dz=\iint(\frac{1}{2}a-x-y)(a-x-y)dx\,dy$ for $0<x+y<a/2$,

i.e. $\frac{1}{12}ca^3=\iint\left\{\left(\frac{a}{2}-x-y\right)^2+\frac{a}{2}\left(\frac{a}{2}-x-y\right)\right\}dx\,dy=\frac{a^4}{64}$; $c=\frac{3a}{16}$.

(ii) *Probability*. A straight line is divided into three parts. Find the chance that these parts form (*a*) a triangle, (*b*) an acute-angled triangle.

(*a*) Let x, y, z be the lengths of the three parts where x, y, z are three positive (or zero) numbers for which $0<x+y<a$, $x+y+z=a$. The set of values x, y is measured by $\iint dx\,dy=\frac{1}{2}a^2=\Delta$. The favourable cases are those for which $x+y>a-x-y$, $a-x>x$, $a-y>y$, i.e. the interior of the triangle Δ_1 determined by $x+y=a/2$, $x=a/2$, $y=a/2$. The chance is $\Delta_1/\Delta=1/4$.

(*b*) For an acute-angled triangle $z^2<x^2+y^2$, $x^2<y^2+z^2$, $y^2<z^2+x^2$ (if these are satisfied the triangle exists since then $x+y>z$, $y+z>x$, $z+x>y$).

One set of unfavourable cases is bounded by the curve of intersection of the cylinder $(x^2+y^2)=(a-x-y)^2$ with the plane $x+y+z=a$. This set is measured by $\int_0^{\frac{a}{2}}\left\{a-\frac{a^2}{2(a-x)}\right\}dx=\frac{1}{2}a^2-\frac{1}{2}a^2\log 2$. By symmetry, there are two other sets measured by the same number (the curves on the plane $x+y+z=a$ intersecting only on the co-ordinate planes).

The number of favourable cases is $\frac{3a^2}{2}\log 2-a^2$ and therefore the chance of an acute-angled triangle is $3\log 2-2$ ($=0{\cdot}08$ approx.).

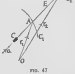


FIG. 47

(iii) *Planimeters*. Let A, B be two points fixed on a rod which moves in the $x-y$ plane so that A, B describe certain closed curves respectively and return to their initial positions. (*Fig. 47.*)

Let the co-ordinates of A, B be (x_1, y_1), (x_2, y_2) respectively, referred to fixed axes through O.

Then $\mathbf{r}_1=x_1\mathbf{i}+y_1\mathbf{j}$, $\mathbf{r}_2=x_2\mathbf{i}+y_2\mathbf{j}$

$\mathbf{r}_1\times d\mathbf{r}_1=(x_1\,dy_1-y_1\,dx_1)\mathbf{k}$ and

$\mathbf{r}_2\times d\mathbf{r}_2=(x_2\,dy_2-y_2\,dx_2)\mathbf{k}$.

Thus $\int_{C_1}\mathbf{r}_1\times d\mathbf{r}_1=2A_1\mathbf{k}$, $\int_{C_2}\mathbf{r}_2\times d\mathbf{r}_2=2A_2\mathbf{k}$, where A_1, A_2 are the areas enclosed by the paths C_1, C_2 of the points A, B respectively.

Let $\mathbf{a}$ be unit vector in the direction $\overrightarrow{AB}$ and $\mathbf{r}_3=\overrightarrow{OE}$ where E is the midpoint of AB.

Then $\mathbf{r}_1=\mathbf{r}_3-\frac{1}{2}l\mathbf{a}$, $\mathbf{r}_2=\mathbf{r}_3+\frac{1}{2}l\mathbf{a}$, where $AB=l$.

Therefore $\mathbf{r}_2\times d\mathbf{r}_2-\mathbf{r}_1\times d\mathbf{r}_1=l(\mathbf{a}\times d\mathbf{r}_3+\mathbf{r}_3\times d\mathbf{a})$.

But $\mathbf{a}\times\mathbf{r}_3=p\mathbf{k}$, where p is the perpendicular from O to AB, i.e.

$$\mathbf{a}\times d\mathbf{r}_3+d\mathbf{a}\times\mathbf{r}_3=dp\,\mathbf{k}.$$

Also $\mathbf{a}\times d\mathbf{r}_3=du\,\mathbf{k}$, where du is the displacement of E perpendicular to the

rod, and therefore $d\mathbf{a} \times \mathbf{r}_2 = (dp - du)\mathbf{k}$. Thus

$$\mathbf{r}_2 \times d\mathbf{r}_2 - \mathbf{r}_1 \times d\mathbf{r}_1 = l(2du - dp)\mathbf{k}$$

or $A_2 - A_1 = lu$, where u is the total displacement at E of the rod perpendicular to itself, since the description of the closed curves restores the value of p.

The above equation gives the theory of such a planimeter as Amsler's, which may be used to determine the area A_2 described by B.

The displacement perpendicular to the rod is measured at some point C by a small wheel whose axis is along the rod. If this displacement is w, then $u = w + c\theta$ where $EC = c$ and θ is the angle through which AB has turned from its initial direction.

Thus $A_2 = A_1 + l(w + c\theta)$.

In Amsler's planimeter, the point A is connected by an arm to a fixed point (which may be taken as O) and two motions are possible.

(i) When A describes an arc of a circle and returns to its original position in such a way that the rod does not make a revolution.

Then $\theta = 0$, $A_1 = 0$, $A_2 = lw$, which is measured by the instrument.

(ii) When A describes a complete circle, and the rod makes one revolution.

Then $A_1 = \pi a^2$, where $OA = a$, $\theta = 2\pi$ and $A_2 = lw + \pi(a^2 + 2lc)$. The instrument measures lw, but to this must be added $\pi(a^2 + 2lc)$, a constant of the instrument.

Examples IX

1. Find the length of the arc of the circle $x^2 + y^2 = 2by$ intercepted by the circles $x^2 + y^2 = 2a_1x$, $x^2 + y^2 = 2a_2x$.

2. Show that the length of the arc of intersection of the surfaces $y = 4\sqrt{x} - x$, $z = x - \frac{1}{3}x^3$ measured from the origin is $x + y - z$.

3. For the curve given by $x = a\cosh t$, $y = b\sinh t$, $z = at$, prove that $bs = y\sqrt{(a^2 + b^2)}$ where s is measured from $t = 0$.

4. Show that the arc of the plane curve given by $x = 3a\sin\theta - a\sin^3\theta$, $y = a\cos^3\theta$, is of length $\frac{3}{2}a\theta + \frac{3}{2}a\sin\theta\cos\theta$ if measured from $(0, a)$.

5. Evaluate $\iint x^2y\,dx\,dy$ over the rectangle $0 < x < 2a$, $b < y < 2b$.

6. Evaluate $\iint \sin\left(\frac{x}{a} + \frac{y}{b}\right)dx\,dy$, $(a, b > 0)$, over $0 < x < \frac{1}{2}\pi a$, $0 < y < \frac{1}{2}\pi b$.

7. Show that $\iint F_{xy}\,dx\,dy$ over the rectangle $x_0 < x < x_1$, $y_0 < y < y_1$, is
$$F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0).$$

8. Evaluate $\iint \sin(2x - 3y)dx\,dy$ over the trapezium whose corners are $(0, 0)$, $(\frac{1}{2}\pi, 0)$, $(\pi, \frac{1}{2}\pi)$, $(\frac{1}{2}\pi, \frac{1}{2}\pi)$.

9. Find $\iint(x^2 + y^2)dx\,dy$ over the interior of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

10. Show that $\iint F_{xy}\,dx\,dy$ over the triangle given by $0 < x/a + y/b < 1$, is
$$\int_0^1 F_x(at, b - bt)dt - F(a, 0) + F(0, 0).$$

11. Prove that $\iint(2x^2 + y^2 + 3x - 2y + 4)dx\,dy$ over the interior of the ellipse $x^2 + 4y^2 - 2x - 8y + 1 = 0$ is $\frac{57\pi}{2}$.

Evaluate the integrals given in *Examples 12–14* over the triangle determined by $0 < x + y < 1$.

12. $\iint e^{2x+3y}\,dx\,dy$. 13. $\iint e^{x+y}\,dx\,dy$. 14. $\iint \cos\pi(x + y)dx\,dy$.

15. Evaluate $\iint x^2\,dx\,dy$ over the interior of the octagon whose vertices, in polar co-ordinates, are given as $(a, r\pi/4)$, $(r = 0$ to $7)$.

16. Find $\iint e^{x+y}\,dx\,dy$ for the interior of the square of corners $(0, \pm a)$, $(\pm a, 0)$.

17. Evaluate $\iint \log(x^2 + y^2)dx\,dy$ over the area between the circles $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$, $(b < a)$ and find its limit when $b \to 0$.

18. Determine the region over which the integral $\int_0^1 dy \int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx$ is to be evaluated and find its value.

19. Evaluate $\int_0^1 dy \int_{y+1}^{2+(1-y)^{\frac{1}{2}}} x^3\,dx$ and state the area in the $x-y$ plane to which the integral refers.

20. Evaluate $\iint x^2y^2\,dx\,dy$ over the area bounded by $y = \pm 3(x-2)$, $y = \pm 3(x-4)$.

21. Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the circular area bounded by

$$(x-2)^2 + (y-3)^2 = 9.$$

22. Calculate the value of $\iint x^2\,dx\,dy$ over that area bounded by $x^2 - y^2 = 1$, $x^2 + y^2 = 4$ which contains $(0, 0)$.

23. Prove that the area of the parallelogram determined by $a_1x + b_1y = k_1$, $a_1x + b_1y = k_2$, $a_2x + b_2y = K_1$, $a_2x + b_2y = K_2$ is $\left|\frac{(K_1 - K_2)(k_1 - k_2)}{a_1b_2 - a_2b_1}\right|$.

24. Prove that for the change of variable given by

$$x^2(a^2 - b^2) = (a^2 + u)(a^2 + v),\quad y^2(a^2 - b^2) = -(b^2 + u)(b^2 + v),\quad (a > b)$$

the curves $u = c_1 (> -b^2)$ and $v = c_2\ (-a^2 < v < -b^2)$ are ellipses and hyperbolas respectively. Show also that u, v are orthogonal co-ordinates and that

$$dS = (u - v)\,du\,dv/4\sqrt{D} \text{ where } D = -(a^2+u)(a^2+v)(b^2+u)(b^2+v).$$

25. If $u = x^2 - y^2$, $v = 2xy$, show that any straight line in the $x - y$ plane not passing through O is transformed into a simple curve.

26. If $u = x^4 - 6x^2y^2 + y^4$, $v = 4xy(x^2 - y^2)$, show that the straight line $y = c\ (\neq 0)$ is transformed into a curve with a loop.

27. If $u = x + y$, $v = y^3 - 3xy - 3y^2 + 6x + 6y$, the region for which $J < 0$ and its transform is $1 - 1$, but not the region for which $J > 0$ and its transform. Show that the region on the left of any tangent to $J = 0$ is $1 - 1$ with its transform.

28. Find the area in the first quadrant determined by $y^m = a_1x^n$, $y^m = a_2x^n$, $y^p = b_1x^q$, $y^p = b_2x^q$ ($a_1, a_2, b_1, b_2 > 0$, $qm - np \neq 0$).

29. Find the area bounded by the curve $r = a + b\cos\theta$ ($b < a$).

30. Show that the area in the first quadrant determined by $xy^2 = 1$, $xy^2 = 4$, $y^2 = 4x$, $y^2 = 9x$ is $\frac{1}{3}(24 - 6\sqrt{2} + 4\sqrt{3} - 8\sqrt{6})$.

31. Prove that the area of the loop of the curve $(x^2 - ay)^2 = a^2y$ is $\frac{1}{3}a^2$.

32. Show that the area of one of the crescents across the x-axis bounded by $x^2 + y^2 = c^2$, $x^2/a^2 + y^2/b^2 = 1$ $(b < c < a)$, is

$$ab \arctan \sqrt{\left(\frac{a^2 - c^2}{c^2 - b^2}\right)} - c^2 \arctan \left\{\frac{b\sqrt{(a^2 - c^2)}}{a\sqrt{(c^2 - b^2)}}\right\}$$

33. If $x = u^m\phi(v)$, $y = u^n\psi(v)$, prove that the area bounded by $u = u_1$, $u = u_2$, $v = v_1$, $v = v_2$ is the numerical value of $(u_1^{m+n} - u_2^{m+n})\int_{v_1}^{v_2} \frac{(m\phi\psi' - n\psi\phi')dv}{(m+n)}$ if $m \neq -n$ and of $m(\phi_1\psi_1 - \phi_2\psi_2)\log\left(\frac{u_1}{u_2}\right)$ if $m = -n$.

34. Show that the area in the first quadrant determined by $a_1y^2 = x^3$, $a_2y^2 = x^3$, $y^2 = b_1x$, $y^2 = b_2x$ is $\frac{1}{10}|(b_2^{5/4} - b_1^{5/4})(a_2^{3/4} - a_1^{3/4})|$, $(a_1, a_2, b_1, b_2 > 0)$.

35. Prove that the area bounded by a simple closed curve C is the line integral

$$\oint_C \frac{x^\alpha y^\beta d(x^{\alpha_1} y^{\beta_1})}{\alpha - \beta} \text{ where } \alpha + \alpha_1 = \beta + \beta_1 = 1,\ \alpha \neq \beta.$$

36. Show that

$$\int_0^{x,y} e^{3x+2y}\{(3\cos 3x + 2\sin 3x)(\sin 2y\,dx + \cos 2y\,dy) + (2\cos 3x - 3\sin 3x)(\cos 2y\,dx - \sin 2y\,dy)\}$$

is independent of the path of integration and find its value.

37. Calculate directly $\int (x^2 + y^2)dx + (x^2 + y^2)dy$ along the boundary of the pentagon whose vertices are $(0, 0)$, $(1, 0)$, $(2, 1)$, $(1, 2)$, $(0, 1)$ and verify the result by Green's formula.

38. Show that $\displaystyle\int\frac{xy(x\,dy - y\,dx)}{x^4+y^4}$ for any closed curve C not passing through the origin (0, 0) is zero.

39. Prove that for a simple closed curve not passing through $(\pm 1, 0)$ the value of the integral $\displaystyle\oint\frac{(x^2-y^2-1)dy - 2xy\,dx}{(x^2+y^2-1)^2+4y^2}$ is $0, \frac{1}{2}\pi$ or π.

Discuss the line integrals given in *Examples 40–1*.

40. $\displaystyle\oint_C\frac{x^2y^2(x\,dy - y\,dx)}{x^6+y^6}$ 41. $\displaystyle\oint_C\frac{(x^2-y^2-x)dx - y(2x-1)dx}{(x^2+y^2)(x^2+y^2-2x+1)}$ (C being closed)

42. Evaluate $\displaystyle\oint_C\frac{(x^3\,dy - y^3\,dx)}{(x^2+y^2)^2}$ when C is (i) the boundary of the square $x = \pm a$, $y = \pm a$; (ii) the circle $x^2+y^2 = a^2$.

43. Show that the line integral $\oint p\,dv$ over the boundary specified by $pv = R\theta_1$, $pv = R\theta_2$, $pv^\gamma = C_1$, $pv^\gamma = C_2$ ($R, \theta_1, \theta_2, C_1, C_2, \gamma\ (> 1)$ being constants and all the symbols representing positive numbers) is equal to $R\left[\dfrac{\theta_1-\theta_2}{\gamma-1}\log\dfrac{C_1}{C_2}\right]$.

44. If a rod ABC moves in a plane and returns to its initial position after having rotated through one complete revolution, show that

$$BC.\Omega_1 + AB.\Omega_3 - AC.\Omega_2 = \pi BC.AC.AB$$

where Ω_1, Ω_2, Ω_3 are the areas enclosed by the curves described by A, B, C respectively.

45. B is a fixed point on a rod AC dividing AC into two segments c_1, c_2. Prove that if the ends A, C move on the same closed curve γ and the rod returns to its initial position after describing one revolution, the area between γ and the curve described by B is πc_1c_2.

46. A square hole of side $2\sqrt{2}$ in. is cut symmetrically through a sphere of radius 2 in. Show that the area of the surface removed is $16\pi(\sqrt{2}-1)$ sq. in.

47. A square hole of side $2b$ is cut symmetrically through a cylinder of radius $a\ (> b)$, the axis of the hole being perpendicular and two of the sides of the section being parallel to the axis of the cylinder. Find the area of the surface removed.

48. Find the portion of the surface of the sphere $x^2+y^2+z^2 = a^2$ within the cylinder $x^2+y^2 = ax$.

49. Find the area of the surface obtained by a complete revolution of one arch of a cycloid about the line joining its extremities.

50. Find the area of the surface $2az = x^2 - y^2$ intercepted by $\rho^2 = a^2\cos\phi$.

51. The ellipse of axes $a, b\ (a > b)$ forms a *prolate* spheroid when it makes a revolution about its *major* axis. Show that the surface of this ellipsoid is $2\pi b^2 + (2\pi ab\arcsin e)/e$ where $b^2 = a^2(1-e^2)$.

52. The ellipse of axes $a, b\ (a > b)$ forms an *oblate* spheroid when it makes a revolution about its *minor* axis. Show that the surface of this ellipsoid is

$$2\pi a^2 + [\pi b^2\log\{(1+e)/(1-e)\}]/e.$$

53. Evaluate $\iiint xy^2z^2\,dx\,dy\,dz$ through the interior of the tetrahedron bounded by $z = 0$, $z = x$, $y = 2z$, $y = 2x$.

54. Show that $\iiint e^{x/\alpha+y/\beta+z/\gamma}\,dx\,dy\,dz$ taken over all positive and zero values of x, y, z that satisfy $0 \leq x/a + y/b + z/c \leq 1$ $(a, b, c > 0)$ is

$$\alpha\beta\gamma\left[-1+\Sigma\frac{e^{a/\alpha}a^2bc}{(a\beta - b\alpha)(a\gamma - c\alpha)}\right]$$

if $a/\alpha \neq b/\beta \neq c/\gamma$.

55. Evaluate $\iiint e^{k(x/a+y/b)+z/\gamma}\,dx\,dy\,dz$ over all positive and zero values of x, y, z that satisfy $0 \leq x/a + y/b + z/c \leq 1$, when $\gamma = c/k$.

56. Evaluate $\iiint e^{k(x/a+y/b+z/c)}\,dx\,dy\,dz$ over all positive and zero values of x, y, z that satisfy $0 \leq x/a + y/b + z/c \leq 1$.

57. Evaluate $\iiint z(x^2+y^2)\,dx\,dy\,dz$ through the volume of the cylinder $x^2+y^2=a^2$, intercepted by the planes $z=0$, $z=h$.

58. Evaluate $\iiint (xy+yz+zx)\,dx\,dy\,dz$ through the interior of the cube determined by $0<x<a$, $0<y<a$, $0<z<a$.

59. Evaluate $\iiint xyz^2\,dx\,dy\,dz$ for $0<4(x-2)^2+(y-1)^2+9z^2<36$.

60. Find the value of $\int_0^a dx\int_0^x dy\int_0^{a-x} xyz\,dz$ and state the region to which the triple integral refers.

61. Evaluate $\int_0^1 dx\int_0^{2-2x} dy\int_{x+\frac{1}{2}y}^{1} z^2\,dz$ and obtain the corresponding forms of the five equivalent repeated integrals.

62 Find the volume of the sphere $x^2+y^2+z^2=a^2$ intercepted by the cylinder $x^2+y^2=ax$.

63. Find the volume enclosed by the surface $r=a(1+e\cos\theta)$ where r,θ are spherical polar co-ordinates and e is constant.

64. Find the volumes bounded by (i) $r=a\sin^2\theta\cos\theta\sin\phi\cos\phi$; (ii) $r^3=27a^3\sin^3\theta\cos\theta\sin\phi\cos\phi$ in the octant for which $x, y, z>0$ (r, θ, ϕ being spherical polar co-ordinates).

65. A square hole of side $2b$ is cut symmetrically through a sphere of radius a ($>b\sqrt{2}$). Find the volume removed.

66. A square hole of side $2b$ is cut symmetrically through a cylinder of radius a ($>b$), two of the sides of the square section being parallel to the axis of the cylinder and the axis of the hole being perpendicular to the axis of the cylinder. Find the volume removed.

67. Show that the volume obtained by one revolution of the loop of the curve $x^3+y^3=3axy$ about OX is $4\pi^2a^3/3\sqrt{3}$.

68. Show that the volume of the surface $x^2+y^2=4az$ cut off by the plane $x+y+2z=4a$ is $25\pi a^3/2$.

69. Show that the volume obtained by a revolution of the circle

$$x^2+(y-c)^2=a^2\ (c>a)$$

about the x-axis is $2\pi^2a^2c$.

70. An arch of a cycloid makes one revolution about the line joining its extremities. Prove that the volume generated is $5\pi^2a^3$.

71. Find the volume of the paraboloid $x^2+y^2=4az$ cut off between the planes $x+2y+z=a$, $2x+y+z=16a$.

72. Calculate the volume determined by $0<\sqrt{x}+\sqrt{y}+\sqrt{z}<1$.

73. Calculate $\iiiint e^u\,dx\,dy\,dz\,du$ for all positive and zero values of x, y, z, u, that satisfy the inequality $0<x+y+z+u<1$.

74. Evaluate $\iiiint (x+y+z+u)^m xy^2z^3u^4\,dx\,dy\,dz\,du$ $(m>0)$ over all positive and zero values of x, y, z, u for which $0<x+y+z+u<1$.

75. Find the value of $\iint\ldots\int e^{x_1+x_2+\cdots+x_n}\,dx_1\,dx_2\ldots dx_n$ over all positive and zero values of $x_1, x_2, \ldots, x_n$ for which $0<x_1+x_2+\cdots+x_n<1$.

76. Evaluate $\iiiint \cos(x+2y+3z+4u)\,dx\,dy\,dz\,du$ over all positive and zero values of x, y, z, u for which $0<x+y+z+u<\pi/2$.

77. Prove that $\iiiint e^{\alpha x+\beta y+\gamma z+\delta u}\,dx\,dy\,dz\,du$ over all positive and zero values of x, y, z, u for which $0<x+y+z+u<1$ is

$$\frac{1}{\alpha\beta\gamma\delta}+\Sigma\frac{e^{\alpha}}{\alpha(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}$$

if $\alpha\beta\gamma\delta(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)\neq 0$.

78. Verify Green's formula by calculating directly the surface integral $\iint z^2(x^2+y^2)\,dS$ over the sphere $x^2+y^2+z^2=a^2$.

79. Use Green's formula to prove that when $ds^2=h_1^2\,du^2+h_2^2\,dv^2+h_3^2\,dw^2$,

$$h_1h_2h_3\nabla^2 V=\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}V_u\right)+\frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}V_v\right)+\frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}V_w\right)$$

80. Calculate directly $\iint(x^3\,dy\,dz + y^3\,dz\,dx + z^3\,dx\,dy)$ over the spherical surface $x^2 + y^2 + z^2 = a^2$, and verify the result by Green's formula.

81. If $\nabla \times \mathbf{F} = -\dot{\mathbf{G}}$; $\nabla \times \mathbf{G} = -\mathbf{F}$, prove that, if S is a simple surface enclosing V, $\iint_S(\mathbf{F} \times \mathbf{G}).d\mathbf{S} = \frac{\partial}{\partial t}\iiint_V \tfrac{1}{2}(\mathbf{F}^2 + \mathbf{G}^2)dx\,dy\,dz$.

82. Show that, in the usual notation, if $\nabla^2 U = 0$, then

$$\iiint_V(U_x^2 + U_y^2 + U_z^2)dx\,dy\,dz = \iint_S U\frac{\partial U}{\partial N}\,dS.$$

83. Find the value of $\int\lambda(y^2 - z^2)dx + \mu(z^2 + x^2)dy + \nu(x^2 + y^2)dz$ along the skew hexagon obtained by joining the following points in order: O (the origin), $A(1, 0, 0)$, $B(1, 1, 0)$, $C(1, 1, 1)$, $D(0, 1, 1)$, $E(0, 0, 1)$, O. Verify the result by Stokes's Theorem by taking a surface integral over the three faces of the unit cube $OAFE$, $ABCF$, $FCDE$ (F being the point 1, 0, 1).

84. Show that if P, Q, R and their derivatives are continuous, the function given by $\int_{x_0, y_0, z_0}^{x, y, z}(P\,dx + Q\,dy + R\,dz)$ is independent of the path of integration if $Q_z = R_y$, $R_x = P_z$, $P_y = Q_x$.

85. Prove that $\int_0^{x, y, z}(x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz$ is independent of the path of integration and find its value.

86. The boundary of an area in polar co-ordinates is given by $r = f(\theta)$. Show that for the sector determined by $\theta_1 < \theta < \theta_2$, r being single-valued,

$$3A\bar{x} = \int_{\theta_1}^{\theta_2} r^3\cos\theta\,d\theta,\ 3A\bar{y} = \int_{\theta_1}^{\theta_2} r^3\sin\theta\,d\theta,\ \text{where } A = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta.$$

87. Find the distance from the y axis of the mean centre of the area bounded by $r = a(3 + 2\cos\theta)$.

88. Show that the mean centre of the area bounded $r^2 = a^2\cos 2\theta$, $y = 0$, in the first quadrant, is given by $8\bar{x} = \pi a\sqrt{2}$, $12\bar{y} = 3\sqrt{2}\log(1 + \sqrt{2}) - 2$.

89. Find the x co-ordinate of the mean centre of the area determined by $x = u^m v^r$, $y = u^n v^s$, $u = u_1$, $u = u_2$, $v = v_1$, $v = v_2$ ($u_1, u_2, v_1, v_2 > 0$, $ms \neq rn$).

90. Show that the mean centre of the area in the first quadrant determined by $xy = c_1^2$, $xy = c_2^2$, $c_3 y = x^2$, $c_4 y = x^2$ ($c_3, c_4 > 0$) is given by

$$4\bar{x}(c_1^2 - c_2^2)\log(c_4/c_3) = 9(c_4^{1/3} - c_3^{1/3})(c_1^{5/3} - c_2^{5/3}),$$
$$5\bar{y}c_3^{1/3}c_4^{1/3}(c_1^2 - c_2^2)\log(c_4/c_3) = 9(c_1^{10/3} - c_2^{10/3})(c_4^{1/3} - c_3^{1/3}).$$

91. Prove that the distance from the centre of a circle of the mean centre of a quadrantal area is $4\sqrt{2}a/3\pi$, where a is the radius of the circle.

92. Prove that the distance from the centre of a circle of radius a of the mean of a quadrantal arc is $2\sqrt{2}a/\pi$, where a is the radius of the circle.

93. Prove that the distance from the centre of a circle of radius a of the mean centre of the complete perimeter of a quadrant is $3a\sqrt{2}/(\pi + 4)$.

94. A circular cone (height h, radius of base a) is divided into two by a plane through the vertex V and a diameter of the base. Find the distance from V of the mean centre G, of one of the halves of the cone and the angle VG makes with the axis.

95. Show that the distance from $z = 0$ of the mean centre of the volume of the sphere $x^2 + y^2 + z^2 = a^2$ intercepted between the planes $z = h_1$, $z = h_2$ ($h_1 > h_2$) is $\tfrac{3}{4}(h_1 + h_2)(2a^2 - h_1^2 - h_2^2)/(3a^2 - h_1^2 - h_1h_2 - h_2^2)$.

96. Deduce from *Example 95* that the distance from the plane face of the mean centre of a spherical cap of height b is $\tfrac{1}{4}b(4a - b)/(3a - b)$ where a is the radius of the sphere.

97. A surface is formed by rotating the arch of a cycloid through 180° about

the line joining its extremities. Find the distance of the mean centre from the plane of its boundary.

Find the co-ordinates of the mean centres of the solids given in *Examples 98–100.*

98. The tetrahedron bounded by $x, y, z = 0$, $2x + 3y + 4z = 9a$.

99. The wedge cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = \pm x \tan \alpha$ $(x > 0)$.

100. The solid determined by $0 \leq \sqrt{(x/a)} + \sqrt{(y/b)} + \sqrt{(z/c)} \leq 1$ $(a, b, c > 0)$.

101. A plane lamina consists of a rectangle $ABCD$ in which $AB = 2b$, $BC = 2a$, on the sides BC, DA of which equilateral triangles BCE, ADF are constructed. Show that if k is the radius of gyration about an axis in the plane of the lamina perpendicular to EF at a distance c from the mean centre of the lamina, then

$$k^2 = \frac{4b(3a^2 + b^2 + 3c^2) + 3a\sqrt{3}(a^2 + 2b^2 + 2c^2)}{6(2b + a\sqrt{3})}.$$

102. Show that the moment of inertia of a uniform solid cylinder (mass M, radius a, height h) about an axis that meets OA, the axis of the cylinder, at a distance c from O and is inclined at an angle θ to OA is

$$M\{\tfrac{1}{4}a^2(1 + \cos^2\theta) + (\tfrac{1}{3}h^2 - ch + c^2)\sin^2\theta\}.$$

Hence show that k^2 for (i) a diameter of one end is $\frac{1}{4}a^2 + \frac{1}{3}h^2$, (ii) its axis is $\frac{1}{2}a^2$, (iii) an axis through the mean centre perpendicular to the axis of the cylinder is $\frac{1}{4}a^2 + \frac{1}{12}h^2$, (iv) an axis joining the centre of one end to a point on the perimeter of the other is $\frac{a^2}{12}(3a^2 + 10h^2)/(a^2 + h^2)$.

103. Prove that the radius of gyration about its axis of an open hollow circular cone made of thin uniform material is unaltered by closing the cone with a thin lid.

104. A framework of thin uniform rods consists of a square $ABCD$ of four rods joined by two others along the diagonals AC, BD and by two others EF, GH where E, F are the midpoints of AB, CD and G, H the midpoints of AD, BC. Show that k^2 for an axis through the centre of the figure perpendicular to the framework is $\frac{4}{21}a^2(23 - 3\sqrt{2})$, where $4a$ is the length of AB.

105. Show that for a uniform elliptic lamina of semi-axes a, b, the moment of inertia about an axis in its plane through the centre of the ellipse making an angle α with the axis of length a, is $\frac{1}{4}M(a^2\sin^2\alpha + b^2\cos^2\alpha)$ where M is the mass of the lamina.

106. Show that the moment of inertia of a uniform ellipsoid (mass M, and semi axes a, b, c) about a diagonal of the rectangular parallelopiped that circumscribes the ellipsoid and whose edges are parallel to the principal axes is

$$\tfrac{2}{5}M\frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2}.$$

107. Show that the moment of inertia of the plane sector determined by $r = f(\theta)$ $\theta_1 < \theta < \theta_2$ (i) about OY, is $\frac{1}{4}\int_{\theta_1}^{\theta_2} r^4\cos^2\theta\, d\theta$, (ii) about OX, is $\frac{1}{4}\int_{\theta_1}^{\theta_2} r^4\sin^2\theta\, d\theta$,

(iii) about the perpendicular to XOY at O, is $\frac{1}{4}\int_{\theta_1}^{\theta_2} r^4\, d\theta$.

108. Find the radii of gyration about $\theta = 0$ and $\theta = \pi/2$ for the area of a loop of the curve $r^2 = a^2\cos 2\theta$.

109. The radii of gyration of a body about two parallel axes distant H apart are k_1, k_2 respectively. If the mean centre of the body is in the plane of these axes show that its distance from the first axis is $\pm(k_2^2 - k_1^2 - H^2)/2H$.

110. AB is the diameter that bounds a semi-circular area of radius a and C is the midpoint of the semi-circular boundary. Prove that k^2 about BC for the area is $a^2\left(\frac{3}{4} - \frac{4}{3\pi}\right)$.

111. Prove that h^2 for the normal to the plane XOY through O of the area in the first quadrant determined by $xy = 1$, $xy = 2$, $y = 2x$, $y = \frac{1}{2}x$ is $9/4 \log 2$.

112. Show that the radii of gyration of a loop of the curve $r = a\cos^2\theta$ about the axes OX, OY are respectively $a\sqrt{7}/8\sqrt{3}$, $a\sqrt{21}/8$.

113. Show that the moment of inertia of a frustum of a circular cone is

$$M\left[\frac{3}{20}\frac{a^4 + a^3b + a^2b^2 + ab^3 + b^4}{a^2 + ab + b^2} + \frac{1}{10}\frac{c^2(a^2 + 3ab + 6b^2)}{(a^2 + ab + b^2)}\right]$$

where a, b are the radii of the ends, c the distance between them, M the mass of the frustum and the axis of gyration is the diameter of the end of radius a.

114. Show that a uniform triangular lamina of mass M has the same radius of gyration for any axis as three masses $\frac{1}{3}M$ at the midpoints of the sides. Deduce that the depth of centre of pressure of a triangle immersed in a liquid is $\frac{1}{2}\frac{a^2 + b^2 + c^2 + ab + bc + ca}{(a + b + c)}$ where a, b, c are the depths of the vertices.

115. $ABCD$ is a parallelogram with the vertex A in the surface of a liquid. The diagonal BD is horizontal. Show that the centre of pressure is on AC at a depth which is seven-twelfths that of C.

116. An area is bounded by two concentric semi-circles on the same side of a common diametral line, which is in the surface of a liquid. Prove that the depth of the centre of pressure is $\frac{3\pi}{16}\frac{(a + b)(a^2 + b^2)}{(a^2 + ab + b^2)}$ where a, b are the radii.

117. A solid sphere rests on a horizontal plane and is just totally immersed in a liquid. If it is divided by two planes through a vertical diameter perpendicular to each other, show that the four parts will not separate if $4\sigma < \rho$, where ρ, σ are the densities of the solid and fluid respectively.

118. The boundary of a vessel full of water consists of $x = 0$, $y = 0$, $z = 0$ and that part of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ for which $x, y > 0$ and $z < 0$. Prove that if the z axis is vertically upwards, the total pressure on the curved surface is $\frac{1}{3}wc(b^2c^2 + c^2a^2 + \frac{1}{4}\pi^2a^2b^2)^{1/2}$ where w is the weight of unit volume of the liquid.

119. A circular area of radius a is partly submerged in a liquid so that an arc subtending an angle 2α at the centre is under the surface. Prove that the depth of the centre of pressure from the centre of the circle is

$$\frac{1}{4}\frac{3\alpha - 3\sin\alpha\cos\alpha - 2\cos\alpha\sin^3\alpha}{3\sin\alpha - \sin^3\alpha - 3\alpha\cos\alpha}.$$

120. Prove that the potential of a circular disc of mass m per unit area and of radius a at a point distant h from the centre on the normal to the disc through the centre is

$$2\pi m\{\sqrt{(a^2 + h^2)} - h\}.$$

121. Show that for a shell of density ρ bounded by two non-intersecting spheres (one inside the other), the potential at a point P is (i) $\frac{4}{3}\pi\rho\left(\frac{a^3}{r_1} - \frac{b^3}{r_2}\right)$, if P is outside both spheres, (ii) $\frac{2}{3}\pi\rho(3a^2 - 3b^2 - r_1^2 + r_2^2)$, if P is inside both spheres, (iii) $\frac{2}{3}\pi\rho\left(3a^2 - r_1^2 - \frac{2b^3}{r_2}\right)$, if P is between the spheres, where r_1, r_2 are the distances of P from the centres of the spheres whose radii are a, b respectively ($a > b$).

122. Show that the integral of the normal component of the attraction at O by a portion of a simple surface S over its area is equal to $m\omega$, where m is the surface density and ω is the solid angle subtended at O by S (i.e. ω is the area intercepted on a unit sphere, centre O, by rays joining O to the periphery of S).

123. Find the attraction at the vertex of a solid right circular cone of mass M, height h and radius of base a.

124. A uniform solid circular cylinder of density ρ is of height a and of radius a.

Show that at a point outside the cylinder on its axis at a distance a from the nearer end, the attraction is $2\pi\rho a\,(1+\sqrt{2}-\sqrt{5})$.

125. Show that the attraction due to a uniform thin rod AB at an external point P is $\dfrac{2m}{p}\sin\frac{1}{2}\alpha$, where m is the mass per unit length, p the perpendicular distance of P from the rod and α the angle subtended by the rod at P.

126. A solid uniform circular cylinder has a given volume. Show that the attraction at the centre of one of the circular ends is a maximum when the ratio of the height of the cylinder to the radius of its end is $(9-\sqrt{17})/8$.

127. The vertical angle of a solid uniform circular cone is $90°$. Prove that the ratio of the attraction at the centre of the base to that at the vertex is approximately 1.29.

128. Two numbers are chosen at random between 0 and 4; show that the chance that their product shall be less than 4 is $\frac{1}{4}+\frac{1}{2}\log 2$.

129. A positive number a is divided at random into three positive parts. Find the chance that none of the parts shall be greater than (i) $\frac{1}{2}a$, (ii) $\frac{5}{12}a$.

130. Show that the mean distance of the points of a circular area (radius a) from the end of a diameter is $32a/9\pi$.

131. A positive number a is divided into three positive parts. What is the mean value of the least?

132. Show that the mean distance of the points of the surface of a sphere of radius a from a point on the surface is $4a/3$.

133. Prove that the mean of the nth powers of the distances of the points of a solid sphere (radius a) from a point distant c from the centre is

$$\frac{3}{2}\cdot\frac{\{c+a(n+3)\}(c-a)^{n+3}-\{c-a(n+3)\}(c+a)^{n+3}}{a^3c(n+2)(n+3)(n+4)},\ (c>a).$$

134. Show that the mean distance of the points of a circular area (radius a) from a point at distance c along the normal to the area through the centre is $\dfrac{2}{3a^2}\{(c^2+a^2)^{\frac{3}{2}}-c^3\}$.

135. Prove that the mean distance of the points of a square area (side a) from one corner is $\frac{1}{3}a\left(\sqrt{2}+\log\tan\dfrac{3\pi}{8}\right)$.

136. Show that the mean distance of the points of a solid sphere (radius a) from a point on the surface is $6a/5$.

137. Prove that the mean squared distances of the points of a solid sphere (radius a) from (i) the centre is $3a^2/5$, (ii) a point on the circumference is $8a^2/5$.

138. Show that the mean distance between two points within a unit circle is approximately 0.9.

139. The density at P of a square lamina (side $4a$) is proportional to the square of the distance of P from a point on a diagonal distant $a\sqrt{2}$ from the centre. Find the mean density.

140. Find the mean density of a sphere (radius a) if its density at any point P is k times the distance of P from a fixed point on the surface.

141. Find the mean value of one of n positive numbers whose sum $\not> 1$.

142. Find the mean value of the product of the three segments into which a line of length a may be divided.

143. A line of unit length is divided into four parts. Find the mean square of one of the parts.

Solutions

1. $2b\left\{\arctan\left(\dfrac{a_1}{b}\right)-\arctan\left(\dfrac{a_2}{b}\right)\right\}$ 2. Take $x=t^2$ 5. $6a^4b^2$

6. $2ab$ 8. $\frac{2}{3}$ 9. $\dfrac{\pi ab}{4}(a^2+b^2)$ 12. $\frac{1}{3}(e-1)^2(2e+1)$

13. 1 14. $-\dfrac{2}{\pi^2}$ 15. $\frac{1}{3}a^4(1+2\sqrt{2})$ 16. $2a \sinh c$

17. $2\pi(a^2 \log a - b^2 \log b - \frac{1}{2}a^2 + \frac{1}{2}b^2) \to 2\pi a^2(\log a - \frac{1}{2})$
18. Region bounded by $y = 0$, $y = x^2$, $y = (x-2)^2$; value $\frac{1}{3}$.
19. $347/60$, area bounded by $y = 0$, $y = x - 1$, $y = 1 - (x-2)^2$
20. Take $x = X + 3$, and use symmetry; result $248\frac{2}{3}$.
21. $1179\pi/4$, use symmetry. 22. $8 \arcsin \dfrac{\sqrt{10}}{4} + \frac{1}{2}\log\left(\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{2}}\right) - 3\dfrac{\sqrt{15}}{4}$
25. A parabola.
26. A quartic with a double point at $v = 0$, $u = -4c^2$, $(x = \pm c)$.
27. $u = 1$, $v = 5$, for example at both points $(1 \pm \sqrt{3}, -\sqrt{3})$; to prove $1-1$ correspondence, use the theorem on $\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}$ § *9.16* (iv).

28 $$\left|\frac{(a_1^{(p+q)/\Delta} - a_2^{(p+q)/\Delta})(b_1^{-(m+n)/\Delta} - b_2^{-(m+n)/\Delta})\Delta}{(m+n)(p+q)}\right|$$
$(\Delta = mq - np,\ (m+n)(p+q) \neq 0)$;
$$\left|\frac{\left(b_1^{\frac{1}{p}} - b_2^{\frac{1}{p}}\right)\log\left(\dfrac{a_1}{a_2}\right)}{(m+n)}\right|,\ (p+q) = 0,\quad \left|\frac{\left(a_1^{\frac{1}{m}} - a_2^{\frac{1}{m}}\right)\log\left(\dfrac{b_1}{b_2}\right)}{p+q}\right|,\ (m+n) = 0.$$
29. $\pi(a^2 + \frac{1}{2}b^2)$
31. By the transformation $u = x$, $v = \sqrt{(ay)}$ $(y > 0)$, the loop becomes the circle $u^2 + v^2 = av$ and area of loop is $\dfrac{2}{a}\iint v\,du\,dv = \frac{1}{8}\pi a^2$.
36. $e^{2x+3y}\cos(3x - 2y) - 1$ 37. 0 38. Take $x = r\cos\theta$, $y = r\sin\theta$.
40. $\frac{2}{3}n\pi$ where n is the number of circuits round 0.
41. $2\pi(m_1 - m_2)$ where m_1, m_2 are the number of circuits round (0, 0) and (1, 0) respectively.
42. (i) $\pi + 2$, (ii) $\frac{3}{2}\pi$
44. See § *9.66* (iii); $\Omega_2 = \Omega_1 + u.AB$, $\Omega_3 = \Omega_1 + (u + \frac{1}{2}BC.2\pi).AC$ from which the result follows.
45. Use *Example 44* 47. $8ab \arcsin\left(\dfrac{b}{a}\right)$
48. $2a^3(\pi - 2)$; use cylindrical co-ordinates.
49. $\frac{64}{15}\pi a^3$ 50. $\frac{2}{9}a^3(20 - 3\pi)$ 53. $\frac{1}{12}\gamma a^5$
55. $\dfrac{aby^2e^{c/y}}{(c-ky)^2} + \dfrac{abcye^{c}(yk^2 - 2yk + c(1-k))}{k^2(c-ky)^2} - \dfrac{aby}{k^2}$
56. $\dfrac{abc}{k^3}\{e^k(1 - k + \frac{1}{2}k^2) - 1\}$ 57. $\frac{1}{8}\pi a^4b^2$ 58. $\frac{2}{3}a^3$ 59. $\frac{32}{5}\pi$
60. $\frac{1}{12}a^4$. Prism bounded by $y = 0$, $y = a$, $x = a$, $z = 0$, $z = a - x$.
61. $\frac{1}{5}$. $\displaystyle\int_0^2 dy \int_0^{1-\frac{1}{2}y} dx \int_{x+\frac{1}{2}y}^{1} z^2\,dz : \int_0^1 dz \int_0^{2z} dy \int_0^{z-\frac{1}{2}y} z^2\,dx$:
$$\int_0^2 dy \int_{\frac{1}{2}y}^1 dz \int_0^{z-\frac{1}{2}y} z^2\,dx : \int_0^1 dz \int_0^z dx \int_0^{2z-2x} z^2\,dy : \int_0^1 dx \int_x^1 dz \int_0^{2z-2x} z^2\,dy;$$
tetrahedron bounded by $z = 1$, $x + \frac{1}{2}y = z$, $x = 0$, $y = 0$.
62. $\frac{2}{9}a^3(3\pi - 4)$; use cylindrical co-ordinates. 63. $\frac{1}{6}\pi a^3(1 + e^2)$
64. (i) $a^3/1440$, (ii) $9a^3/8$
65. $\frac{4}{3}b^2\sqrt{(a^2 - 2b^2)} + \frac{2}{3}b(3a^2 - b^2) \arcsin\left(\dfrac{b}{\sqrt{(a^2 - b^2)}}\right)$
$$-\tfrac{8}{3}a^3 \arctan\left(\frac{b^2}{a\sqrt{(a^2 - 2b^2)}}\right)$$

66. $4b^2\sqrt{(a^2 - b^2)} + 4a^2b \arcsin\left(\frac{b}{a}\right)$

67. Take $u = x^2/y$, $v = y^2/x$

69. Use Pappus's Theorem. 71. $310\pi a^3$ 72. $1/90$

73. $e - 8/3$ 74. $\dfrac{288}{(13)!(m + 14)}$

75. $(-1)^n\left[1 - e\left\{\frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^{n-1}\frac{1}{(n-1)!}\right\}\right]$

76. $-1/6$ 78. $\frac{8}{15}\pi a^5$ 80. $\frac{12}{5}\pi a^5$ 83. $2r - 2l$

85. $\frac{1}{3}(x^3 + y^3 + z^3) - xyz$ 87. $20a/11$

89. $$\bar{z} = \frac{(m+n)(r+s)}{(2m+n)(2r+s)}\left(\frac{u_1^{2m+n} - u_2^{2m+n}}{u_1^{m+n} - u_2^{m+n}}\right)\left(\frac{v_1^{2r+s} - v_2^{2r+s}}{v_1^{r+s} - v_2^{r+s}}\right)$$

if $(2m + n)(2r + s)(m + n)(r + s) \neq 0$

and $\bar{z} = \dfrac{n(r+s)}{2(2r+s)} \cdot \dfrac{\{\log(u_1/u_2)\}(v_1^{2r+s} - v_2^{2r+s})}{(u_1^{n/2} - u_2^{n/2})(v_1^{r+s} - v_2^{r+s})}$ if $n + 2m = 0$, and

$(2r + s)(r + s) \neq 0$

94. $\sqrt{\left(\frac{a^2}{\pi^2} + \frac{9h^2}{16}\right)}$; arc tan $(4a/3h\pi)$

97. $16a/5\pi$, the cycloid being given by $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

98. $\frac{2}{3}a$, $\frac{1}{2}a$, $\frac{2}{15}a$ 99. $\frac{2}{15}\pi a$, 0, 0 100. $\frac{4}{3\pi}a$, $\frac{4}{3\pi}b$, $\frac{3}{8}c$

108. $a^3(\pi/16 - 1/6)$, $a^3(\pi/16 + 1/6)$ 123. $\dfrac{2M}{a^2}\left\{1 - \dfrac{h}{\sqrt{(a^2 + h^2)}}\right\}$

128. Total number of cases is measured by $\iint dx\,dy$ for $0 < x < 4$, $0 < y < 4$. Number for which $xy < 4$ is given by $\iint dx\,dy$ for $0 < x < 4$, $0 < y < 4$, $0 < xy < 4$, i.e. $4 + \int_1^4\left(\frac{4}{x}\right)dx$

129. Take x to be the greatest part, and y one of the others; the third is $a - x - y < x$. Total number is measured by the quadrilateral $A(\frac{1}{3}a, 0)$, $B(a, 0)$, $C(\frac{1}{2}a, \frac{1}{2}a)$, $D(\frac{1}{3}a, \frac{1}{3}a)$. If c lies between $\frac{1}{3}a$ and a, the line $x = c$ divides the quadrilateral into two parts, one for which $x > c$ and one for which $x < c$. Results (i) 2/3, (ii) 1/16

131. $\frac{1}{3}a$, the mean value of $(a - x - y)$ over the triangle $2x + y = a$, $2y + x = a$, $x + y = a$.

139. $\frac{14}{3}ka^5$, where k is the constant of proportionality. 140. $6ak/5$

141. $1/(n + 1)$, the mean value of x_1 for the region $0 < x_1 + x_2 + \dots + x_n < 1$.

142. $a^2/60$, the mean value of $xy(a - x - y)$ for $0 < x + y < a$.

143. $1/10$, the mean value of x^2 for the volume $0 < x + y + z < 1$.

CHAPTER X

FUNCTIONS OF A COMPLEX VARIABLE. CONTOUR INTEGRALS.

10. Introduction of Complex Numbers. The inadequacy of the real domain of numbers for all the purposes of analysis is most simply shown by considering the general quadratic equation $ax^2 + 2bx + c = 0$; for this equation has no *real* solution when $b^2 < ac$. In solving such an equation as $x^2 + 4x + 13 = 0$, for example, we are led to the formal result $x + 2 = \pm\sqrt{(-9)}$, which is meaningless, if complex numbers have not been defined. If, however, we assume for the moment that there exists a number i obeying the laws of algebra and also the law of multiplication $i \times i = -1$, we can write the above solution in the form $x = -2 \pm 3i$. This suggests that the notion of number may be extended in such a way that it will be consistent to use for its representation the symbol $x + iy$, where x, y are real and where $i \times i$ may be replaced by -1. It is easy then to deduce that the numbers must obey the following rules for addition and multiplication

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$
$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

10.01. Definition of Complex Numbers. A complex number is *defined* to be a number-pair (x, y) (where x, y are *real*), obeying the laws of algebra and the following laws of *addition* and *multiplication*:

Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

Multiplication: $(x_1, y_1) \times (x_2, y_2) = \{(x_1x_2 - y_1y_2), (x_1y_2 + x_2y_1)\}$.

Thus only the notion of real number is used in this definition.

The number $(x, 0)$ is called a *real* number and corresponds to (but is not logically identical with) the real number x of the unextended domain. No ambiguity arises by using the same symbol x for its representation. In particular (from the definition),

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0); \quad (x_1, 0) \times (x_2, 0) = (x_1x_2, 0)$$

and these equations are consistent with the operations in the original domain of real numbers. Again, $k \times (x, y) = (k, 0) \times (x, y) = (kx, ky)$.

In particular $(-1)(x, y) = (-x, -y)$, and it is consistent to write $-(x, y)$ for $(-1)(x, y)$.

The idea of *subtraction* may be included in that of addition by defining $(x_1, y_1) - (x_2, y_2)$ to mean

$$(x_1, y_1) + \{-(x_2, y_2)\} = \{(x_1 - x_2), (y_1 - y_2)\}.$$

Let the number $(0, 1)$ be denoted by i; then $yi = y(0, 1) = (0, y)$, and such a number is said to be *purely imaginary*.

Now $(x, y) = (x, 0) + (0, y) = x + yi$ (or $x + iy$).

Again, $(x, y) \times (x, y) = (x^2 - y^2, 2xy)$ and this number may be written $(x, y)^2$.

It follows therefore that $i^2 = (0, 1)^2 = (-1, 0) = -1$. The use of the symbol $x + iy$, where $i^2 = -1$ is therefore justified.

Notes. (i) If $z = x + iy$, x is called the *real part* of z and written $\mathbf{R}(z)$, whilst y is called the *imaginary part* of z and written $\mathbf{I}(z)$.

(ii) If $x_1 + iy_1 = x_2 + iy_2$, then $x_1 = x_2$, $y_1 = y_2$. This is implied in the definition and may be verified by squaring each side of the relation $x_1 - x_2 = (y_1 - y_2)i$. In particular if $x + iy = 0$, then $x = y = 0$.

(iii) The number $\bar{z} = x - iy$ is called the *conjugate* of z $(= x + iy)$. If $z = 0$, then $\bar{z} = 0$ (and conversely).

(iv) The number $(x_1 + iy_1)/(x_2 + iy_2)$ is defined to be the number $x_3 + iy_3$, if it exists, that satisfies the relation

$$(x_3 + iy_3)(x_2 + iy_2) = (x_1 + iy_1).$$

But if $x_2 + iy_2 \neq 0$, then $x_2 - iy_2 \neq 0$ and therefore

$$(x_3 + iy_3)(x_2^2 + y_2^2) = (x_1 + iy_1)(x_2 - iy_2)$$

i.e. $x_3 + iy_3$ exists and has the value $\dfrac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$ whilst $x_3 + iy_3$ does not exist if $x_2 + iy_2 = 0$. In this way, *division* by a non-zero number is defined.

Example. Express $\dfrac{(2 + i)(3 - 2i)}{(i - 1)(11i + 3)}$ in the form $A + iB$ where A, B are real.

The above number is $\dfrac{8 - i}{-14 - 8i} = -\dfrac{(8 - i)(7 - 4i)}{2(7 + 4i)(7 - 4i)} = -\frac{2}{5} + \frac{3}{10}i$.

10.02. Geometrical Representation of Complex Numbers. A complex number z $(\equiv x + iy)$ obeys the *vector* law of addition, when x, y are regarded as *components*. It may therefore be represented by the displacement-vector $\overrightarrow{OP}$ where (x, y) are the co-ordinates of P referred to (rectangular) axes through O. (*Fig. 1.*)

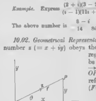


FIG. 1

10.03. Modulus and Amplitude. The length (absolute) of OP is called the *modulus* of z and is usually written $|z|$ or *mod* z. Thus

$$|z| = \sqrt{(x^2 + y^2)} \; (= r).$$

The angle that $\overrightarrow{OP}$ makes with $\overrightarrow{OX}$ is called the *amplitude* (or *argument* or *phase*) of z and is written *amp* z (or *am* z or *arg* z).

If the number $(x + iy)$ is given, the amplitude is many valued, but that value θ that satisfies the inequality $-\pi < \theta \leq \pi$ is often called the *principal value*; and therefore amp z is, in general, equal to $\theta + 2k\pi$ (k being integral ($\pm$) or zero).

The principal value is one of the values of arc tan (y/x) but is precisely given by the equations $\cos\theta = x/r$, $\sin\theta = y/r$. $(-\pi < \theta \leq \pi.)$

Notes. (i) *Fig. 1* is called the *Argand Diagram*. The phrase 'the point z' is used for 'the point P when $\overrightarrow{OP}$ represents z.'

(ii) If the point z moves in a prescribed path in the plane from A to B, and amp z is prescribed at A (say the principal value), then the value of amp z at B is obviously determinate. Thus if P describes a closed curve surrounding O, its amplitude increases by 2π, whilst if it describes a simple curve not containing O within it, its amplitude is unchanged.

(iii) It is sometimes more convenient to take the principal value θ to be the angle that satisfies the inequality $0 \leq \theta < 2\pi$.

10.04. Addition and Subtraction. If P_1, P_2 are the points z_1, z_2, then Q the fourth vertex of the parallelogram determined by OP_1, OP_2 as adjacent sides is the point $z_1 + z_2$ (*Fig. 2.*)

Thus $\overrightarrow{OQ} = z_1 + z_2$ and $\overrightarrow{P_2P_1} = z_1 - z_2$ so that the sum and difference are represented by the diagonals of the parallelogram.

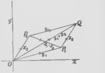


FIG. 2

From the figure we obtain respectively from the inequalities:

$$OQ \leq OP_1 + P_1Q; \quad P_1P_2 \leq OP_1 + OP_2$$
$$OQ \geq OP_1 \sim P_1Q; \quad P_1P_2 \geq OP_1 \sim OP_2$$

the corresponding inequalities:

$$|z_1 + z_2| \leq |z_1| + |z_2|; \quad |z_1 - z_2| \leq |z_1| + |z_2|$$
$$|z_1 + z_2| \geq \big|\,|z_1| - |z_2|\,\big|; \quad |z_1 - z_2| \geq \big|\,|z_1| - |z_2|\,\big|$$

(these of course being variations of the *same* inequality

$$|\alpha| + |\beta| \geq |\alpha - \beta|).$$

10.05. Multiplication and Division. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

i.e. $\quad |z_1z_2| = r_1r_2 = |z_1|\,|z_2|$ and $\operatorname{amp}(z_1z_2) = \theta_1 + \theta_2 + 2k\pi$.

Thus one of the values of amp (z_1z_2) is amp z_1 + amp z_2.

Again, $(z_1/z_2) = (r_1/r_2)(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$, so that

$$|z_1/z_2| = |z_1|/|z_2|$$

and one of the values of amp (z_1/z_2) is amp z_1 − amp z_2.

By repeated applications of these results, we find that

$$\left|\frac{z_1z_2 \ldots z_n}{w_1w_2 \ldots w_m}\right| = \frac{|z_1|\,|z_2| \ldots |z_n|}{|w_1|\,|w_2| \ldots |w_m|}$$

and that one of the values of amp $((z_1z_2 \ldots z_n)/(w_1w_2 \ldots w_m))$ is $\sum_1^n \operatorname{amp} z_r - \sum_1^m \operatorname{amp} w_r$.

Example. Formulae for $\cos(\theta_1 + \theta_2 + \ldots + \theta_n)$, $\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$, $\tan(\theta_1 + \theta_2 + \ldots + \theta_n)$ in terms of the circular functions of $\theta_1, \theta_2, \ldots, \theta_n$ may be deduced from the above result.

For $\cos(\theta_1 + \theta_2 + \ldots + \theta_n) = \mathbf{R}\prod_1^n (\cos\theta_r + i\sin\theta_r)$

and $\sin(\theta_1 + \theta_2 + \ldots + \theta_n) = \mathbf{I}\prod_1^n (\cos\theta_r + i\sin\theta_r)$.

Also
$$\tan(\theta_1 + \theta_2 + \ldots + \theta_n) = \frac{\mathbf{I}\prod_1^n (1 + i\tan\theta_r)}{\mathbf{R}\prod_1^n (1 + i\tan\theta_r)} = \frac{s_1 - s_3 + s_5 \ldots}{1 - s_2 + s_4 \ldots}$$

where s_r is the sum of the products of $\tan\theta_1, \tan\theta_2, \ldots, \tan\theta_n$ taken r at a time.

For example,

$\cos 5A = \mathbf{R}(\cos A + i\sin A)^5 = \cos^5 A - 10\cos^3 A\sin^2 A + 5\cos A\sin^4 A$.

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B}$$

10.06. Demoivre's Theorem. (*Integral Exponent.*) From the previous paragraph we deduce that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, when n is a positive integer. This result is known as *Demoivre's Theorem* (for a positive integer).

When n is not a positive integer, we interpret $(\cos\theta + i\sin\theta)^n$ by means of the Laws of Indices. Thus if $n = -m$, where m is a positive integer,

$$(\cos\theta + i\sin\theta)^n = \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}$$
$$= \cos m\theta - i\sin m\theta$$
$$= \cos n\theta + i\sin n\theta.$$

The theorem is therefore true for a negative integer, and is obviously true for $n = 0$.

Examples. (i) Find the modulus and amplitude of $\dfrac{(2 + i)(3 - i)}{(4 + i)(1 + 2i)}$. Modulus is

$$\frac{\sqrt{5}.\sqrt{10}}{\sqrt{17}.\sqrt{5}} = \sqrt{\left(\frac{10}{17}\right)}$$

Amplitude is $2k\pi + \text{arc tan}\left(\tfrac{1}{2}\right) - \text{arc tan}\left(\tfrac{1}{3}\right) - \text{arc tan}\left(\tfrac{1}{4}\right) - \text{arc tan } 2 = 2k\pi - 69°\ 20'$ (approx.) or, simplifying the original number to $(23 - 61i)/85$, we find that the amplitude is $2k\pi - \text{arc tan}\left(\tfrac{61}{23}\right) = 2k\pi - 69°\ 20'$ (approx.).

(ii) Prove that $2\,\text{arc tan}\left(\tfrac{1}{3}\right) + \text{arc tan}\left(\tfrac{1}{7}\right) = \tfrac{1}{4}\pi$.

The left-hand side is the principal value of amp $(3 + i)^2(7 + i)$, i.e. of amp. $50(1 + i)$ and is therefore $\tfrac{1}{4}\pi$.

(iii) $ir(\cos\theta + i\sin\theta) = r\{\cos(\theta + \tfrac{1}{2}\pi) + i\sin(\theta + \tfrac{1}{2}\pi)\}$.

Therefore, multiplication of a complex number by i is equivalent to *rotation* of the vector through 90° in the same direction as that from $\overrightarrow{OX}$ to $\overrightarrow{OY}$.

(iv) If $z_1\ (= x_1 + iy_1)$, $z_2\ (= x_2 + iy_2)$ are the opposite corners of a square, find the co-ordinates of the other corners. The centre of the square is $\tfrac{1}{2}(z_1 + z_2)$ and the given diagonal is represented by $(z_1 - z_2)$. The other corners are therefore given by $\tfrac{1}{2}(z_1 + z_2) \pm i(z_1 - z_2)$.

The other corners are therefore $\tfrac{1}{2}(x_1 + x_2 \mp y_1 \pm y_2)$, $\tfrac{1}{2}(\pm x_1 \mp x_2 + y_1 + y_2)$.

(v) Show that $\cos n\theta - \cos n\alpha = 2^{n-1}\prod_0^{n-1}\left\{\cos\theta - \cos\left(\alpha + \dfrac{2r\pi}{n}\right)\right\}$.

$$\cos n\theta = \mathbf{R}(\cos\theta + i\sin\theta)^n = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots$$

This may be expressed as a polynomial in $\cos\theta$, of degree n, the coefficient of $\cos^n\theta$ being $1 + {}^nC_2 + {}^nC_4 + \ldots$
i.e.
$$\tfrac{1}{2}\{(1+1)^n + (1-1)^n\} = 2^{n-1}.$$

But $\cos n\theta = \cos n\alpha$ when $\theta = \pm\alpha,\ \pm\left(\alpha + \dfrac{2\pi}{n}\right),\ \pm\left(\alpha + \dfrac{4\pi}{n}\right),\ \ldots$ and therefore the different values of $\cos\theta$ are included in the set $\cos\left(\alpha + \dfrac{2r\pi}{n}\right)$, ($r = 0$ to $n-1$), which are all different (except possibly when $\cos r\alpha = \pm 1$).

Thus $\cos n\theta - \cos n\alpha = 2^{n-1}\displaystyle\prod_{0}^{n-1}\left\{\cos\theta - \cos\left(\alpha + \frac{2r\pi}{n}\right)\right\}$.

(vi) Show that $\sin\dfrac{\pi}{n}\sin\dfrac{2\pi}{n}\ldots\sin\dfrac{(n-1)\pi}{n} = \dfrac{n}{2^{n-1}}$.

From *Example* (v) above, when $\alpha \to \theta$, we find
$$\lim_{\alpha\to\theta}\frac{\cos n\theta - \cos n\alpha}{\cos\theta - \cos\alpha} = \frac{n\sin n\theta}{\sin\theta} = 2^{2n-2}\prod_{1}^{n-1}\sin\frac{r\pi}{n}\sin\left(\theta + \frac{r\pi}{n}\right).$$

Let $\theta \to 0$, then $n^2 = 2^{2n-2}\displaystyle\prod_{1}^{n-1}\sin^2\left(\frac{r\pi}{n}\right)$, from which the result follows since $\sin\dfrac{r\pi}{n} > 0$ ($r = 1$ to $n-1$).

(vii) Express $\sin^6\theta\cos^4\theta$ and $\sin^3\theta\cos^2\theta$ as linear combinations of the sines and cosines of multiples of θ.

Let $z = \cos\theta + i\sin\theta$, then $2\cos\theta = z + 1/z$, $2i\sin\theta = z - 1/z$. Also
$$2\cos n\theta = z^n + z^{-n} \text{ and } 2i\sin n\theta = z^n - z^{-n}.$$

Thus
$$(2^6 i^6\sin^6\theta)(2^4\cos^4\theta) = \left(z - \frac{1}{z}\right)^6\left(z + \frac{1}{z}\right)^4$$
$$= (z^{10} + z^{-10}) - 2(z^8 + z^{-8}) - 3(z^6 + z^{-6}) + 8(z^4 + z^{-4}) + 2(z^2 + z^{-2}) - 12$$
i.e. $512\sin^6\theta\cos^4\theta = 6 - 2\cos 2\theta - 8\cos 4\theta + 3\cos 6\theta + 2\cos 8\theta - \cos 10\theta$

Similarly, $16\sin^3\theta\cos^2\theta = 2\sin\theta + \sin 3\theta - \sin 5\theta$.

10.061. Demoivre's Theorem. (*Rational Exponent.*) From the laws of indices, the function $z^{1/m} = w$ is interpreted as one that satisfies the relation $w^m = z$ (m is a positive integer).

Let $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\phi + i\sin\phi)$. Then
$$\rho^m(\cos m\phi + i\sin m\phi) = r(\cos\theta + i\sin\theta).$$
Since ρ is real and positive, $\rho = r^{1/m}$; and we have also $m\phi = 2k\pi + \theta$.

Thus *one* value of ϕ is θ/m and there are $(m-1)$ other values that lead to *different* values of w; these may be taken as $(\theta + 2k\pi)/m$ ($k = 1$ to $m - 1$). *One* value of $z^{1/m}$ is therefore
$$r^{1/m}(\cos(\theta/m) + i\sin(\theta/m))$$
and there are $(m-1)$ other values obtained by adding multiples (integer) of $2\pi/m$ to the amplitude. The function $z^{1/m}$ must therefore be regarded as m-valued, unless its amplitude is specified. It is convenient in practice, however, to regard the symbol $r^{1/m}$ (when r is real and positive) as one-valued and as meaning the *real positive* value.

More generally, if p, q are integers, prime to each other, and $q > 0$, the function $z^{p/q}$ is $r^{p/q}(\cos p\theta + i\sin p\theta)^{1/q}$ and therefore *one* of its values is $r^{p/q}\{\cos(p\theta/q) + i\sin(p\theta/q)\}$; it has $(q-1)$ other values obtained by adding multiples of $2\pi/q$ to the amplitude of the above value. The

symbol $r^{p/q}$ is in this case used for the determinate number (real and positive) $1/(r^{-p/q})$ if $p < 0$.

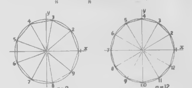
Taking $r = 1$, we can therefore say that one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ (*n rational*).

Note. When α is irrational, $(\cos\theta + i\sin\theta)^\alpha$ may be defined as the limit, if it exists, of $(\cos\theta + i\sin\theta)^{\alpha_n}$ where α_n is any sequence of rational numbers that tends to α. This has an infinite number of values given by
$\lim\{\cos\alpha_n(\theta + 2k\pi) + i\sin\alpha_n(\theta + 2k\pi)\}$, i.e. $\cos\alpha(\theta + 2k\pi) + i\sin\alpha(\theta + 2k\pi)$ since $\cos\phi$, $\sin\phi$ are continuous functions of ϕ.

When such a function occurs, however, we are usually concerned with the particular value $\cos\alpha\theta + i\sin\alpha\theta$, and in any case, it is more appropriate to use the exponential (and logarithmic) function in its expression.

10·062. The n nth Roots of Unity. The roots of the equation $z^n = 1$ are called the n nth roots of unity. By the previous paragraph, they are

$$\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n} \quad (r = 0 \text{ to } n - 1)$$



$z^n = 1$

FIG. 3

i.e. $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ where $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. They are given by the vertices of a regular polygon of n sides inscribed in the circle $x^2 + y^2 = 1$. (*Fig. 3.*) It is useful to note that

$$1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} = 0.$$

This follows from the fact that the coefficient of z^{n-1} in the equation $z^n - 1 = 0$ is zero (or because $(\alpha^n - 1)/(\alpha - 1) = 0$, $\alpha \neq 1$, or by the vector law of addition).

The three cube roots of unity are often denoted by $1, \omega, \omega^2$ where

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \omega^2 = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$

i.e. $\omega, \omega^2 = \frac{1}{2}(-1 \pm i\sqrt{3})$ and $1 + \omega + \omega^2 = 0$

Note. If γ is any root of the equation $z^n = \beta$, then the other roots are $\gamma\alpha$, $\gamma\alpha^2, \dots, \gamma\alpha^{n-1}$ where $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Examples. (i) The three roots of $z^3 + a^3 = 0$ are $-a$, $-a\omega$, $-a\omega^2$.
(ii) Solve the equation $z^4 + 1 = 0$.

One root is $\cos \pi/4 + i \sin \pi/4$, i.e. $(1 + i)/\sqrt{2}$, and so the four roots are given by $\pm (1 \pm i)/\sqrt{2}$.

(iii) Solve the equation $z^6 = 1 - i\sqrt{3}$.

$1 - i\sqrt{3} = 2(\cos \pi/3 - i \sin \pi/3)$. The roots have modulus $2^{\frac{1}{6}}$ and their amplitudes may be taken as $-\pi/18 + r\pi/3$, ($r = 0$ to 5). Thus, the six roots may be written

$\pm 2^{\frac{1}{6}}(\cos \pi/18 - i \sin \pi/18)$, $\pm 2^{\frac{1}{6}}(\sin \pi/9 - i \cos \pi/9)$, $\pm 2^{\frac{1}{6}}(\sin 2\pi/9 + i \cos 2\pi/9)$

(iv) Solve the equation $z^4 + (z + 2)^4 = 0$.

Here $\dfrac{z + 2}{z} = \dfrac{\pm 1 \pm i}{\sqrt{2}}$ (*see Example* (ii) above).

The roots are $-1 \pm (1 \pm \sqrt{2})i$.

10.07. Sequences of Complex Numbers. If x_n, y_n are convergent sequences of *real* numbers that tend respectively to a, b then the sequence of complex numbers z_n ($= x_n + iy_n$) is said to tend to the limit $a + ib$.

This is consistent with the definition of limit for real sequences, since it may be shown that the necessary and sufficient condition that z_n should tend to a limit is that $|z_m - z_n|$ should be ultimately small. The condition is necessary because $|z_m - z_n| < |x_m - x_n| + |y_m - y_n|$ and it is sufficient because $|x_m - x_n| < |z_m - z_n|$ and $|y_m - y_n| < |z_m - z_n|$. Thus if $z_n \to \alpha$, then, given ε (> 0) we can find n_0 such that $|z_n - \alpha| < \varepsilon$ for all $n > n_0$; i.e. *all* the points z_n for $n > n_0$ are within the circle centre α and radius ε.

10.08. The Sequence z^n. If $z = r(\cos \theta + i \sin \theta)$,

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

Therefore $z^n \to 0$ when $|z| < 1$ (since $|\cos n\theta|$, $|\sin n\theta|$ are $\leqslant 1$). If $|z| > 1$, z^n oscillates infinitely (except when $\theta = 0$ or $2k\pi$, when z is real and positive and $z^n \to +\infty$). If $|z| = 1$, z^n oscillates finitely on the unit circle $|z| = 1$ (except when $\theta = 0$ or $2k\pi$, when $z^n \to 1$).

Notes. (i) If $|z_n|$ is bounded and z_n does not tend to a limit, the sequence z_n is said to oscillate finitely. If $|z_n|$ is unbounded, then z_n, in general, will oscillate infinitely. If, however, amp z_n tends to a definite value α, we may say that z_n tends to ∞ in the direction α. For example if $z = c + iy_n$, where c, y_n are real and y_n tends to $+\infty$, we may say that z_n tends to ∞ in the direction of the y-axis and write $z_n \to c + i\infty$.

(ii) If $z_n \to \alpha$, $z_n' \to \beta$, $z_n'' \to \gamma$, . . ., then by proofs similar to those for real sequences, we deduce that $R(z_n, z_n', z_n'' \ldots) \to R(\alpha, \beta, \gamma, \ldots)$ where R denotes a finite number of fundamental operations on its arguments and where there is no division by zero.

10.1. Functions of a Complex Variable. When x, y are continuous real variables, we may regard z ($= x + iy$) as the *continuous complex variable* since $z \to a + ib$ when x, y tend continuously to a, b respectively. A function w of the form $u(x, y) + iv(x, y)$ is a function of z (in the general sense), since, when z is given, x and y are known and w is determined. Actually, however, as we shall see later, the function that is important in this theory is a particular type of this general class. It should be noted that for the general class, w is a *continuous* function of z if u, v are continuous functions of x, y (continuity being defined in an

obvious way). For if $u \to \alpha$, $v \to \beta$ when $x \to a$, $y \to b$, then

$$w \to \alpha + i\beta$$

when $z \to a + ib$.

Thus $(x + iy)^2$, $x - iy$, $\sqrt{(x^2 - y^2)}$, $\sqrt{|xy|}$ are all continuous functions of z for all x, y.

Also the fundamental theorems on *limits* of functions of a real variable may be extended to functions of a complex variable. More definitely, if $w_1 \to \rho$, $w_2 \to \sigma$ when $z \to z_0$, then (by the method used for real variables), $w_1 + w_2 \to \rho + \sigma$, $w_1 w_2 \to \rho\sigma$, $(w_1/w_2) \to (\rho/\sigma)$ $(\sigma \neq 0)$, when $z \to z_0$.

10.11. The Polynomial and Rational Function. A function of the form $a_0 z^n + a_1 z^{n-1} + \ldots + a_n$, where n is a positive integer and $a_0, a_1, \ldots, a_n$ are complex is called a *Polynomial* of degree n in the complex variable z. It is obviously continuous for all values of z.

A function $R(z)$ of the form $P(z)/Q(z)$ where $P(z)$, $Q(z)$ are polynomials is called a *Rational Function* of z and is continuous for all values of z except those that make $Q(z)$ zero.

Example. The polynomial $1 + nz + \frac{n(n-1)}{1.2} z^2 + \ldots + z^n$ is equal to $(1 + z)^n$, the proof (by induction, for example), being the same as for the real variable.

10.12 Series of Complex Numbers. If $w_n = u_n + iv_n$, where u_n, v_n are real, then $S_n \left(= \sum_1^n w_r\right) = \sum_1^n u_r + i\sum_1^n v_r$, and therefore S_n converges to $u + iv$, if $\sum_1^n u_r$, $\sum_1^n v_r$ converge respectively to u, v, and we may write $S = \sum_1^\infty w_r = u + iv$.

It is necessary and sufficient for convergence that, given $\epsilon\ (> 0)$, we can find a suffix n_0 such that

$$\left|\sum_{m+1}^{m+p} w_r\right| < \epsilon \text{ for all } m > n_0 \text{ and all positive integers } p$$

10.13. Absolute Convergence of Complex Series. Since

$$\left|\sum_{m+1}^{m+p} w_r\right| \leq \sum_{m+1}^{m+p} |w_r|,$$

the convergence of $\sum_1^\infty |w_r|$ implies that of $\sum_1^\infty w_r$. In such a case $\sum_1^\infty w_r$ is said to be *absolutely* convergent. The convergence of $\sum_1^\infty |w_n|$ implies that of $\sum_1^\infty |u_n|$ and $\sum_1^\infty |v_n|$ where $w_n = u_n + iv_n$ (and conversely): and therefore the value of $\sum_1^\infty w_n$, when the series is absolutely convergent, is independent of the order of summation of the terms.

Example. $1 + \frac{\alpha.\beta}{1.\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}z^2 \ldots$ (*Hypergeometric Series*)

Here $\left|\frac{w_n}{w_{n+1}}\right| = \left|\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)}\right| \cdot \frac{1}{|z|}$ which $\longrightarrow \frac{1}{|z|}$ as $n \longrightarrow \infty$.

There is *absolute* convergence when $|z| < 1$ (γ not being a negative integer). If $|z| > 1$, the series cannot be convergent since the nth term does not tend to zero.

If $$|z| = 1, \left|\frac{w_n}{w_{n+1}}\right| = \left|1 + \frac{1+\gamma-\alpha-\beta}{n} + \frac{k_n}{n^2}\right|, \ (k_n \text{ bounded})$$

$$= 1 + \frac{\rho}{n} + \frac{A_n}{n^2}, \ (A_n \text{ bounded})$$

where $\rho = \mathbf{R}(1+\gamma-\alpha-\beta)$.

There is, therefore, absolute convergence when $|z| = 1$ if $\mathbf{R}(\gamma-\alpha-\beta) > 0$ and the series cannot otherwise be *absolutely* convergent. It may be proved (*Chapter XI, § 11.07*) that the series converges (not absolutely) when $0 > \mathbf{R}(\gamma-\alpha-\beta) > -1$ except when $z = 1$ and that the series does not converge when $\mathbf{R}(\gamma-\alpha-\beta) < -1$.

10.14. Power Series. (Complex Variable.) Let

$$F(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

Suppose that $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists and has the value R. The series is absolutely convergent if $|z| < R$ and is not convergent when $|z| > R$. Thus $F(z)$ is defined at all points within the circle $|z| = R$ and possibly for points on this circle. The circle is called the *Circle of Convergence* and R the Radius of Convergence.

Note. The series need not converge at any point of the circle. It may converge absolutely at *all* points of the circle (e.g. $\Sigma z^n/n^2$); and series can be constructed that converge (not absolutely) at all points of the circle. (*Pringsheim, Math. Ann., Bd 25, (419)*)

In many cases a_n/a_{n+1} can be expressed in the form

$$(1 + \mu/n - k_n/n^2)\rho$$

(ρ, μ independent of n and $|\rho| = R$), where k_n is bounded, from which it follows that $|a_n/a_{n+1}|$ is of the form $(1 + \sigma/n + A_n/n^2)R$ where A_n is bounded and $\sigma = \mathbf{R}(\mu)$. There is therefore absolute convergence for $|z| = R$ when $\mathbf{R}(\mu) > 1$.

Notes. (i) It may be shown (*Chapter XI, § 11 10*) that there is convergence (not absolute) when $0 < \mathbf{R}(\mu) < 1$ on $|z| = R$ except when $z = R$ and that the series is not convergent on $|z| = R$ when $\mathbf{R}(\mu) < 0$.

(ii) In many cases a_n is real and the problem of convergence is simplified. Suppose for simplicity that $R = 1$ (a case to which the general case is reduced ($R \neq 0$ or ∞) by the substitution $z = R\zeta$).

When $|z| = 1$, $z = \cos\theta + i\sin\theta$, and the series becomes

$$(\Sigma a_n \cos n\theta) + i(\Sigma a_n \sin n\theta).$$

This converges when both the series $\Sigma a_n \cos n\theta$, $\Sigma a_n \sin n\theta$ converge. Thus there is *absolute* convergence when Σa_n is absolutely convergent. More generally, there is convergence (not necessarily absolute) when $a_n \longrightarrow 0$ ($\theta \neq 0$ or a multiple of 2π); if $\theta = 0$, $\Sigma a_n \cos n\theta$ converges or diverges with Σa_n, whilst $\Sigma a_n \sin n\theta = 0$. (*Chapter XI, § 11 08.*)

Many of the properties of power series in the *real* variable can be extended to power series in the complex variable, appropriate modifications being made in the meanings of the terms employed. The pro-

perties involving integration will be given later in this chapter (§ *10.13 et foll.*), but in the meantime we note the following

I. $\sum\limits_0^\infty a_n z^n$ is a *continuous* function of z within its region of convergence

II. $\left(\sum\limits_0^\infty a_n z^n\right) \times \left(\sum\limits_0^\infty b_n z^n\right) = \sum\limits_0^\infty (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0) z^n$ at least when z is within the region of convergence of both the series $\Sigma a_n z^n$, $\Sigma b_n z^n$.

III. $\lim\limits_{z \to z_0} \sum\limits_0^\infty a_n z^n = \sum\limits_0^\infty a_n z_0^n$ when the series on the right converges where $|z_0| = R$ and $z \to z_0$ along a radius (*Abel's Theorem, simplified*). The properties I, II may be proved by the same method as that used for the real variable, and property III in this simple form is an immediate consequence of Abel's Theorem for the real variable.

For let $z = t(\cos\alpha + i\sin\alpha)$ where $z_0 = R(\cos\alpha + i\sin\alpha)$, then on the radius through z_0,

$\Sigma a_n z^n = \Sigma a_n(\cos n\alpha + i\sin n\alpha)t^n$. ($t$ real with $0 < t < R$).

But it is given that $\Sigma a_n \cos n\alpha\, R^n$ and $\Sigma a_n \sin n\alpha\, R^n$ are convergent. Therefore by Abel's Theorem for the real variable, it follows that

$$\Sigma a_n z^n \to \Sigma a_n z_0^n$$

when $t \to R$.

Notes.—(i) The result may be proved true when $z \to z_0$ by any path in the circle that lies between two chords passing through z_0, e.g. when the path cuts the circle at z_0 at a finite angle. (*See Picard, Traité d'Analyse, II, 73; Titchmarsh, Theory of Functions, 7.6.*)

(ii) The converse of Abel's Theorem, viz. that if $\Sigma a_n z^n \to s$ as $z \to z_0$ along a suitable path (where $|z_0| = R$), then $\Sigma a_n \to s$ is not true in general; but this converse is true if (i) $a_n = o(1/n)$ (*Tauber*) (ii) $a_n = O(1/n)$ (*Littlewood*).

10.15. Derivatives. If $\lim\limits_{\delta z \to 0} \dfrac{f(z + \delta z) - f(z)}{\delta z}$ exists, the limit is called the derivative of $f(z)$ and is written $f'(z)$ or $\dfrac{d}{dz}f(z)$. Thus since

$$(z + \delta z)^n - z^n = nz^{n-1}\,\delta z + O(|\delta z|^2)$$

the derivative of z^n is nz^{n-1} when n is a positive integer

Note. We use the symbol $O(|G(z)|)$ for $F(z)$ when $|F(z)/G(z)|$ is bounded in some neighbourhood.

Now consider the continuous function $f(z) = x^2 + iy^2$

Here $$\frac{f(z + \delta z) - f(z)}{\delta z} = \frac{2x\,\delta x + 2iy\,\delta y + (\delta x)^2 + i(\delta y)^2}{\delta x + i\,\delta y}$$

and this does not tend to a unique limit when δx, δy tend independently to zero.

In particular, if $\delta x = h\cos\alpha$, $\delta y = h\sin\alpha$, then $\delta z \to 0$ when $h \to 0$ along a fixed direction (α constant). The limit is

$$\lim_{h \to 0} \frac{2h(x\cos\alpha + iy\sin\alpha) + O(h^2)}{h(\cos\alpha + i\sin\alpha)} = \frac{2(x + iy\tan\alpha)}{1 + i\tan\alpha}$$

which depends on the value of α

It is therefore indicated by the above that the type of function that is important in this theory is one that possesses a derivative independent of the way in which $\delta z \to 0$. It is obvious that such a function as a polynomial has this property, since a polynomial consists of certain simple operations on the variables x, y when these variables occur only in the particular combination $x + iy$. The property itself is, however, independent of the way in which the function possessing it is expressed or determined and provides a suitable definition of the class of such functions.

10.16. Analytic Functions. (*Cauchy.*) The previous paragraph suggests the definition of a class of functions and these are called *analytic*. If w is a function of z such that $\lim \delta w/\delta z$ exists at a given point when $\delta z \to 0$ in any way, the function w is said to be *analytic* at that point; and if w is analytic at every point of a given domain D, it is said to be analytic throughout D.

Let $w = f(z)$ and let z, z_0 be two points of the domain in which $f(z)$ is defined. If w is analytic at z_0, $\dfrac{f(z) - f(z_0)}{z - z_0}$ tends to a limit which may be denoted by $f'(z_0)$ as $z \to z_0$; i.e. given ε (> 0), we can find δ (> 0) such that

$$f(z) - f(z_0) = (z - z_0)f'(z_0) + \lambda(z - z_0) \text{ and } |\lambda| < \varepsilon$$

for all points z within the circle $|z - z_0| = \delta$.

10.17. Elementary Analytic Functions. The rules for the differentiation of $w_1 \pm w_2$, $w_1 w_2$, w_1/w_2 obviously apply to functions w_1, w_2 of the complex variable; and in particular the derivative of z^n is proved to be nz^{n-1} (at least for n integral) by the same method as that used for the real variable. Thus the rational function is analytic except for those values that make a denominator vanish; and its derivative is obtained by the application of the above rules and by the use of the derivative of z^n. Again, a function defined by a power series is analytic for the interior of its circle of convergence, and its derivative is obtained by differentiating the series term-by-term. It has been shown in the proof for power series in the real variable that if $F(x) = \sum\limits_0^\infty a_n x^n$ then

$$F(x_0 + h) = F(x_0) + hF_1(x_0) + \ldots + \frac{h^r}{r!}F_r(x_0) + \ldots$$

where $F_r(x_0)$ is the series obtained by differentiating $F(x)$ r times term-by term with respect to x, and where $|h| < R - |x|$, and that by using the property of continuity, $F_r(x_0) = F^{(r)}(x_0)$. The same proof is applicable to the series $F(z) = \sum\limits_0^\infty a_n z^n$ so that not only is $F(z)$ analytic within its circle of convergence but it possesses analytic derivatives of all orders and is capable of expansion in an infinite Taylor series at $z = z_0$ given by

$$F(z) = F(z_0) + (z - z_0)F'(z_0) + \ldots + \frac{(z - z_0)^r}{r!}F^{(r)}(z_0) + \ldots$$

this series being convergent at least within the circle determined by $|z - z_0| = R - |z_0|$.

Thus the rational function and functions given by power series are not only analytic within a certain domain; they possess analytic derivatives of all orders within that domain. This property is true of all analytic functions, but to prove it generally we must introduce the notion of *integration*.

10.2. Contours. The integral of a function of a complex variable is an integral along a curve on the x–y plane, but we shall deal here only with simple curves of an elementary type.

It will be assumed, therefore, that (i) when the curve is given parametrically by the equations $x = x(t)$, $y = y(t)$ ($t_0 \leq t \leq T$) the functions $x(t)$, $y(t)$ are continuous and possess derivatives $x'(t)$, $y'(t)$ which are continuous except possibly at a finite number of points where the discontinuities are finite; (ii) two different values of t do not lead to the same point (x, y); (iii) the curve can be divided up into a finite number of parts in each of which y (or x) can be expressed as a continuous function $y(x)$ (or $x(y)$) possessing a derivative $y'(x)$ ($x'(y)$) continuous except at a finite number of points when the discontinuities are finite.

If $x(t_0) = x(T)$ and $y(t_0) = y(T)$, the curve is *closed*, thus providing a single exception to assumption (ii) above. In this theory, it is usual to call a closed simple curve a closed *Contour*, but this must not be confused with the other use of the term as a *level curve*. We shall regard it as obvious that a closed contour divides the points of the x–y plane into three categories (i) those on the curve, (ii) a set forming the *interior*, (iii) a set forming the *exterior*; and if z_0 is interior to the curve, the change in amp $(z - z_0)$ when z describes the curve once in the counter clockwise direction is 2π, whilst if z_0 is exterior, the change is zero.

The curve we consider here therefore does not cross itself, has a finite length, covers zero area and encloses a finite area, and may have a finite number of corners.

We shall find, however, in subsequent applications that only arcs of circles or segments of straight lines are used to form the contours and therefore no difficulties relating to the general theory of curves need arise.

10.21 The Process of Dissection for an Area. Suppose that $f(x, y)$ possesses a property in the domain consisting of a contour C and its interior A. Let the domain be divided into any two parts and suppose that the property is of such a kind that it must be satisfied by $f(x, y)$ for at least one of the parts. Then by a process of subdivision analogous to the process of bisection for an interval, it is possible to find one point of the domain in the neighbourhood of which $f(x, y)$ possesses the property.

A neighbourhood of an interior point $P(x_0, y_0)$ may be defined as the domain specified by $|x - x_0| < \delta$, $|y - y_0| < \delta$ ($\delta > 0$), and δ can be taken sufficiently small (but not zero) to ensure that all points of this neighbourhood belong to the domain. A neighbourhood of a point $P(x_0, y_0)$ on the boundary C may be defined as the domain common to

the given domain and that specified by $|x - x_0| < \delta$, $|y - y_0| < \delta\ (\delta > 0)$; and (since a line parallel to an axis meets C in a finite number of points), δ can be taken sufficiently small to ensure that this neighbourhood is a single area (i.e. is bounded by a simple contour).

Take a square whose sides are parallel to the axis and is such that C is interior to it (*Fig. 4*); the sides of the square being given by $x = a_1$, A_1, $y = b_1$, B_1, $(A_1 - a_1 = B_1 - b_1 = c > 0)$. Divide this square into 4 quarter squares by the lines $2x = a_1 + A_1$, $2y = b_1 + B_1$. Then, c having been chosen sufficiently small, the lines of subdivision divide A into a finite number of parts, and $f(x, y)$ must possess the property in

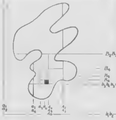


FIG. 4

one of these sub-regions (including its boundary). Let the quarter square in which such a sub-region lies be specified by $x = a_2$, A_2; $y = b_2$, B_2 $(A_2 - a_2 = B_2 - b_2 = c/2)$. Divide this quarter square in 4 quarters and let the process be continued. After $(n - 1)$ steps in this continued subdivision, we have a square specified by $x = a_n$, A_n; $y = b_n$, B_n $(A_n - a_n = B_n - b_n = c/2^{n-1})$ and this square contains a finite number of sub-regions (belonging to the original domain), within one of which at least $f(x, y)$ possesses the property. The monotones a_n, A_n obviously tend to the same limit x_0, and the monotones b_n, B_n to the same limit y_0. The point (x_0, y_0) is interior (in the broad sense) to every selected quarter-square, and ultimately every selected quarter-square is interior to C, if (x_0, y_0) is interior to C (since a line parallel to an axis meets C in a finite number of points). Thus, when (x_0, y_0) is interior to C, a neighbourhood (complete) of (x_0, y_0) exists in which the property is satisfied. Similarly, if (x_0, y_0) is on C, a neighbourhood (partial) if (x_0, y_0) exists for which the function possesses the property.

10.22. Uniform Differentiability. If a function $f(z)$ is analytic in a domain D and z_0 is any point of D, then given ε (> 0), we can find δ (> 0) such that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon \text{ in } |z - z_0| < \delta.$$

The value of δ that is suitable at z_0 will not, however, be suitable, in general, for *all* points of D. We can, however, show that a value of δ (> 0) exists that is suitable for *all* points of D. An analytic function is therefore said to be *uniformly* differentiable in D. Let D be divided into a finite number of parts $D_1, \ldots, D_m$. If δ_r is suitable for all the points of D_r ($r = 1$ to m), then $\min(\delta_r)$ is obviously suitable for all points of D. Suppose that δ does not exist for D, then by the process of dissection, there exists a point z_0 in the neighbourhood of which a δ cannot be found. But the neighbourhood of z_0 can be chosen sufficiently small to lie entirely within any circle $|z - z_0| = \delta$ and we thus arrive at a contradiction.

10.23. Conjugate Functions. Let $w = f(z)$ be analytic and let its derivative be $f'(z)$. Then w may be expressed in the form $u + iv$, where u, v are functions of x, y. The functions u, v are called *Conjugate*. Now $\delta w = f'(z)\,\delta z + \lambda$ where $|\lambda| < \varepsilon|\delta z|$ in the neighbourhood of a point z.

Let $f'(z) = U + iV$ where U, V are functions of x, y.

Then $\delta u = U\,\delta x - V\,\delta y + \rho$; $\delta v = V\,\delta x + U\,\delta y + \sigma$

where $|\rho| = |\mathbf{R}(\lambda)| \leq |\lambda| < \varepsilon|\delta z|$

and $|\sigma| = |\mathbf{I}(\lambda)| \leq |\lambda| < \varepsilon|\delta z|$.

Thus u, v are differentiable functions for which

$$u_x = U = v_y \text{ and } u_y = -V = -v_x$$

The equations $u_x = v_y$; $u_y = -v_x$ are called the *Riemann-Cauchy* conditions.

Conversely, if u, v are differentiable functions in a domain D, such that $u_x = v_y$ and $u_y = -v_x$ then $u + iv$ ($= w$) is an analytic function of z. For $\delta u = u_x\,\delta x + u_y\,\delta y + o(|\delta z|)$, $\delta v = v_x\,\delta x + v_y\,\delta y + o(|\delta z|)$ and therefore $\delta u + i\,\delta v = (u_x + iv_x)(\delta x + i\,\delta y) + o(|\delta x + i\,\delta y|)$

i.e. $$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y} \text{ exists and has the value}$$

$$u_x + iv_x = -i(u_y + iv_y)$$

It follows from Cauchy's Theorem (proved later) that u, v possess derivatives for all orders; and if we assume this result for the moment, we find that

$$u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}.$$

Thus u, v are solutions of Laplace's equation $\nabla^2 V = 0$ and are therefore called *Harmonic Functions* (in two dimensions).

If one of the functions is known, the other may be determined (except for an arbitrary constant): for

$$u = \int(u_x\,dx + u_y\,dy) = \int(v_y\,dx - v_x\,dy).$$

For example, $v = 2xy$ satisfies the equation $\nabla^2 v = 0$, and therefore $u = \int (2x\,dx - 2y\,dy) = x^2 - y^2 + c$.

The gradient m_1 of the curve $u(x, y) =$ constant, at the point (x, y), is given by $m_1 = -u_x/u_y$; and the gradient m_2 of $v(x, y) =$ constant, at (x, y), is given by $m_2 = -v_x/v_y$. But $m_1 m_2 = u_x v_x/u_y v_y = -1$ and therefore the curves $u(x, y) =$ constant, $v(x, y) =$ constant, if they intersect in real points, do so at right angles.

Conjugate functions are important in applications where it is required to determine solutions of Laplace's equation $\nabla^2 V = 0$, satisfying certain boundary conditions. For example, if u is a harmonic function $V\ (= u - c)$ is a solution that vanishes on the boundary $u = c$, and if u were a *potential function*, $v =$ constant would represent a *line of force*.

10.3. Complex Integration. Let the equations giving a simple curve of an elementary type connecting two points $A(z_0)$, $B(Z)$ be

$$x = x(t),\ y = y(t).$$

Suppose that $\dot{x}\left(= \frac{dx}{dt}\right)$, $\dot{y}\left(= \frac{dy}{dt}\right)$ are continuous in $t_0 \leq t \leq T$ (except possibly at a finite number of corners), where $x_0 = x(t_0)$, $y_0 = y(t_0)$, $X = x(T)$, $Y = y(T)$, $z_0 = x_0 + iy_0$, and $Z = X + iY$. It can be assumed that the parameter t is chosen so that $T - t_0$ is finite.

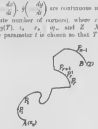


FIG. 5

The line integral $\int_A^B (P\,dx + Q\,dy)$ where P, Q are continuous functions of x, y is defined to be $\int_{t_0}^T (P\dot{x} + Q\dot{y})dt$. If R, S are two other continuous functions of x, y, the expression $\int_A^B (P + iR)dx + (Q + iS)dy$ is defined to be $\int_A^B (P\,dx + Q\,dy) + i\int_A^B (R\,dx + S\,dy)$ and is called a *complex integral*. The expression $\int_{z_0}^Z f(z)dz$, hitherto undefined, is defined to be

$\int_A^B (u + iv)(dx + i\,dy)$, i.e. $\int_A^B (u\,dx - v\,dy) + i\int_A^B (v\,dx + u\,dy)$ where $f(z)$ is continuous and equal to $u + iv$.

To justify this notation, let the interval (t_0, T) be divided into n sub-intervals by the values $t_1, t_2, \ldots, t_{n-1}$ where

$$t_0 < t_1 < t_2 < \ldots < t_{n-1} < T$$

and let P_r correspond to the value t_r, with $P_0 = A$, $P_n = B$. Let $z_r = x_r + iy_r$. Also let t_r' be *any* point in the interval $t_r < t < t_{r+1}$ and $z_r' = x_r' + iy_r'$ be the corresponding point P_r.

Suppose, for the moment, that $\dot{x}$, $\dot{y}$ are continuous functions of t in $t_0 < t < T$. They are therefore uniformly continuous, and given $\varepsilon_1\ (> 0)$, the interval (t_0, T) can be divided into n parts at t_r such that in every sub-interval

$$|\dot{x}(t_r'') - \dot{x}(t_r')| < \varepsilon_1$$

where t_r', t_r'' are any two values in the sub-interval $t_r < t < t_{r+1}$. By the mean value theorem, $x_{r+1} - x_r = \{\dot{x}(t_r'')\}(t_{r+1} - t_r)$ where t_r'' is some value in the interval $t_r < t_r'' < t_{r+1}$ and therefore

$$x_{r+1} - x_r = \{\dot{x}(t_r') + \lambda_r\}(t_{r+1} - t_r)$$

where $|\lambda_r| < \varepsilon_1$ and t_r' is any point of the interval $t_r < t < t_{r+1}$. Similarly $y_{r+1} - y_r = \{\dot{y}(t_r') + \mu_r\}(t_{r+1} - t_r)$ where $|\mu_r| < \varepsilon_1$ (the number of sub-intervals being finitely increased, if necessary, to ensure that $|\dot{y}(t_r'') - \dot{y}(t_r')| < \varepsilon_1$).

Consider the sum $S_n = \sum_0^{n-1} (z_{r+1} - z_r) f(z_r')$.

Let $u_r' = u(x_r', y_r')$ and $v_r' = v(x_r', y_r')$; then

$$S_n = \sum_0^{n-1} \{(x_{r+1} - x_r) + i(y_{r+1} - y_r)\}(u_r' + iv_r') = E_n + F_n$$

where $E_n = \sum_0^{n-1} \{\dot{x}(t_r') + i\dot{y}(t_r')\}(u_r' + iv_r')(t_{r+1} - t_r)$

and $F_n = \sum_0^{n-1} (\lambda_r + i\mu_r)(u_r' + iv_r')(t_{r+1} - t_r)$.

If the number of sub-intervals $(t_{r+1} - t_r)$ tends to infinity in such a way that $\max(t_{r+1} - t_r)$ tends to zero, then

$$E_n \to \int_{t_0}^T (u + iv)(\dot{x} + i\dot{y})\,dt$$

i.e. to $\int_{z_0}^Z f(z)\,dz$.

Also $|F_n| < 2M\varepsilon_1(T - t_0)$ where $M = \max|f(z)|$ on the curve. The function $f(z)$ is continuous and therefore M is finite and so $F_n \to 0$,

i.e.
$$\sum_0^{n-1}(z_{r+1} - z_r)f(z'_r) \to \int_{z_0}^{Z} f(z)dz$$

thus justifying the use of the symbol on the right.

We infer also that in the continued subdivision, an integer n_0 exists such that for a given ε (> 0), the inequality

$$\left|\int_{z_0}^{Z} f(z)dz - \sum_0^{n-1}(z_{r+1} - z_r)f(z'_r)\right| < \varepsilon \text{ for all } n \geq n_0.$$

Since $\int_{z_0}^{Z} f(z)dz$ has been defined in terms of ordinary integrals it follows that $\int_A^B f(z)dz = \int_A^C f(z)dz + \int_C^B f(z)dz$ where C is any point of the curve AB, and in particular $\int_A^B f(z)dz = -\int_B^A f(z)dz$.

Also (i) for any curve AB, $\int_{z_0}^{Z} dz = \lim \sum_0^{n-1}(z_{r+1} - z_r) = Z - z_0$.

(ii) for any curve AB,

$$\int_{z_0}^{Z} z\,dz = \lim \Sigma(z_{r+1} - z_r)z_{r+1} = \lim \Sigma(z_{r+1} - z_r)z_r$$

taking z'_r successively at the ends of the interval,

i.e.
$$\int_{z_0}^{Z} z\,dz = \tfrac{1}{2}\lim \Sigma(z_{r+1} - z_r)(z_{r+1} + z_r) = \tfrac{1}{2}(Z^2 - z_0^2).$$

(iii) $\int_C \frac{dz}{z}$ where C is the circle $x^2 + y^2 = R^2$. Take $x = R\cos t$, $y = R\sin t$ where t varies from 0 to 2π; then

$$\dot{x} + i\dot{y} = R(-\sin t + i\cos t) = iz$$

and the integral is $\int_0^{2\pi} i\,dt = 2\pi i$. It is the same for every circle whose centre is O.

Notes. (i) These results are unaffected when the curve AB has a finite number of corners at the points C_r ($r = 1$ to m), since we may define

$$\int_A^B f(z)dz \text{ as } \int_A^{C_1} f(z)dz + \sum_1^{m-1}\int_{C_r}^{C_{r+1}} f(z)dz + \int_{C_m}^{B} f(z)dz$$

(ii) When the curve is a closed contour (C), the integral round this contour is written $\int_C f(z)dz$. This, however, does not indicate the *direction* in which the contour is described, but we shall always assume, unless otherwise indicated, that the direction is counter-clockwise (i.e. in the same sense as the direction from OX to OY). When there is any likelihood of ambiguity, we can use the notation $\circlearrowleft_C f(z)dz$ for counter-clockwise description and $\circlearrowright f(z)dz$ for clockwise.

10.31. An Upper Bound to the Modulus of a Complex Integral. The length l of the arc of the curve AB has been defined (*Chap. IX*) as the limit of the sum of the lengths of the chords $AP_1, P_1P_2, \ldots, P_{n-1}B$ when n tends to infinity in such a way that every $(t_{r+1}-t_r)$ tends to zero; this limit has been shown to be equal to

$$\int_{t_0}^{T}\sqrt{(\dot{x}^2+\dot{y}^2)}\,dt$$

Thus
$$\lim\sum_{0}^{n-1}|z_{r+1}-z_r| = l.$$

Then
$$\left|\int_{z_0}^{Z}f(z)dz\right| = \left|\lim\sum_{0}^{n-1}(z_{r+1}-z_r)f(z_r)\right|$$

$$\leq M\lim\sum_{0}^{n-1}|z_{r+1}-z_r|,$$ where M is the upper bound of $|f(z)|$ on AB,

i.e.
$$\left|\int_{z_0}^{Z}f(z)dz\right| \leq Ml$$

10.32. Cauchy's Theorem. This is the fundamental theorem of the subject and is usually stated in the following form:

If $f(z)$ is analytic and one-valued inside and on a contour C then

$$\int_C f(z)dz = 0.$$

Let a square be drawn, with its sides parallel to the axes, and containing C entirely within it. Let this square be divided into n^2 smaller equal squares by equidistant lines parallel to the axes. (*Fig 6.*) If n

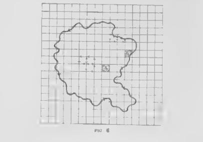


FIG 6

is large enough, these lines of subdivision divide the domain into m smaller squares S_r $(r = 1$ to $m)$ and p irregular areas T_s $(s = 1$ to $p)$, bounded by parts of the subdividing lines and by parts of C. Also, since the contour is elementary, no small square contains more than one irregular area, if n is taken large enough. Let S_r, T_s denote the boundaries of the sub-regions. Then $\int_C f(z)dz = \sum_{r=1}^{m}\int_{S_r} f(z)dz + \sum_{s=1}^{p}\int_{T_s} f(z)dz$,
because in the total summation on the right the integrals along the parts of the subdividing lines that occur in S_r and T_s appear twice and are described once in each direction.

Now since $f(z)$ is uniformly differentiable, a number δ (> 0) exists, independent of (x', y'), any point of the domain, such that

$$\left|\frac{f(z) - f(z')}{z - z'} - f'(z')\right| < \varepsilon$$

for all z within the circle $|z - z'| = \delta$.

The greatest distance between any two points of S_r or T_s is $\leq c\sqrt{2}/n$, where c is the length of the side of the large square, and therefore n can be chosen sufficiently large to ensure that $|z - z'| < \delta$ for every two points z, z' of S_r or T_s. Thus if z' is a point within S_r,

$$\int_{S_r} f(z)dz = \int_{S_r} \{f(z') + (z - z')f'(z') + \lambda\}dz, \text{ where } |\lambda| < \varepsilon|z - z'|$$

i.e.
$$\int_{S_r} f(z)dz = \int_{S_r} \lambda\, dz, \text{ since } \int_{S_r} dz = 0 = \int_{S_r} z\, dz.$$

Now $|\lambda| = \varepsilon|z - z'| \leq \varepsilon c\sqrt{2}/n$, and the length of S_r is $4c/n$,

i.e.
$$\left|\int_{S_r} f(z)dz\right| \leq 4\varepsilon c^2\sqrt{2}/n^2.$$

Similarly
$$\int_{T_s} f(z)dz = \int_{T_s} \lambda\, dz$$

where $|\lambda| \leq \varepsilon c\sqrt{2}/n$ and the length of T_s is $\leq (l_s + 4c/n)$, l_s being that part of C that belongs to T_s,

i.e.
$$\left|\int_{T_s} f(z)dz\right| \leq \frac{\varepsilon c\sqrt{2}}{n}\left(l_s + \frac{4c}{n}\right).$$

Thus
$$\left|\int_C f(z)dz\right| \leq \frac{4\varepsilon c^2\sqrt{2}}{n^2}(s + m) + \frac{\varepsilon c\sqrt{2}}{n}\Sigma l_s$$

i.e.
$$< 4\varepsilon c^2\sqrt{2} + \varepsilon c\sqrt{2}l/n, \text{ since } s + m < n^2$$

where l is the length of the contour C. Since ε is *any* number (> 0), however small, the value of $\int_C f(z)dz$ must be zero.

Notes. (i) It is *sufficient* for the truth of Cauchy's Theorem that $f(z)$ should be analytic inside C and *continuous* merely on C. For suppose that C is such that every line through some point O interior to C meets C in two points only (on opposite ?

sides of the point). Let $0 < k < 1$: then as z describes C, kz describes a contour C_1 entirely within C, if the origin is taken at O.

Then $$\int_C f(z)dz - \int_{C_1} f(z)dz = \int_C \{f(z) - f(kz)k\}dz.$$

But k can be taken sufficiently near 1 to ensure that $|f(z) - f(kz)| < \varepsilon$ for all z on C, since $f(z)$ is continuous on (and within) C.

i.e. $$\left|\int_C f(z)dz - \int_{C_1} f(z)dz\right| = \left|k\int_C (f(z) - f(kz))dz - (1-k)\int_C f(z)dz\right|$$
$$< k\varepsilon l + (1-k)Ml^{\dagger}$$

where l is the length of C and $M = \max|f(z)|$ on C

i.e. $$\lim_{k \to 1} \left[\int_C f(z)dz - \int_{C_1} f(z)dz\right] = 0.$$

But $\int_{C_1} f(z)dz = 0$, since $f(z)$ is analytic inside and on C_1 and therefore $\int_C f(z)dz = 0$.

The result may be extended to a contour, the interior of which can be divided up into a finite number of parts bounded by contours similar to that used in the proof.

(ii) Cauchy's Theorem may be proved by Green's Theorem in two dimensions, if we assume that $f'(z)$ is continuous, i.e. that u_x, u_y, v_x, v_y are continuous.

For the line integral
$$\int_C (u + iv)(dx + i\,dy)$$
is equal to the double integral
$$\iint \{i(u_x + iv_x) - (u_y + iv_y)\}dx\,dy = 0$$
by the Cauchy-Riemann conditions.

This proof (called Riemann's) assumes more than is necessary, although we shall prove (by means of Cauchy's Theorem) that the derivative $f'(z)$ actually is continuous (and analytic) in the domain.

10.33. Multiple Contours. The integral round a contour that crosses itself (*Fig. 7* (*i*), (*ii*), (*iii*), (*iv*)) or round the *boundary* of an area, that

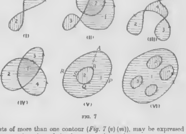


FIG. 7

consists of more than one contour (*Fig. 7* (*v*) (*vi*)), may be expressed as a linear combination of simple contour integrals.

In the examples illustrated, areas bounded by simple contours are marked with the numerals 1, 2, 3, . . . and the multiple closed curve is assumed to be described in a *given* direction (indicated by an arrow). If C_r is the boundary of area r, then we can express $\int f(z)dz$ round the curves shown in *Fig.* 7 (*i*), (*ii*), (*iii*), (*iv*) in terms of simple contour integrals as follows:

(i) $\int_C f(z)dz = \int_{C_2} f(z)dz + \int_{C_1} f(z)dz$

(ii) $\int_C f(z)dz = \int_{C_1} f(z)dz - 2\int_{C_2} f(z)dz + \int_{C_3} f(z)dz$

(iii) $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz$

(iv) $\int_C f(z)dz = 2\int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz$

and if $f(z)$ is analytic inside or in any area r, the corresponding integral is zero.

Now consider an area bounded externally by a simple contour C_1 and internally by another contour C_2 (*Fig.* 7 (*v*)); and suppose $f(z)$ is single-valued (but not necessarily analytic) in this domain (including C_1, C_2). Join a point A of C_1 to a point B of C_2 by a line lying within the domain. Let P, R be any two other points on C_1 and Q, S any two other points on C_2, where $\overrightarrow{PAR}$, $\overrightarrow{QBS}$ are counter-clockwise. Let z_0 be any point between C_1, C_2; then if z describes the single contour $PABQSBARP$ in this order amp $(z - z_0)$ *increases* by 2π, and therefore this is the correct description for the corresponding contour integral. The integral round this contour may therefore be written

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz$$

since

$$\int_A^B f(z)dz + \int_B^A f(z)dz = 0.$$

If then $f(z)$ is analytic on C_1, C_2 and in the area between, we have

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

This result is important for the *evaluation* of the contour integral $\int_{C_1} f(z)dz$, when $f(z)$ is not analytic at all points within C_1; for we can choose C_2 to be a simple curve (a circle for example) and evaluate the integral $\int_{C_2} f(z)dz$ which is equivalent if $f(z)$ is analytic in the area between C_1, C_2.

Similarly, if there are n contours $C_1, \ldots, C_n$ within a given contour C, and $f(z)$ is analytic between these and C then

$$\int_C f(z)dz = \sum_1^n \int_{C_r} f(z)dz. \quad (Fig\ 7\ (rt.))$$

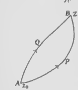


FIG. 8

10.34. *The Indefinite Integral.* Consider the integral $\int_{z_0}^{Z} f(z)dz$, where $f(z)$ is analytic in a domain D and the path of integration is a curve APB lying in D. (*Fig. 8.*) This integral may be regarded as a function of its upper limit Z which may be denoted by $F(Z)$. Let AQB be another path joining AB lying in D. Then $APBQA$ is a simple contour and therefore by Cauchy's Theorem.

$$\int_{\overrightarrow{APB}} f(z)dz - \int_{\overrightarrow{AQB}} f(z)dz = 0$$

i.e. $F(Z) = \int_{z_0}^{Z} f(z)dz$ along *any* path joining AB that lies in D.

Now
$$F(Z + \delta Z) - F(Z) = \int_{Z}^{Z+\delta Z} f(z)dz.$$

But since $f(z)$ is continuous, δZ can be chosen sufficiently small to ensure that $|f(z) - f(Z)| < \varepsilon$ for all values of z in the region $|z - Z| \leq |\delta Z|$.

Thus $F(Z + \delta Z) - F(Z) = f(Z)\delta Z + \lambda$, where $|\lambda| < \varepsilon|\delta Z|$.

i.e.
$$\lim \frac{F(Z + \delta Z) - F(Z)}{\delta Z}$$

exists when $\delta Z \to 0$ and its value is $f(Z)$. Therefore, with a change of notation the integral $F(z) = \int_{z_0}^{z} f(t)dt$ is an analytic function of z whose derivative is $f(z)$, any path of integration being drawn in the domain within which $f(z)$ is analytic. Now the only analytic function w that satisfies the relation $dw/dz = 0$ is a constant since $u_x = v_x = u_y = v_y = 0$. Therefore $\int_{z_0}^{z} f(z)dz = G(z) - G(z_0)$ since the integral vanishes when $z = z_0$, where $G(z)$ is *any* function whose derivative is $f(z)$.

A point where $f(z)$ ceases to be analytic is called a *singularity*. If one path of integration can be deformed into another without crossing a singularity, the corresponding integrals are equal; but if there is a singularity in the domain bounded by the two paths, the integrals are, *in general*, different. The value of the integral $\int_{z_0}^{z} f(z)dz$ is therefore in general many-valued, but ($f(z)$ being single-valued) its different values differ by constants (the *periods* of the integral). In evaluating an integral, the relationship between the route and the disposition of the singu-

larities must be prescribed in order to give a definite result; and although a particular functional value (or branch) may be chosen for $G(z)$, it is the difference between the values of that branch that must be evaluated as z describes the prescribed path.

Examples. (i) Find $\int_1^z z^n\,dz$ when n is an integer positive or negative but not equal to -1.

$n > 0$. z^n is analytic all finite z.

Therefore $\int_1^z z^n\,dz = (z^{n+1} - 1)/(n+1)$ for all paths.

$n \leq -2$; z^n is analytic all finite z except $z = 0$.

But $\frac{d}{dz}(z^{n+1}) = (n+1)z^n$ $(z \neq 0)$ and z^{n+1} is single valued.

Thus $\int_1^z z^n\,dz = (z^{n+1} - 1)/(n+1)$ provided the path does not pass through O.

Therefore if n is an integer, positive, negative or zero but not -1, $\int_C z^n\,dz = 0$ except that when n is negative, C must not pass through O.

(ii) Find $\int_C \frac{dz}{z}$ where C does not pass through O.

If O is exterior to C, $\int_C \frac{dz}{z} = 0$ by Cauchy's Theorem.

If O is interior, $\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z}$ where C_1 is any circle centre O
$= 2\pi i$ (§ *10.3*).

Similarly $\int_C \frac{dz}{z-a} = 0$ if the point a is exterior to C and its value is $2\pi i$ if a is interior.

10.4. Functions expressed as Contour Integrals. Let C be a simple contour drawn in a domain D within which $f(z)$ is analytic, and let a be any point within C.

Then $\int_C \frac{f(z)dz}{(z-a)} = \int_{C_1} \frac{f(z)dz}{z-a}$ where C_1 is a circle centre a and radius ρ lying in D, since C can be deformed into C_1 without crossing the point a, the only singularity of the integrand.

Since $f(z)$ is continuous, ρ can be chosen sufficiently small to ensure that $|f(z) - f(a)| < \varepsilon$ at all points of C_1.

i.e. $\int_C \frac{f(z)dz}{z-a} = \int_{C_1} \frac{f(a)+\lambda}{(z-a)}\,dz$, where $|\lambda| < \varepsilon$ on C_1

But $$\int_{C_1} \frac{f(a)dz}{z-a} = 2\pi i f(a) \quad (\S\ 10.34)$$

and $$\left|\int_{C_1} \frac{\lambda\,dz}{z-a}\right| < \varepsilon\left|\int_{C_1} \frac{dz}{z-a}\right| < 2\pi\varepsilon.$$

Thus $\int_{\Gamma} \frac{f(z)dz}{z-a} = 2\pi i f(a) + \mu$, where $|\mu| < 2\pi\epsilon$

i.e. $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)} \quad \textit{(Cauchy's Integral.)}$$

10.41. Derivatives of Analytic Functions. Let C be the contour of the previous paragraph and let a, $a+h$ be two neighbouring points within C. Then

$$f(a+h) - f(a) = \frac{1}{2\pi i}\int_C \left(\frac{1}{z-a-h} - \frac{1}{z-a}\right) f(z)dz.$$

The identity $(z-a)^2 \equiv (z-a-h)(z-a+h) + h^2$ gives

$$\frac{1}{z-a-h} - \frac{1}{z-a} = \frac{h}{(z-a)^2} + \frac{h^2}{(z-a)^2(z-a-h)}$$

and therefore $$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2} + I$$

where $$I = \frac{h}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2(z-a-h)}.$$

The points a, $a+h$ are within C and therefore $|z-a|$ has a lower bound $\delta\,(>0)$ and $|z-a-h|$ a lower bound $\delta - |h|$. Thus if max $|f(z)|$ on C is M

$$|I| < \frac{Ml|h|}{2\pi\delta^2(\delta - |h|)},$$ and therefore $I \to 0$ when $h \to 0$

Thus $$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}.$$

Again, differentiating the identity

$$\frac{1}{(z-a-h)} - \frac{1}{(z-a)} = \frac{h}{(z-a)^2} + \frac{h^2}{(z-a)^2(z-a-h)}$$

with respect to z, we obtain

$$\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2} = \frac{2h}{(z-a)^3} + h^2R_1(z)$$

where $|R_1(z)|$ is obviously bounded on C. And therefore

$$\lim_{h\to 0}\frac{1}{h}\int_C \left(\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2}\right)f(z)dz = 2\int_C \frac{f(z)dz}{(z-a)^3}$$

i.e. $$f''(a) = \lim_{h\to 0}\frac{f'(a+h)-f'(a)}{h} = \frac{1}{2\pi i h}\int_C \left(\frac{f(z)}{(z-a-h)^2} - \frac{f(z)}{(z-a)^2}\right)dz$$

$$= \frac{2}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^3}.$$

It may be noted therefore that $f'(a)$, $f''(a)$ are obtained by differentiating the integrand with respect to a.

Let us assume that $f^{(n)}(a)$ is obtained in this way, i.e. that

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}.$$

Taking the nth derivative with respect to z of the identity given above, we obtain

$$\frac{1}{(z-a-h)^{n+1}} = \frac{1}{(z-a)^{n+1}} + \frac{(n+1)h}{(z-a)^{n+2}} + h^2R_n(z)$$

where $|R_n(z)|$ is obviously bounded on C, so that $\left|\int_C f(z)R_n(z)dz\right|$ is finite.

Thus
$$\begin{aligned} f^{(n+1)}(a) &= \lim_{h\to 0} \frac{f^{(n)}(a+h) - f^{(n)}(a)}{h} \\ &= \lim_{h\to 0} \frac{n!}{2\pi i h}\int_C \left(\frac{f(z)}{(z-a-h)^{n+1}} - \frac{f(z)}{(z-a)^{n+1}}\right)dz \\ &= \lim_{h\to 0} \frac{n!}{2\pi i}\int_C \left(\frac{n+1}{(z-a)^{n+2}} + hR_n(z)\right)f(z)dz \\ &= \frac{(n+1)!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+2}} \end{aligned}$$

That the formula is correct follows by induction.

10·42. Taylor's Expansion for an Analytic Function. Let $f(z)$ be analytic inside and on a simple contour C and let a be a point within C whose distance from the nearest point of C is δ (> 0).

For any point z within C

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z}.$$

Now

$$\frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \ldots + \frac{(z-a)^n}{(w-a)^{n+1}} = \frac{1}{(w-z)}\left\{1 - \left(\frac{z-a}{w-a}\right)^{n+1}\right\}$$

But $\int_C \frac{f(w)dw}{(w-a)^{n+1}} = \frac{2\pi i}{n!}f^{(n)}(a)$ and $\int_C \frac{f(w)dw}{w-a} = 2\pi i f(a)$,

i.e.

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots + \frac{(z-a)^n}{n!}f^{(n)}(a) + R_n$$

where
$$R_n = \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(w)dw}{(w-z)(w-a)^{n+1}}.$$

Let $|z-a| = \rho$; then $|w-z| > \delta - \rho$ (> 0) on C and also $|w-a| > \delta$, $|f(w)| \le M$ on C,

i.e. $|R_n| \le \frac{\rho^{n+1}}{2\pi}\cdot\frac{Ml}{(\delta-\rho)\delta^{n+1}}$ which $\to 0$ as $n \to \infty$ since $\rho < \delta$. Thus

$$f(z) = f(a) + (z-a)f'(a) + \ldots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \ldots$$

the series being convergent if $|z-a| < \delta$.

An analytic function is therefore always expansible in an infinite power series in $(z-a)$, when a is a point within the domain of the function. We deduce therefore (i) that the radius of convergence of this

series is the distance from a to the nearest singularity of the function, (ii) an analytic function given by a power series in $(z - a)$ with radius of convergence R must have a singularity on the circle $|z - a| = R$. This, of course, does not mean that the power series is not convergent there.

A function of the real variable need not possess derivatives of all orders and the corresponding expansion in powers of $(x - a)$, considered in Chapter II, is finite, involving the values of $f(a), f'(a), \ldots, f^{(n)}(a)$ and a remainder term involving the $(n + 1)$th derivative. A result corresponding to this for the complex variable has been given by Darboux in which the remainder term is that given by Lagrange (for the real variable) multiplied by λ where $|\lambda| \leq 1$, but the determination of the remainder R_n in this form has not the same importance here since it is sufficient to note that $|R_n| \leq |z - a|^{n+1}M/(n + 1)!$, where $M \to f^{(n+1)}(a)$ as $z \to a$. Thus for a fixed n $|R_n| = O(|z - a|^{n+1})$, (which is true for the real variable when $f^{(n+1)}(a)$ exists); and also $R_n \to 0$ when $n \to \infty$ ($|z - a| < \delta$), (a result that is not necessarily true for functions of a real variable x that possess all derivatives with regard to x at $x = a$).

10.43. Integration of Power Series. Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

have a radius of convergence equal to R.

The series $F(z) = a_0 z + a_1 \frac{z^2}{2} + a_2 \frac{z^3}{3} + \ldots + \frac{a_n z^{n+1}}{n+1} + \ldots$ obtained by integrating term-by-term also defines an analytic function for at least $|z| < R$. But $F'(z) = f(z)$ and therefore

$$\int_0^z f(z)dz = F(z)$$

(since $F(0) = 0$), when the path of integration is any curve within $|z| < R$.

10.44. Cauchy's Inequality for a Power Series. If $M(r)$ is the upper bound of $|f(z)|$ on the circle $|z| = r$ lying within the circle of convergence $|z| = R$ of power series for

$$f(z) = \sum_0^\infty a_n z^n$$

then $|a_n| \leq \dfrac{M(r)}{r^n}$ for all n.

For $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)dz}{z^{n+1}}$, where C is the circle $|z| = r$.

i.e. $$|a_n| < \frac{1}{2\pi}\,\frac{M(r)}{r^{n+1}}\,2\pi r\left(= \frac{M(r)}{r^n}\right).$$

10.45. Liouville's Theorem. If $|f(z)|$ is bounded for all finite z and also as $z \to \infty$, then $f(z)$, if analytic for all finite z, is constant. For $f(z)$ is expressible as a power series $\sum_0^\infty a_n z^n$ for all finite z; and $|a_n| \leq Mr^{-n}$, all r and n, where M is the upper bound of $|f(z)|$ (independent of r). Let $r \to \infty$; then $a_n \to 0$ if $n > 0$ and therefore $f(z)$ is constant.

Notes. (i) It is obviously sufficient that $f(z)$ should be bounded on a sequence of contours that tend wholly to infinity.

(ii) If $|f(z)|$ is analytic for all finite z and $f(z) = O(|z^m|)$ as $|z| \to \infty$, then $f(z)$ is a polynomial of degree $\leq m$.

For $[f(z) - \sum_{0}^{m-1} a_n z^n] \div z^m = a_m + a_{m+1} z + \ldots$ is obviously bounded for all z

and therefore $a_{m+1} = a_{m+2} = \ldots = 0$

i.e.
$$f(z) = a_0 + a_1 z + \ldots + a_m z^m.$$

(iii) Every equation $a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$ has a root (and therefore n roots).

For the polynomial $f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ is analytic for all finite z and if it never vanishes, $|f(z)|$ must have a lower bound $m > 0$; therefore $1/f(z)$ is analytic for all finite z and is bounded (obviously) as $z \to \infty$; i.e. $f(z)$ reduces to a constant and we thus arrive at a contradiction.

If α_1 is a root, then $f(z)/(z - \alpha_1)$ is a polynomial of degree $(n - 1)$ and by continued application of the theorem we deduce that
$$f(z) = a_0(z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n)$$
where the numbers α_r are not necessarily different.

This theorem is sometimes called '*The Fundamental Theorem of Algebra.*'

(iv) Liouville's theorem is important in the theory of elliptic functions, which are defined to be analytic for all z (except for poles (§ *10.48*), and to have two periods $2\omega_1$, $2\omega_2$ (ω_2/ω_1 not real). for all the possible values of an elliptic function must occur in the parallelogram (cell) whose corners are a, $a + 2\omega_1$, $a + 2\omega_2$, $a + 2\omega_1 + 2\omega_2$. To prove that a given relation $E(z) = 0$ is true, it is sufficient therefore to show that $E(z)$ is elliptic and possesses no infinities in a cell.

10.46. Singularities and Zeros. If $f(a) = 0$, $f'(a) \neq 0$, a is called a simple zero of $f(z)$, and if $f(a) = 0 = f'(a) = \ldots = f^{(n-1)}(a)$; $f^{(n)}(a) \neq 0$, a is called a zero (multiple) of order n. In the latter case, the expansion of $f(z)$ at $z = a$ takes the form
$$f(z) = (z - a)^n \{A_0 + A_1(z - a) + A_2(z - a)^2 + \ldots\}.$$
If a is a *singularity* of $f(z)$ and a circle $|z - a| = \rho$ ($\rho > 0$) can be drawn so as to include no other singularity but a, the point a is called an *isolated* singularity. In the next paragraph we obtain the expansion of an analytic function in the neighbourhood of an isolated singularity a.

FIG. 9

10.47. Laurent's Series. Let C_1, C_2 be two concentric circles centre a and radii R_1, R_2 respectively where $R_1 < R_2$ and let C_1, C_2 be both within the domain in which $f(z)$ is analytic except at a. (*Fig. 9.*) Then $f(z)$ is analytic within the ring-shaped region lying between these circles, and also on C_1, C_2. If γ is a circle centre z, lying entirely within this region, and AB a line joining a point A of C_1 to a point B of C_2 not passing through z, the circle γ can be deformed into the contour consisting of C_2 described counter-clockwise, C_1 described clockwise, and AB

described once in each direction (§ *10.33*). Therefore, $f(z)$ being single-valued

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)dw}{w-z} = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)dw}{w-z} - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)dw}{w-z}.$$

By a proof similar to that given for the Taylor expansion, the first integral may be proved to be

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_n(z-a)^n + \ldots$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_2} \frac{f(w)dw}{(w-a)^{n+1}}$ $(n = 0, 1, 2, \ldots)$

In the second integral use the identity

$$1 - \left(\frac{w-a}{z-a}\right)^{n+1} = (z-w)\left\{\frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \ldots + \frac{(w-a)^n}{(z-a)^{n+1}}\right\}$$

since here $|w-a| < |z-a|$

The second integral therefore is

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(z-a)}\left\{1 + \frac{w-a}{z-a} + \ldots + \frac{(w-a)^n}{(z-a)^n} + \frac{(w-a)^{n+1}}{(z-w)(z-a)^n}\right\}dw$$

$$= \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \ldots + \frac{A_n}{(z-a)^n} + E_n$$

where $\qquad A_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_1}(w-a)^{n-1}f(w)dw \ (n = 1, 2, 3, \ldots)$

and $\qquad E_n = \dfrac{1}{2\pi i(z-a)^{n+1}}\displaystyle\int_{C_1}\frac{f(w)(w-a)^{n+1}dw}{(z-w)}.$

Let max $|f(w)|$ on C_1 be M_1, let $|z-a| = \rho\ (> R_1)$.

Then since $|w-a| = R_1$, $|z-w| > \rho - R_1$

Thus $|E_n| < \dfrac{M_1R_1}{\rho - R_1}\left(\dfrac{R_1}{\rho}\right)^{n+1}$ which $\to 0$ as $n \to \infty$ since $R_1 < \rho$

Thus $\qquad f(z) = \displaystyle\sum_1^\infty A_n(z-a)^{-n} + \sum_0^\infty a_n(z-a)^n$

where $A_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_1}(w-a)^{n-1}f(w)dw$; $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_2}\frac{f(w)dw}{(w-a)^{n+1}}$.

Notes. (i) If α is the singularity of $f(z)$ nearest to a, C_1 or C_2 may be taken to be *any* circles of radii $|\alpha - a|$ (and centre a).

(ii) Any simple contour may be taken for C_1 or C_2 that can be deformed into one of these circles without crossing a singularity.

(iii) By writing $A_n = a_{-n}$, the formula for a_n will be correct for all n positive, negative or zero.

10.48. Poles and Essential Singularities. Residue. If

$$A_{n+1} = A_{n+2} = \ldots$$

are all zero, so that $f(z)$ takes the form

$$\frac{A_n}{(z-a)^n} + \frac{A_{n-1}}{(z-a)^{n-1}} + \ldots + \frac{A_1}{(z-a)} + a_0 + a_1(z-a) + \ldots$$

the point a is called a *pole of order n*. If the part involving negative powers of n is infinite, a is called an *essential singularity*.

In the case of a pole, the part $\dfrac{A_n}{(z-a)^n} + \dots + \dfrac{A_1}{(z-a)}$ is called the *principal part* of $f(z)$ at $z = a$, and in all cases, the coefficient A_1 is called the *residue* of $f(z)$ at a since $A_1 = \dfrac{1}{2\pi i}\displaystyle\int_{C_1} f(w)dw$.

10.49. The Residue Theorem. Let C be a simple contour within and on which $f(z)$ is analytic except at a finite number of singularities (isolated) at $a_1, a_2, \dots a_s$. (If there is a finite number only, they must be isolated.)

Draw small closed contours $C_1, C_2, \dots C_s$ enclosing $a_1, a_2, \dots a_s$ so that each contour is external to every other. (*Fig. 10.*) Then the contour C may be deformed into the s contours $C_1, \dots C_s$ (§ *10.33*) and therefore

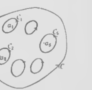


FIG. 10

$$\int_C f(w)dw = \sum_1^s \int_{C_r} f(w)dw = 2\pi i(A_1 + A_2 + \dots + A_s)$$

where A_r is the residue at a_r ($r = 1$ to s). Later in this chapter we shall apply this theorem to the calculation of different types of real integrals.

10.5. Conformal Representation. If $w = u + iv$, we may suppose that u, v are the co-ordinates of a point w which may for convenience be represented on a plane different from the z-plane. Sometimes it may be more useful to mark the point w on the z-plane itself.

If w is a single-valued function, to each point (x, y) there corresponds a single point (u, v) (the converse not being usually true); and we may obtain some idea of the nature of the functional relationship (or transformation) by finding the paths described by (u, v) when (x, y) describes a given path such as a circle or a straight line. Conversely we may consider the path (not usually simple) in the z plane corresponding to a circle or straight line in the w-plane.

Let w_0 correspond to z_0 (i.e. $w_0 = f(z_0)$) and let z_1, z_2 be two points near z_0 with the corresponding values w_1, w_2. If P_r, Q_r denote the points z_r, w_r, the triangle $Q_0Q_1Q_2$ corresponds to the triangle $P_0P_1P_2$. (*Fig. 11.*)

Now $\dfrac{w_1 - w_0}{z_1 - z_0}$ and $\dfrac{w_2 - w_0}{z_2 - z_0}$ both tend to the same limit $\left(\dfrac{dw}{dz}\right)_{z_0}$ when $z_1, z_2 \to z_0$. Thus, near z_0, $\dfrac{w_1 - w_0}{z_1 - z_2}$ is nearly equal to $\dfrac{w_2 - w_0}{z_2 - z_0}$ and, therefore, if $\dfrac{dw}{dz} \neq 0$, $\left|\dfrac{w_1 - w_0}{w_2 - w_0}\right| = \left|\dfrac{z_1 - z_0}{z_2 - z_0}\right|$ and

$$\text{amp}\,(w_1 - w_0) - \text{amp}\,(w_2 - w_0) = \text{amp}\,(z_1 - z_0) - \text{amp}\,(z_2 - z_0)$$

(ignoring terms of the order $|z_1 - z_0|^2$, $|z_2 - z_0|^2$)

In the figure, these results are equivalent to $Q_1Q_0/Q_2Q_0 = P_1P_0/P_2P_0$ and $\angle Q_1Q_0Q_2 = \angle P_1P_0P_2$; i.e. the small triangles $P_0P_1P_2$, $Q_0Q_1Q_2$ are *similar*.

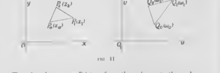


FIG. 11

Thus the relation $w = f(z)$ *transforms* the z plane into the w plane in such a way that corresponding neighbourhoods are similar. A transformation of this kind is said to be *conformal*.

Again let δw correspond to δz; then

$$\delta w = \frac{dw}{dz}\,\delta z + O(|\delta z|^2).$$

Thus $|\delta w|$ is nearly $\left|\frac{dw}{dz}\right| |\delta z|$ and amp $(\delta w) - $ amp $(\delta z) \to$ amp $\left(\frac{dw}{dz}\right)$.

The displacement δw is therefore obtained (approximately) from δz by a *magnification* of amount $|dw/dz|$ and a *rotation* of amount amp (dw/dz).

In particular, it follows that if two curves in the z-plane intersect at an angle α, the corresponding curves in the w-plane intersect at the same angle. For example, the curves $|w| =$ constant in the z-plane are orthogonal to the curves amp $w =$ constant since those curves are respectively circles, centre origin, and the radii of these circles in the w-plane. The curves $|w| =$ constant are appropriately called *level curves* and the orthogonal system amp w are called *Lines of Slope*. We verify also that the curves $u\,\{= \mathbf{R}(w)\} =$ constant are orthogonal to the curves $v\ (= \mathbf{I}(w)) =$ constant, since these are obviously orthogonal in the w-plane.

Notes (i) The conformal representation breaks down at a point where $dw/dz = 0$.

(ii) For any transformation given by $u = u(x, y)$, $v = v(x, y)$ (where u, v are differentiable functions and $J = \dfrac{\partial(u, v)}{\partial(x, y)} \neq 0$), if ds_1 is the element of length in the u-v plane corresponding to ds of the x-y plane

$$ds_1^2 = (u_x^2 + v_x^2)dx^2 + 2(u_x u_y + v_x v_y)dx\,dy + (u_y^2 + v_y^2)dy^2.$$

To secure conformal representation we must have $u_x^2 + v_x^2 = u_y^2 + v_y^2 = \lambda$; $u_x u_y + v_x v_y = 0$ since $ds^2 = dx^2 + dy^2$ and λ is the magnification. Assuming that none of these derivatives vanishes (and so ignoring a trivial solution), we find that if $u_x = \theta v_y$, then $u_y = -\theta v_x$ and $v_y^2(\theta^2 - 1) = v_x^2(\theta^2 - 1)$. The only non-trivial

solutions are given by $\theta = \pm 1$. The solution $\theta = 1$ gives $u_x = v_y$, $u_y = -v_x$, i.e. $w = f(x + iy)$. The solution $\theta = -1$ gives $w = f(x - iy)$. In the former the *direction* of rotation is preserved and in the latter it is reversed. More generally, when a *surface* is transformed into another in such a way that corresponding surface elements are similar, the transformation is called conformal.

10.51. The Polynomial. We have already seen that the polynomial $w = f(z) = a_0z^n + a_1z^{n-1} + \ldots + a_n$ can be expressed in the form $a_0(z - z_1)(z - z_2) \ldots (z - z_n)$, where some of the numbers $z_1, z_2, \ldots z_n$ (zeros) may be equal.

Consider the change in amp w when z describes a simple contour C not passing through any z_r, but containing s zeros within it.

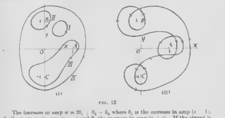
The increase in amp w is equal to the increase in $\sum\limits_1^n \text{amp}\,(z - z_r)$. If z_r is within C, the increase in amp $(z - z_r)$ is 2π and if it is not within C, the increase is zero.

Thus the increase in amp w when z describes C is $2\pi N$, where N is the number of zeros within C.

For a multiple contour (that can be deformed without passing over a zero into a finite number of simpler contours $C_1, C_2, \ldots C_p$) the increase in amp w must be $2\pi N$ where N is an integer ($\pm$) or 0.

Example. $w = 2(z - 1)^2(z^2 + 1)$.

For definiteness, suppose that the initial value of z is z_0 and that z describes a closed path not passing through $A(1)$, $B(i)$, $C(-i)$ (the zeros of w) and return to z_0. (*Fig. 12.*)



(i) (ii)

FIG. 12

The increase in amp w is $2\theta_1 + \theta_2 + \theta_3$ where θ_1 is the increase in amp $(z - 1)$, θ_2 the increase in amp $(z - i)$ and θ_3 the increase in amp $(z + i)$. If the circuit is simple (and counter-clockwise), there are eight possibilities since A, B, C may or may not be enclosed.

Thus in *Fig. 12* (i), for I (none enclosed), the increase is zero; for II (B enclosed), it is 2π; for III (A, C enclosed), 6π; and for IV (all enclosed), 8π. For a circuit that is not simple but equivalent to a finite number of simple circuits ($\pm$), the increase is $2k\pi$ where k is an integer positive or negative, or zero. Thus in *Fig. 12* (ii), the circuit shown is equivalent to a double circuit (+) round B, a single circuit (−) round the *double* zero at A, and a double (+) circuit round C. The total increase is $4\pi - 4\pi + 4\pi = 4\pi$.

10.52. The Disposition of the Zeros of a Polynomial. We have seen that the increase in amp w for a simple counter-clockwise circuit C is $2N\pi$ where N is the number of zeros within C. The following propositions give a rough idea of the disposition of the zeros.

I. If θ_0 is the *positive root* of the equation

$$F(\theta) = |a_0|\theta^n - |a_1|\theta^{n-1} - |a_2|\theta^{n-2} - \ldots - |a_n| = 0$$

all the roots of $w = 0$ lie within or on the circle $|z| = \theta_0$ (i.e. the modulus of every root is $\leq \theta_0$).

By Descartes' Rule of Signs, the equation in θ has only one positive root. That there is at least one, is obvious since $F(\infty)$ is $+$ and $F(0)$ is $-$. Now $w = a_0 z^n(1 + \rho)$ where

$$\rho = \frac{a_1}{a_0 z} + \frac{a_2}{a_0 z^2} + \ldots + \frac{a_n}{a_0 z^n}.$$

The change in amp w is the increase in amp z^n + the increase in amp $(1 + \rho)$. Let z describe the circle $|z| = R$ where $R > \theta_0$.

The change in amp z^n is $2n\pi$.

The change in amp $(1 + \rho)$ is zero if $|\rho| < 1$.

$$\text{But} \qquad |\rho| \leq \frac{|a_1|}{|a_0 z|} + \frac{|a_2|}{|a_0 z^2|} + \ldots + \frac{|a_n|}{|a_0 z^n|} < 1$$

$$\text{if} \qquad a_0 R^n > |a_1|R^{n-1} + \ldots + |a_n|.$$

This is true since $R > \theta_0$.

The number of roots inside is therefore n.

I (*a*). If ϕ_0 is the positive root of the equation

$$G(\phi) = |a_0|\phi^n + |a_1|\phi^{n-1} + \ldots + |a_{n-1}|\phi - |a_n| = 0$$

then all the roots of $w = 0$ lie outside (or on) the circle $|z| = \phi_0$.

This follows from I by writing $z = 1/\zeta$ and considering the equation $\zeta^n f(1/\zeta) = 0$.

II. The change in amp w when z describes an arc θ of the circle $|z| = R$ tends to $n\theta$ as $R \to \infty$. In the notation of I, $w = a_0 z^n(1 + \rho)$, where $\rho \to 0$ as $|z| \to \infty$. Therefore the change in amp $(1 + \rho)$ must tend to zero, i.e. the change in amp w tends to $n\theta$ (the change in amp z^n)

III. If a_r is real (all r) and $p + iq$ is a root of $w = 0$, then $p - iq$ is a root (p, q being real). For if $f(p + iq) = A + iB$ (A, B real), then $f(p - iq) = A - iB$ and if $f(p + iq) = 0$, $A = 0 = B$ and therefore $f(p - iq) = 0$. Thus the imaginary roots of an equation $f(z) = 0$ with *real* coefficients occur in *conjugate pairs.*

Note. A complex number $p + iq$ is often called *imaginary* when $q \neq 0$. It is called *purely imaginary* if $p = 0$, $q \neq 0$.

Examples. (i) The equation $z^6 + 6z^5 - 3000 = 0$.

(*a*) The positive real root of $R^6 - 6R^5 - 3000 = 0$ is easily found to be 6·302 approx., by taking $R = 6 + h$ and using Newton's approximation

(*b*) The real roots of $x^6 + 6x^5 = 3000$ are similarly shown to be 3·183 and $-6·302$ approx.

Therefore the 6 roots all lie between the circles $|z| = 3·18$ and $|z| = 6·31$, two only being real

(*c*) Let $z = iy$ (the imaginary axis $+\infty > y > -\infty$)

The corresponding curve in the $(u\text{-}v)$ plane is $v^6 = -6^6(u + 3000)^5$ described

from (∞, ∞) to $(-\infty, -\infty)$, and a rough sketch of the curve shows that change in amp w is zero.

(d) When z describes the semicircle of $|z| = R$, $\mathbf{R}(z) \geq 0$, the change in amp w as $R \to \infty$ is 6π. There are therefore 3 roots on the right of the y-axis. Thus, since imaginary roots occur in conjugate pairs, there is an imaginary root in each quadrant.

(ii) The equation $w \equiv z^8 + 2iz^5 + i + 1 = 0$.

There are *no* real roots and no roots purely imaginary. The x-axis is transformed into the curve $u = x^8 + 1$, $v = 2x^5 + 1$ (i.e. $2^8(u - 1)^5 = (v - 1)^8$), and the y axis into the line

$$u = y^8 - 2y^5 + 1, \quad v = 1.$$

As y varies from $-\infty$ to 0, u decreases from $+\infty$ to 1 and as y varies from 0 to $+\infty$, u decreases from 1 to $-0{\cdot}09$ (approx.) and then increases to $+\infty$ (*Fig. 13*). The change in amp w when z describes YOX is zero and so the change in amp w when z describes the first (infinite) quadrant is 4π. There are two roots in the first quadrant. Similarly there are two roots in each of the other quadrants.

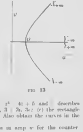


FIG 13

Solving the equations $R^8 = 2R^5 + \sqrt{2}$, $R^8 + 2R^5 = \sqrt{2}$ for the positive roots, we find that the eight roots lie between the circles $|z| = 0{\cdot}8$ and $|z| = 1{\cdot}4$.

(iii) Discuss the change in amp w where $w = z^2 - 4z + 5$ and z describes (a) the square $0, 1, 1 + i, i$; (b) the square $0, 3, 3 + 3i, 3i$; (c) the rectangle $-2i, 3 - 2i, 3 + 3i, 3i$; (d) the circle $|z| = 3$. Also obtain the curves in the w-plane corresponding to these contours.

Since $w = (z - 2 - i)(z - 2 + i)$, the changes in amp w for the counter clockwise circuits are

(a) 0; (b) 2π; (c) 4π; (d) 4π.

These results are verified when we determine the corresponding circuits in the w-plane. (*Fig. 14.*)

Since $u = x^2 - y^2 - 4x + 5$, $v = 2y(x - 2)$, any straight line parallel to an axis in the $(x\text{-}y)$ plane is transformed into a parabola in the $(u\text{-}v)$ plane. The arcs of the parabolas that correspond to the sides of the squares and rectangles are shown in *Figs. 14* (i), (ii), (iii).

With regard to the circle $|z| = 3$, take $z = 3(\cos\phi + i\sin\phi)$ and its transformation is

$$u = 9\cos 2\phi - 12\cos\phi + 5, \quad v = 9\sin 2\phi - 12\sin\phi.$$

Take a new origin at $u = -4$, $v = 0$ and the initial line as $v = 0$. Then the equation in polar co-ordinates will be found to be $r = 18\cos\theta - 12$. (*Fig. 14* (iv).)

10·53. The Rational Function. A rational function w can be written in the form

$$w = A\frac{(z - a_1)(z - a_2)\ldots(z - a_n)}{(z - b_1)(z - b_2)\ldots(z - b_m)}$$

where A, a_r, b_s are constant. A simple closed contour described counter-clockwise produces an increase in amp w of amount $2(k_1 - k_2)\pi$ where k_1 is the number of points a_r enclosed and k_2 is the number of points b_s enclosed.

Example. Let $w = (z+i)/(z-i)$; and let z describe the perimeter of the square whose corners are $(1, 0)$, $(1, 2)$, $(-1, 2)$, $(-1, 0)$. Find the corresponding boundary in the w-plane and verify that the change in amp w is -2π.

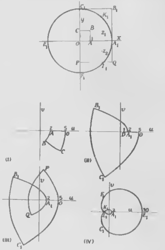


Fig. 14

The point i is within the square but not the point $-i$, and therefore there is a decrease of 2π when z describes the boundary of the square counter-clockwise.

If $w = \dfrac{z+i}{z-i}$, then $z = i\dfrac{u+1+iv}{u-1+iv}$, i.e. $x = -\dfrac{2v}{(u-1)^2+v^2}$, $y = \dfrac{u^2+v^2-1}{(u-1)^2+v^2}$.

The side DOA becomes the arc of the circle $u^2+v^2=1$ from $(0, -1)$ to $(0, 1)$ passing through $(-1, 0)$. Similarly the other three sides become semi-circles as shown in *Fig. 15*.

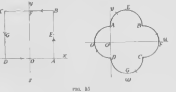


FIG. 15

10.54. The Point at Infinity. Let $w = 1/z$, then $|w| \to \infty$ as $z \to 0$ (in every direction); also as $z \to \infty$ in every direction $w \to 0$ (for $|w| \to 0$). We may therefore regard ∞ as a single point of the Argand Diagram. With this assumption we can give, for descriptive purposes, a convenient representation of the z-plane by means of the surface of a sphere. There are various ways of doing this, but the one chosen here is known as the Stereographic Projection of a spherical surface.

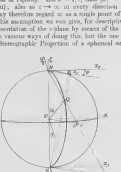


FIG. 16

Take a sphere centre O and radius R referred to rectangular axes Ox, Oy, $O\zeta$, the co-ordinates being x, y, ζ. (*Fig. 16.*) Using an obvious analogy, we may refer to the points $N(0, 0, R)$, $S(0, 0, -R)$ as the north and south poles and $\zeta = 0$ as the equator. Let the co-latitude of a point Q on the sphere be α and let the longitude of Q be θ measured west

of the meridian that passes through $(R, 0, 0)$. Take Oy in the meridian plane $x = 0$ where $\theta = \pi/2$. Let NQ meet $z = 0$ in P. Then

$$OP = ON \tan \angle ONQ = R \cot \tfrac{1}{2}\alpha.$$

The polar co-ordinates of P in xOy are therefore

$$\rho \ (= R \cot \tfrac{1}{2}\alpha),\ \theta.$$

If $z = x + iy$, then $z = R \cot \frac{1}{2}\alpha(\cos\theta + i \sin\theta)$ and the points Q on the sphere are in $1-1$ correspondence with the points of the plane. In particular, N corresponds to ∞ and S to the origin. Circles of latitude, $\alpha =$ constant, are transformed into concentric circles $|z| = R \cot \frac{1}{2}\alpha$ and meridians of longitude $\theta =$ constant into the radii of these circles ($\operatorname{amp} z =$ constant).

If ds_1 is the element of length on the sphere

$$ds_1^2 = R^2(d\alpha^2 + \sin^2\alpha\, d\theta^2)$$

whilst the corresponding element of length ds in the z-plane is given by

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2\, d\theta^2 \\ &= \tfrac{1}{4}R^4 \operatorname{cosec}^4(\tfrac{1}{2}\alpha)\,(d\alpha^2 + \sin^2\alpha\, d\theta^2). \end{aligned}$$

Thus $|ds/ds_1| = \frac{1}{2} \operatorname{cosec}^2(\frac{1}{2}\alpha)$ and since this does not depend on the direction of ds_1 at (α, θ), the transformation is *conformal* with magnification $\frac{1}{2} \operatorname{cosec}^2(\frac{1}{2}\alpha)$.

Note. Let NQ meet the tangent plane at S in P_1 and let SQ meet the tangent plane at N in P_2. Also take $R = \frac{1}{2}$. Regard the tangent plane at S as a z_1-plane in which $\overrightarrow{Ox_1}$ is parallel to $\overrightarrow{Ox}$ and $\overrightarrow{Oy_1}$ parallel to $\overrightarrow{Oy}$, and regard the tangent plane at N as a z_2 plane in which $\overrightarrow{Ox_2}$ is parallel to $\overrightarrow{Ox}$ and $\overrightarrow{Oy_2}$ parallel to $\overrightarrow{yO}$ (not $\overrightarrow{Oy}$). Then $z_1 = 2R \cot \frac{1}{2}\alpha(\cos\theta + i \sin\theta)$, $z_2 = 2R \tan \frac{1}{2}\alpha(\cos\theta - i \sin\theta)$ so that if $R = \frac{1}{2}$, $z_1 z_2 = 1$.

This is Neumann's method of representing the points of a spherical surface on a z_1 plane and illustrates the transformation $z_1 z_2 = 1$.

10.55. Bilinear Transformations (also called Linear). If w is the rational function $(az + b)/(cz + d)$ (where $ad - bc \neq 0$), there is a $1-1$ correspondence between the z-plane and the w-plane, and the transformation is therefore called *bilinear* (or linear). Since there are 3 independent constants, the transformation is made definite if we know three sets of corresponding points. Thus if $z = 0, 1, \infty$ corresponds to $w = -1, 0, 1$, the relation is $w = (z - 1)/(z + 1)$, and, more generally, z_1, z_2, z_3 are transformed into w_1, w_2, w_3 if

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

This transformation has the special property of transforming circles (or straight lines) into circles (or straight lines).

Consider the locus determined by $PA = kPB$ where A, B are fixed points and k is a constant. The locus of P is a circle (if $k \neq 1$) for which A, B are inverse (the circle of Apollonius), and a diameter of this circle is the line joining the points that divide AB internally and externally in the ratio $k : 1$. Any circle, therefore, can be expressed in the form $|(z - z_1)/(z - z_2)| = k$ $(k \neq 1)$. If $k = 1$, the locus of z is the right bisector of the line joining z_1, z_2. If w_1, w_2 are the points in the w-plane,

corresponding to z_1, z_2 in the z-plane, for a bilinear transformation, this relation must be of the form $\frac{w - w_1}{w - w_2} = \lambda \frac{z - z_1}{z - z_2}$ (when w_1, w_2 are in the finite part of the plane).

The circle $\left|\frac{z - z_1}{z - z_2}\right| = k$ transforms into the circle $\left|\frac{w - w_1}{w - w_2}\right| = k|\lambda|$ so that a circle and every pair of inverse points is, in general, transformed into a circle and a pair of inverse points. If one of the points, w_2 say, is at ∞, the transformation must be of the form $w - w_1 = \lambda \frac{z - z_1}{z - z_2}$ so that the circle $\left|\frac{z - z_1}{z - z_2}\right| = k$ with its inverse points z_1, z_2 becomes the circle of centre w_1 and radius $k|\lambda|$.

If in the first case $|\lambda|k = 1$, the circle becomes a straight line.

Examples. (i) If $w = \frac{2z + 3}{z - 4}$, find the transforms of the circles (a) $x^2 + y^2 = 4y$, (b) $x^2 + y^2 = 4x$.

(a) The circle is $|z - 2i| = 2$ and this becomes $\left|\frac{4w + 3}{w - 2} - 2i\right| = 2$

i.e. $$|4u + 2v + 3 + i(2u - 4v - 4)| = 2|u - 2 + iv|$$
or $$16u^2 + 16v^2 + 24u + 44v + 9 = 0$$

(b) The circle is $|z - 2| = 2$ and this gives $|2w + 7| = 2|w - 2|$

i.e. $$(2u + 7)^2 + 4v^2 = 4(u - 2)^2 + 4v^2$$
or the straight line $4u + 3 = 0$.

(ii) Find a relation that transforms the upper half of the z plane into the interior of the circle $|w| = 1$.

As the two corresponding boundaries are described, the rotation from the tangent (in the direction of motion) to the *inward-drawn* normal to corresponding areas must be the same for each plane.

Thus it is sufficient to make the points $z = 0, 1, \infty$ in this order correspond to $1, i, -1$ for w.

Thus $\frac{w - 1}{w + 1} \cdot \frac{i + 1}{i - 1} = z$, i.e. $-i\frac{w - 1}{w + 1} = z$ or $w = -\frac{z - i}{z + i}$.

(iii) Find the general transformation that will make the circle $|w| = B$ correspond to $|z| = A$. Any two inverse points for $|z| = A$ are $a, \frac{A^2}{\bar{a}}$ and the corresponding transformation may be taken as $w = \lambda \frac{z - a}{z - A^2/\bar{a}}$ or $\mu \frac{z - a}{\bar{a}z - A^2}$ (thus allowing for $a = \bar{a} = 0$).

$z = 0$ gives $w = \mu a/A^2$, and $z = \infty$ gives $w = \mu/\bar{a}$.

If these are inverse for $|w| = B$, $|\mu| = BA$, i.e. we may take

$$w = AB\frac{z - a}{\bar{a}z - A^2}(\cos\phi + i\sin\phi).$$

(iv) Consider the simple transformations (a) $w = z + b$, (b) $w = \lambda z$, (c) $w = 1/z$.

(a) $w = z + b$. Here w is obtained from z by a simple *translation* determined by the vector b. Straight lines and circles are unaltered except in position.

(b) $w = \lambda z$. In this case $|w| = |\lambda|\,|z|$ and amp w = amp z + amp λ, i.e. w is obtained from z by a *magnification* of amount $|\lambda|$ and a *rotation* of amount amp λ. A straight line is transformed into another straight line and a circle into another circle

(c) $w = 1/\bar{z}$. Here $|w| \cdot |z| = 1$ and amp $w = -$ amp z, so that w is obtained from z by *inverting* z with respect to the unit-circle $|z| = 1$ and taking the *image* in the x-axis. A straight line or a circle becomes a circle (or a straight line).

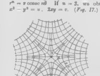
A straight line γ becomes a circle C (or a straight line if γ passes through O). A circle C' becomes a circle C'' (or a straight line if C' passes through O). The *general* transformation $w = \dfrac{az + b}{cz + d}$ is a combination of these three transformations for

$w = z_3 + a/c$, if $z_3 = (bc - ad)z_2/c^2$, $z_2 = 1/z_1$, $z_1 = z + \dfrac{d}{c}$ $(c \neq 0)$ and $w = z_1 + b/d$, where $z_1 = az/d$ when $c = 0$.

10.56. Examples of other Transformations

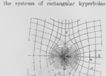
(i) $w = z^n = r^n(\cos n\theta + i \sin n\theta)$

The conjugate systems $u =$ constant, $v =$ constant are given by $r^n = u \sec n\theta$, $r^n = v \operatorname{cosec} n\theta$. If $n = 2$, we obtain the systems of rectangular hyperbolas $x^2 - y^2 = u$, $2xy = v$. (*Fig. 17.*)



$w = z^2$

FIG. 17



$w = z - a^2/z$

FIG. 18

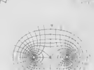


FIG. 19

(ii) $w = a^2/z = a^2(\cos\theta - i \sin\theta)/r$.

The conjugate systems are the two systems of circles

$$ru = a^2 \cos\theta, \quad rv = -a^2 \sin\theta$$

or $u(x^2 + y^2) = a^2x$, $v(x^2 + y^2) = -a^2y$.

We have already seen that circles and straight lines are in general transformed into circles.

(iii) $w = z - a^2/z = (r - a^2/r)\cos\theta + i(r + a^2/r)\sin\theta$.

The systems $u =$ constant, $v =$ constant can be expressed in the form

$$y^2 = \frac{x}{x - u}(x^2 - ux - a^2), \quad x^2 = \frac{y}{y - v}(y^2 - vy - a^2) \quad (\textit{Fig. 18.})$$

The circles $r = \rho$ are transformed into the confocal ellipses

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$$

where $$A = \rho - a^2/\rho, \quad B = \rho + a^2/\rho.$$

Since the circles $r = \rho$, $r = a^2/\rho$ are transformed into the same ellipse, the whole w-plane is represented by the circle $|z| = a$ and its interior (or exterior).

(iv) $w = z^2 - 1$.

Consider the orthogonal systems given by $|w| =$ constant (the level curves), and amp $w =$ constant (lines of slope).

Let A, B be the points $(1, 0)$ and $(-1, 0)$ respectively. (*Fig. 19.*) The curves $|w| =$ constant are given by $AP.BP =$ constant, where P is a variable point. These are *Cassinian Ovals* and are given in Cartesian co-ordinates by the equation

$$(x^2 + y^2 + 1)^2 - 4x^2 = k$$

Their shapes may be determined by using the formula $y = \{(4x^2 + k)^{\frac{1}{2}} - 1 - x^2\}^{\frac{1}{2}}$ and completing by symmetry. When $k > 1$, there is one oval and when $0 < k < 1$, there are two. In the limiting case $k = 1$, the locus is the lemniscate $r^2 = 2 \cos 2\theta$. The lines of slope, amp $(z - 1) +$ amp $(z + 1) =$ constant, are rectangular hyperbolas $x^2 - 2xy \cot \alpha - y^2 = 1$ and they all pass through A, B.

10.57. Saddle Points. The conjugate systems $u =$ constant, $v =$ constant (where $w = f(z) = u + iv$) being orthogonal, one of them, say the former, may be regarded as the level lines of the surface $Z = u(x, y)$ and then the other represents the lines of slopes (or lines of steepest descent). The stationary values of $u(x, y)$ are given by $u_x = 0 = u_y$, and since $u_x = v_y$, $u_y = -v_x$, these equations determine also the stationary values of v. If (x_0, y_0) is a stationary value

$$2\{u(x, y) - u(x_0, y_0)\} = (x - x_0)^2 \frac{\partial^2 u}{\partial x_0^2} + 2(x - x_0)(y - y_0)\frac{\partial^2 u}{\partial x_0 \partial y_0}$$
$$+ (y - y_0)^2 \frac{\partial^2 u}{\partial y_0^2} + O(\delta\rho^3)$$

where $$\delta\rho = |z - z_0|.$$

But $\dfrac{\partial^2 u}{\partial x_0^2} + \dfrac{\partial^2 u}{\partial y_0^2} = 0$ and therefore all the stationary values of a function harmonic in a region D are saddle points. Its maximum or minimum value can occur only on the boundary of D.

Since $f'(z) = u_x - iu_y$, the saddle points are obtained by solving the equation $f'(z) = 0$.

Thus, in the above examples, when (i) $w = z^n$, the only saddle point is $z = 0$, and (ii) when $w = z + a^2/z$, the saddle points are $\pm ia$.

10.58. Residue at Infinity. If $\zeta = 0$ is an isolated singularity of $f(1/\zeta)$ then $z = \infty$ is called an isolated singularity of $f(z)$. Consider the integral $\dfrac{1}{2\pi i}\displaystyle\int_C f(z)dz$, where C is a contour exterior to which $z = \infty$ is the only singularity. A consideration of the representation of the z-plane on the sphere shows that when C is described counter-clockwise to O, it is described clockwise for ∞. For this reason $-\dfrac{1}{2\pi i}\displaystyle\int_C f(z)dz$ is defined to be the residue of $f(z)$ at ∞. Thus the sum of the residues

of a function for all its singularities (if all isolated) including infinity is zero.

Again, a Laurent series exists near $\zeta = 0$ for $f(1/\zeta)$ in the form
$$\ldots + A_n\zeta^{-n} + \ldots + A_1\zeta^{-1} + a_0 + a_1\zeta + \ldots + a_n\zeta^n + \ldots$$
so that the series near $z = \infty$ is
$$\ldots A_n z^n + \ldots + A_1 z + a_0 + a_1/z + \ldots + a_n/z^n + \ldots$$

The residue for ∞ is $\dfrac{1}{2\pi i}\displaystyle\int_C f(z)dz$ which is equal to
$$-\frac{1}{2\pi i}\int_{C_1} f\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta^2}$$
(C_1 being described counter-clockwise for $\zeta = 0$) must therefore be $-a_1$. It should be noted that $\dfrac{1}{2\pi i}\displaystyle\int_C f(z)dz$ where ∞ is the only singularity exterior to C is the coefficient of $1/z$ in the expansion of z near $z = \infty$.

If in the above expansion $A_{n+1} = A_{n+2} = \ldots = 0$, then $z = \infty$ is a pole of order n. If $A_1 = A_2 = \ldots = A_n = \ldots = 0$, ∞ is not a singularity; whilst if also $a_0 = a_1 = \ldots = a_{m-1} = 0$, then the expansion is of the form $a_m\zeta^m + a_{m+1}\zeta^{m+1} + \ldots$ and ∞ is a zero of order m.

10.59. The Zeros and Poles of a Rational Function. Let
$$f(z) = \frac{a_0(z-a_1)(z-a_2)\ldots(z-a_m)}{b_0(z-b_1)(z-b_2)\ldots(z-b_n)} = \frac{P_m(z)}{Q_n(z)}$$
where no a_r is equal to any b_r.

The zeros are $a_1, a_2, \ldots a_m$, in the finite part of the plane and the poles are $b_1, b_2, \ldots b_n$.

If $n > m$, ∞ is a zero of order $n - m$, and if $n < m$, ∞ is a pole of order $m - n$.

If $n = m$, ∞ is neither a zero nor a pole.

Thus in all cases the number of zeros is equal to the number of poles (this number being the degree of the equation $P(z) - cQ(z) = 0$).

If we take account of multiple poles in the expression for $f(z)$, we may write ($b_0 = 1$)
$$f(z) = \frac{P_m(z)}{(z-b_1)^{r_1}(z-b_2)^{r_2}\ldots(z-b_s)^{r_s}}$$
where $r_1 + r_2 + \ldots + r_s = n$. The expression of this in partial fractions gives the principal parts for each of the poles (and ∞ if this is a pole); for this expression is
$$A_p z^p + \ldots + A_0 + \sum_{q=1}^{s}\left(\frac{{}_qA_1}{z-b_q} + \frac{{}_qA_2}{(z-b_q)^2} + \ldots + \frac{{}_qA_{r_q}}{(z-b_q)^{r_q}}\right)$$
the numbers $A_0, \ldots A_p$ being zero if $m < n$ and we verify that the residue at ∞ is $-\sum_1^s({}_qA_1) = -$ (sum of the residues at the other poles).

The only singularities of a Rational Function are poles, and con-

versely if the singularities of an analytic function are poles (for the whole plane including ∞) it must be rational; for when the sum of all the principal parts are subtracted from the function, the remainder must be constant by Liouville's Theorem.

10.6. Algebraic and Transcendental Functions. The *algebraic function* w is one that satisfies an equation reducible to the form

$$P_0(z)w^m + P_1(z)w^{m-1} + \dots + P_m(z) = 0 \; (= F(w, z))$$

where m is a positive integer and P_r a polynomial.

The theory of algebraic functions is beyond our scope but we can give a rough indication of their nature by the consideration of some simple types of explicit functions. When $z = z_0$, there are m values of w, $(w_{01}, w_{02}, \dots, w_{0m})$, some of which may be equal and some may be infinite (when $P_0(z_0) = 0$). A point where two values (at least) are equal is (*in general*) called a *Branch Point*. It can be shown that for the variable z, the equation determines m functional values (or *branches*) $w_1, w_2, \dots, w_m$ that can be expressed as analytic functions in a domain limited by a finite number of isolated singularities. (*Ref Appell and Goursat, Fonctions Algébriques d'une Variable, IV.*) To determine, in practice, the approximate forms for the branches, we can use the *method* of Newton's Polygon. The branch points are formed by solving the equations $F = 0 = F_w$ although every solution of these equations is not necessarily a branch point. The points where a value of w becomes infinite are determined by solving the equation $P_0(z) = 0$, and such a point may or may not be a branch point. We infer therefore that

(i) If $P_0(z_0) \neq 0$ and z_0 is not a branch point. All the branches $w_1, \dots, w_m$ are expressible in power series in $(z - z_0)$, the radius of convergence being the distance of z_0 from the nearest singularity.

(ii) If $P_0(z_0) = 0$, and z_0 is not a branch point. At least one branch has a pole at z_0; and a branch that is not infinite at z_0 is expansible in a power series.

Thus if z_0 is not a branch point, $w_1, \dots, w_m$ have expansions like rational functions.

(iii) If (z_0, w_0) is a 'point' for which $F = 0 = F_w$, the *first* approximations to the branches will be given by a set of relations of the type

$$(w - w_0)^n = A(z - z_0)^s$$

where $n \leqslant m$, s integral $(\pm)$ or zero.

In general, of course, the value z_0 for z will give other values to w besides w_0. Thus for the relation

$$F(w, z) \equiv w^4 + w^3 - 3wz - z^2 = 0$$

the pair of values $z = 0$, $w = 0$ satisfies $F = 0 = F_w$. When $z = 0$, w has a triple root 0 and a single root -1. By using Newton's polygon, we easily find that the approximations for $z = 0$ are

$$w = -\tfrac{1}{3}z^2 + \dots; \quad w^2 = 3z + \dots; \quad w = -1 + 3z + \dots$$

Now consider the relation $w^n = z^s$. Let $z = r(\cos\theta + i\sin\theta)$ where θ is prescribed initially, say the principal value of amp z. Take $n > 2$ and let s be an integer $(\pm)$ or zero.

The values of w are therefore $w_1, w_2, \ldots, w_n$, where

$$w_p = r^{\frac{s}{n}} (\cos \theta_p + i \sin \theta_p)$$

and $$\theta_p = \frac{s\theta}{n} + \frac{2(p-1)\pi}{n}.$$

Let z describe a small closed circuit round $z = 0$; the increase in amp w_p is $2\pi s/n$.

If s is an integral multiple of n (including unity), the values of $w_1, \ldots, w_n$ are *unaltered* by the circuit. Otherwise s is of the form $qn + \alpha$ where q is an integer ($\pm$) or zero and α is one of the integers $1, 2, \ldots, (n-1)$. The description of the circuit therefore changes $w_1, w_2, \ldots, w_n$ into $w_{\alpha+1}, w_{\alpha+2}, \ldots, w_n, \ldots, w_\alpha$. Also, in this particular example, a positive circuit round 0 is equivalent to a negative circuit round ∞. Therefore a positive circuit round ∞ (i.e. clock-wise round a large circle, for example) changes $w_1, w_2, \ldots, w_\alpha, w_{\alpha+1}, \ldots w_n$ into $w_{n-\alpha+1}, w_{n-\alpha+2}, \ldots, w_n, w_1, \ldots, w_{n-\alpha}$. If z_0 is a point for which $F = 0 = F_w$ and a small circuit described by z round z_0 (i.e. one containing no other point for which $F = 0 = F_w$), changes the value of a branch, z_0 is then called a branch point. The branch points must be finite in number (and therefore isolated) and there must be at least *two*. For if z_0 were the only branch point in the finite part of the plane, a circuit round z_0 (which changes therefore two branches at least) is equivalent to a circuit round ∞. Thus ∞ must be a branch point. The aggregate of branches may be called the *algebraic function* w and two methods have been devised for removing the ambiguity that arises in the value of w when z describes all possible paths.

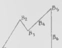


FIG. 20

(*a*) (*Cauchy*). Let the branch points be $B_1, \ldots, B_r$; it is then sufficient to cut the plane along the lines $B_1B_2 \ldots B_r$ (*Fig. 20*), and to regard these lines as impassable barriers. The lines B_sB_{s+1} may be deformed, if necessary, provided they do not pass through some other branch point. On such a plane, the various branches are obviously single-valued. The above method of cutting the plane is not unique, and, as we shall see in the examples below, may be simplified.

(*b*) (*Riemann*). This method consists in representing w on n z-planes (occupying the same position in space, with axes coinciding, but assumed for the moment to be unconnected). Each plane is associated with one of the branches. The planes are cut along lines B_1B_s so that no line B_1B_s contains any branch point other than B_1 or B_s. One edge of the cut B_1B_s in any particular plane is connected to the opposite edge of the corresponding cut in another plane, the connexion being made to secure the correct interchange of the branches; i.e. in such a way that, for example, when the variable point z (with value w_1) meets the line B_1B_s, the point passes into the plane appropriate for the point B_s. It

is simpler, for descriptive purposes, to take the spherical representation of the m planes. We have then m spherical surfaces, coinciding in space, but connected only along the branch lines B_1B_s. Such a surface is, in general, not simply-connected. (*Ref. Appell and Goursat, Fonctions Algébriques.*)

Examples. (i) $w^3 = z^7$. $z = 0$ and $z = \infty$ are branch points.

$$w_p = r^{7/3}\left[\cos\left\{\frac{7\theta}{3} + \frac{2(p-1)\pi}{3}\right\} + i\sin\left\{\frac{7\theta}{3} + \frac{2(p-1)\pi}{3}\right\}\right],\ p = 1, 2, 3$$

i.e. amp $w_1, w_2, w_3 = \frac{7}{3}\theta,\ \frac{7}{3}\theta + \frac{2}{3}\pi,\ \frac{7}{3}\theta + \frac{4}{3}\pi$.

One circuit round O changes w_1, w_2, w_3 into w_2, w_3, w_1 and three circuits restore their values.

To make the branches single-valued, we can draw a semi-infinite line through O (e.g. the positive half of the x-axis).

If the relation had been $w^3 = z^8$, with a corresponding notation, one circuit changes w_1, w_2, w_3 into w_3, w_1, w_2.

(ii) $w^{12} = z^9$.

$$w_p = r^{3/4}[\cos x_p + i\sin x_p] \text{ where } x_p = \tfrac{3}{4}\theta + \tfrac{1}{6}(p-1)\pi,\ (p = 1 \text{ to } 12)$$

A single circuit round O changes $w_1, w_2, \ldots, w_{12}$ into $w_{10}, w_{11}, \ldots, w_9$; and *three* circuits restore their values.

(iii) $w^2 = z^2(z+1)$.

The branch points are $-1, \infty$. $z = 0$ is not a branch point although two values of w are equal there. The plane may be cut along the real axes from -1 to $+\infty$.

(iv) $w^2 = z(z-1)(z-2)(z-3)$; $z = 0, 1, 2, 3$ are branch points, but ∞ is not a branch point.

A circuit round any of these points changes w_1 into w_2, and therefore it is sufficient to cut the plane along the real axes between 0 and 1 and between 2 and 3.

(v) If $w^n = z$ and $w_1 = |z|^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ be denoted by $z^{\frac{1}{n}}$, obtain the derivative of $z^{\frac{1}{n}}$ directly

$$d\{r(\cos\theta + i\sin\theta)\} = (\cos\theta + i\sin\theta)(dr + ir\,d\theta)$$

$$d\left\{r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)\right\} = \left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)\left(\frac{1}{n}r^{\frac{1}{n}-1}dr + \frac{i}{n}r^{\frac{1}{n}}d\theta\right)$$

$$= \frac{1}{n}r^{\frac{1}{n}-1}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)(dr + ir\,d\theta).$$

Thus $\dfrac{dw_1}{dz}$ exists and is equal to $\dfrac{1}{n}r^{\frac{1}{n}-1}\left\{\cos\left(\dfrac{1}{n}-1\right)\theta + i\sin\left(\dfrac{1}{n}-1\right)\theta\right\}$ i.e. may be denoted by $\dfrac{1}{n}z^{\frac{1}{n}-1}$.

(vi) $w^6 = z^4(z+1)^3(z+i)^2(z-i)$.

Take the initial value of z to be 1 and its initial amplitude zero. The branch points are $0, -1, -i, i, \infty$. When $z = 1$, $w^6 = 2^{9/2}(\cos\pi/4 + i\sin\pi/4)$ and the initial values of the six branches $w_1, w_2, w_3, w_4, w_5, w_6$ are $w_{10}, w_{20} \ldots w_{60}$

where $$w_{10} = 2^{3/4}\left(\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right)$$

$$w_{20} = \omega w_{10},\ w_{30} = \omega^2 w_{10},\ w_{40} = -w_{10},\ w_{50} = -w_{20},\ w_{60} = -w_{30}$$

and $$\omega = \cos\tfrac{1}{3}\pi + i\sin\tfrac{1}{3}\pi.$$

The increase in amp w when z describes a given circuit is $\frac{2}{3}\theta_1 + \frac{1}{2}\theta_2 + \frac{1}{3}\theta_3 + \frac{1}{6}\theta_4$ where $\theta_1, \theta_2, \theta_3, \theta_4$ are respectively the increases in amp z, amp $(z+1)$, amp $(z+i)$ and amp $(z-i)$. Suppose that the suffixes of branches are written in the order

(123456) initially. A circuit round O (and no other branch point) increases amp w by $4\pi/3$, and the suffixes take the order (561234). The following table gives all the possibilities for simple circuits: where A is -1, B is i, C is $-i$. (*Fig. 21.*)

Branch points enclosed	*Order of suffixes*
(None), (O, C); (A, B, C)	1 2 3 4 5 6
(B); (O, A); (O, B, C)	2 3 4 5 6 1
(C); (O, A, B)	3 4 5 6 1 2
(A), (B, C); (O, A, C)	4 5 6 1 2 3
(O), (A, B); (O, A, B, C)	5 6 1 2 3 4
(O, B), (A, C)	6 1 2 3 4 5

It is sufficient therefore to cut the plane along BA, AC and along the real axis from O to ∞. On the cut plane a circuit round A, B, C is possible (as it should be), but the circuit round O, C is not possible. If the latter were required, we could cut the original plane along OC, along BA and from A to $-\infty$ along the real axis; but since this makes a circuit round A, B, C impossible, this method therefore is not so effective as that of Riemann.

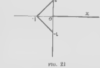


FIG. 21

10.61. The Elementary Transcendental Functions. Functions of the real variable defined as power series may obviously be defined by these series for the complex variable; they are analytic within the circle of convergence, and possess those properties that have a meaning for the complex variable and can be proved by analogous methods (such as by the differentiation and integration of series or the multiplication of series).

10.62. The Exponential Function.

$$F(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

is taken as the definition of the exponential function of z and is written e^z (or $\exp z$). It is defined thus for all finite z, has the derivative e^z, and has the property $e^{z_1} \times e^{z_2} = e^{z_1+z_2}$.

10.63. The Trigonometric (or Circular) Functions.

The function $\cos z$ is defined to be $1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} \ldots$, and the function $\sin z$ to be $z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} \ldots$. They satisfy the relation $\cos^2 z + \sin^2 z = 1$, have an addition theorem and their derivatives are $-\sin z$, $\cos z$ respectively.

It is easily verified that

$$\cos z + i \sin z = e^{iz} \text{ and } \cos z - i \sin z = e^{-iz};$$

thus $\cos z$, $\sin z$ are related to the exponential function by the equations

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}); \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

10.64 Hyperbolic Functions. Similarly $\cosh z$ is defined to be $1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots$, and $\sinh z$ to be $z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots$

Thus $\cos iz = \frac{e^{-z} + e^{z}}{2} = \cosh z$; $\cosh iz = \frac{e^{iz} + e^{-iz}}{2} = \cos z$ and $\sin iz = \frac{e^{-z} - e^{z}}{2i} = i \sinh z$; $\sinh iz = \frac{e^{iz} - e^{-iz}}{2} = i \sin z$.

The other circular and hyperbolic functions are defined in an obvious way; thus $\tan z = \sin z/\cos z$, $\sec z = 1/\cos z$, $\operatorname{cosech} z = 1/\sinh z$, $\coth z = \cosh z/\sinh z$, &c.

10.641. The Conjugate Functions for e^z, $\sin z$, $\sinh z$, &c. (i) Let $w = u + iv = e^z$; then $u + iv = e^x e^{iy} = e^x(\cos y + i \sin y)$.

Thus $\mathbf{R}(e^z) = e^x \cos y$; $\mathbf{I}(e^z) = e^x \sin y$,

$$|e^z| = e^x; \quad \operatorname{amp}(e^z) = y + 2n\pi.$$

Thus $x =$ constant are transformed into circles $u^2 + v^2 = e^{2x}$ and $y =$ constant to the radii of these circles.

(ii) Let $w = \sin z = \sin x \cos iy + \cos x \sin iy$

$$= \sin x \cosh y + i \cos x \sinh y$$

so that $u = \sin x \cosh y$, $v = \cos x \sinh y$.

The lines $x =$ constant become the confocal hyperbolas

$$u^2/(\sin^2 x) - v^2/(\cos^2 x) = 1$$

whilst the lines $y =$ constant become the orthogonal system of confocal ellipses $u^2/(\cosh^2 y) + v^2/(\sinh^2 y) = 1$. (The foci are $\pm 1, 0$.)

A zero of $\sin z$ must make $\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = 0$ and this can only be satisfied if $\sin x \cosh y = 0$ and $\cos x \sinh y = 0$.

Now $\cosh y \neq 0$ and therefore the zeros must satisfy the equations $\sin x = 0$, $\sinh y = 0$, i.e. are $n\pi$ (as for the real variable).

Similarly if $w = \cos z$, the lines $x =$ constant, $y =$ constant are transformed into the same confocal systems as the above, since $\cos z = \sin(z + \frac{1}{2}\pi)$. Also $\cos z$ has the same zeros as for the real variable.

(iii) Similarly if $w = \sinh z$ or $\cosh z$, the lines $x =$ constant, $y =$ constant are transformed into the same confocals as the above, with the hyperbolas and ellipses interchanged.

10.65. The Logarithmic Function. If $z = e^w$ and $w = u + iv$, then $z = e^u(\cos v + i \sin v)$, i.e. $e^u = |z|$ $(= r)$ and $v = \operatorname{amp} z$ $(= \theta + 2n\pi)$, where θ is the principal value of amp z and n is an integer, positive or negative or zero.

Thus the equation defines w as the many-valued function

$$\log r + i(\theta + 2n\pi).$$

It is thus defined for all z (except $z = 0$) and one value ($n = 0$) agrees with the definition of $\log x$ for a real variable x (> 0). It is therefore called the logarithm of z and its general value is often written $\operatorname{Log} z$. The principal value of $\operatorname{Log} z$ is defined to be $\log r + i\theta$ and is usually written $\log z$.

Thus $\operatorname{Log} z - \log z = 2n\pi i$.

Examples. (i) $\operatorname{Log} i = \frac{1}{2}\pi i + 2n\pi i$; $\log i = \frac{1}{2}\pi i$.
(ii) $\operatorname{Log}(1 + i) = \frac{1}{2}\log 2 + i(\frac{1}{4}\pi + 2n\pi)$.

10.651. Conjugate Functions for Log z. Let

$$w = u + iv = \operatorname{Log} z = \log r + i(\theta + 2n\pi).$$

The lines $u =$ constant correspond to the circles $r = e^u$ and the lines $v =$ constant to the radii $\theta =$ constant. The whole z-plane is determined by $0 < r < \infty$, $-\pi < \theta < \pi$ and is therefore represented by the *infinite strip* of the w-plane given by

$$-\infty < u\ (= \log r) < \infty; \quad -\pi < v < \pi. \quad (\textit{Fig. 22.})$$

The whole of the z-plane is also represented by *any* such infinite strip of breadth 2π parallel to $v = 0$.

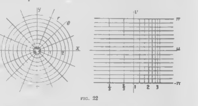


FIG. 22

10.66. The Function a^z. The function a^z $(a \neq 0)$ is defined to be $e^{z \operatorname{Log} a}$ and is sometimes called the *generalised power*. It is therefore many-valued (except when z is a positive or negative integer). The function $e^{z \log a}$ is called the principal value of a^z and is therefore equal to

$$\exp\{z(\log|a| + i \operatorname{amp} a)\}$$

where amp a is the principal value.

Examples. (i) Determine the conjugate functions for a^z.

$$\begin{aligned} a^z &= \exp[(x + iy)\{\log|a| + i(\operatorname{amp} a + 2n\pi)\}] \\ &= \exp[\{x \log|a| - y(\operatorname{amp} a + 2n\pi)\} + i\{x(\operatorname{amp} a + 2n\pi) + y\log|a|\}] \\ &= |a|^x e^{-y(\operatorname{amp} a + 2n\pi)}(\cos\phi + i\sin\phi) \end{aligned}$$

where $\qquad \phi = x(\operatorname{amp} a + 2n\pi) + y\log|a|$.

(ii) Find the principal value of $(1 + i)^{1-i}$.

$$\begin{aligned} \exp\{(1 - i)\log(1 + i)\} &= \exp\{(1 - i)(\tfrac{1}{2}\log 2 + \tfrac{1}{4}\pi i)\} \\ &= \sqrt{2}e^{\frac{1}{4}\pi}\{\cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\log 2) + i\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}\log 2)\} \\ &= e^{\frac{1}{4}\pi}\{(1 + i)\cos(\tfrac{1}{2}\log 2) + (1 - i)\sin(\tfrac{1}{2}\log 2)\} \end{aligned}$$

10.67. The Inverse Circular and Hyperbolic Functions. If $z = \sin w$, we can determine w as a many-valued function of z, which is denoted by $\operatorname{Sin}^{-1} z$ (or Arc sin z).

For $2iz = e^{iw} - e^{-iw}$ from which we find that $e^{iw} = iz \pm (1 - z^2)^{\frac{1}{2}}$, where $(1 - z^2)^{\frac{1}{2}}$ is used to denote the value that tends to 1, when $z \to 0$.

Thus $w = \operatorname{Sin}^{-1} z = -i \operatorname{Log}\{iz \pm (1 - z^2)^{\frac{1}{2}}\}$.

The value that tends to zero when $z \to 0$ is denoted by $\sin^{-1} z$, and is therefore equal to $-i \log(iz + (1 - z^2)^{\frac{1}{2}})$.

Now $\{iz + (1 - z^2)^{\frac{1}{2}}\}\{iz - (1 - z^2)^{\frac{1}{2}}\} = -1$,

and therefore

$$-i \operatorname{Log}\{iz - (1 - z^2)^{\frac{1}{2}}\} = -i[-\operatorname{Log}\{iz + (1 - z^2)^{\frac{1}{2}}\} - i\pi]$$
$$= \pi + i \operatorname{Log}\{iz + (1 - z^2)^{\frac{1}{2}}\}.$$

Thus, taking into account the various values of the Log function we have $\operatorname{Sin}^{-1} z = m\pi + (-1)^m \sin^{-1} z$ (m integral or zero) agreeing with the result for the real variable.

Similarly we may find that if $z = \tan w$

$$w = \operatorname{Tan}^{-1} z = -\tfrac{1}{2} i \operatorname{Log}\left(\frac{1 + iz}{1 - iz}\right)$$
$$= -\tfrac{1}{2} i \log\left(\frac{1 + iz}{1 - iz}\right) + m\pi$$

and $\tan^{-1} z = -\frac{1}{2} i \log\left(\frac{1 + iz}{1 - iz}\right)$ being the value that tends to zero when z tends to zero.

$\operatorname{Cos}^{-1} z$ is defined to be $\pi/2 - \operatorname{Sin}^{-1} z$ and is easily deduced to be $2p\pi \pm \cos^{-1} z$ where $\cos^{-1} z = \pi/2 - \sin^{-1} z$.

The general value of $\operatorname{Cos}^{-1} z$ may also be written

$$-i \operatorname{Log}\{z \pm (z^2 - 1)^{\frac{1}{2}}\}.$$

$\operatorname{Sinh}^{-1} z$ may be defined as $-i \operatorname{Sin}^{-1}(iz)$ and is $\operatorname{Log}\{z \pm (z^2 + 1)^{\frac{1}{2}}\}$. $\operatorname{Cosh}^{-1} z$ similarly is $\operatorname{Log}\{z \pm (z^2 - 1)^{\frac{1}{2}}\}$ and

$$\operatorname{Tanh}^{-1} z \text{ is } \tfrac{1}{2} \operatorname{Log}\frac{1 + z}{1 - z} = -i \operatorname{Tan}^{-1}(iz).$$

Example. Solve the equations (i) $\sin z = 2$, (ii) $\tan 2z = 2i$.

(i) If $\sin z = 2$, $e^{iz} - e^{-iz} = 4i$ and therefore $e^{2iz} - 4ie^{iz} - 1 = 0$, i.e. $e^{iz} = i(2 \pm \sqrt{3})$ giving $iz = \log(2 \pm \sqrt{3}) + i(2m + \frac{1}{2})\pi$.

Thus $z = -i \log(2 \pm \sqrt{3}) + (2m + \frac{1}{2})\pi$ or $z = n\pi + (-1)^n \alpha$ where

$$\alpha = \tfrac{1}{2}\pi - i \log(2 + \sqrt{3}).$$

(ii) If $\tan 2z = 2i$, $z = -\frac{1}{4} i \operatorname{Log}(-\frac{1}{3}) = \frac{1}{4} i \log 3 + \frac{1}{4}(2m + 1)\pi$.

10.68. The Logarithmic Series. If $z = e^w$, $\dfrac{dz}{dw} = e^w$ and therefore

$$\frac{d}{dz}(\operatorname{Log} z) = \frac{d}{dz}(\log z) = \frac{1}{z}.$$

Now $\dfrac{1}{1 + z} = 1 - z + z^2 \ldots$, when $|z| < 1$.

Integration from 0 to z gives

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots, \quad (|z| < 1)$$

since $\log(1 + z)$ is the value that tends to 0 when z tends to 0. We have already seen that this series is valid for $z = 1$ and that it is not convergent for $z = -1$.

It can be shown to be convergent at all other points of the circle (*Chap. XI, § 11.08*), and therefore (by Abel's Theorem) the series is equal to log $(1 + z)$ at all points of the circle except $z = -1$.

10.681. The Series for $\tan^{-1} z$. If $z = \tan w$, $\frac{dz}{dw} = 1 + z^2$ and therefore

$$\frac{d}{dz}(\mathrm{Tan}^{-1} z) = \frac{1}{1 + z^2}.$$

Now

$$\frac{1}{1 + z^2} = 1 - z^2 + z^4 \ldots \quad (|z| < 1)$$

and therefore by integration since $\tan^{-1}(z) \to 0$ when $z \to 0$, we have

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} \ldots \quad (|z| < 1).$$

We have already seen that the series are valued for $z = \pm 1$; it is obviously not convergent for $z = \pm i$, but it can be proved convergent for all other values on the circle. (*Chap. XI, § 11.08.*)

10.682. The Binomial Series. The function $F(z)$ defined by the general binormal series

$$1 + \nu z + \frac{\nu(\nu - 1)}{1.2} z^2 + \ldots + \frac{\nu(\nu - 1) \ldots (\nu - n + 1)}{n!} z^n + \ldots$$

where ν, z are complex, has a radius of convergence unity since

$$\lim \left|\frac{n + 1}{n - \nu}\right| = 1.$$

The method for the real variable (§ 5.72) is applicable to show that $F(z) = (1 + z)^\nu$ at least for $|z| < 1$ where $(1 + z)^\nu$ means

$$\exp \{\nu \log (1 + z)\}$$

i.e. $\to 1$ when $z \to 0$.

Now $\left|\frac{a_n}{a_{n+1}}\right| = 1 + \frac{\alpha + 1}{n} + O\left(\frac{1}{n^2}\right)$, where a_n is the coefficient of z^n in the series and $\nu = \alpha + i\beta$. There is therefore absolute convergence on $|z| = 1$ if $\mathbf{R}(\nu) > 0$, but if $\mathbf{R}(\nu) < 0$, there cannot be absolute convergence on $|z| = 1$. When $z \neq -1$, $\exp \{\nu \log (1 + z)\}$ is continuous and therefore $F(z) = (1 + z)^\nu$ when $|z| = 1$, $\mathbf{R}(\nu) > 0$ (except possibly at $z = -1$). But when $z = -1 + \rho e^{i\phi}$, ρ small and $|\phi| < \frac{1}{2}\pi$ (so that points near -1 of the domain are under consideration), we have

$$|\exp \{\nu \log (1 + z)\}| = \rho^\alpha e^{-\beta\phi}$$

which $\to 0$ as $\rho \to 0$, if $\alpha > 0$. Thus the value of the series is zero when $z = -1$ and $\mathbf{R}(\nu) > 0$.

When $\mathbf{R}(\nu) < -1$, the terms do not decrease in absolute value and therefore the series does not converge. (For the case $-1 < \mathbf{R}(\nu) < 0$ see *Chap. XI, § 11.19.*)

Note. Actually, when $z = -1$, we can find a simple expression S_n for the sum of the first $(n + 1)$ terms, for t_n, the $(n + 1)$th term is $P_n - P_{n-1}$, where

$$P_n = (1 - \nu)\left(1 - \frac{\nu}{2}\right) \ldots \left(1 - \frac{\nu}{n}\right), \quad (n > 2).$$

Also $t_2 = P_2 - P_1$, where $1 + t_1 = P_1$

i.e. $$S_n = P_n = (1 - \nu)\left(1 - \frac{\nu}{2}\right) \dots \left(1 - \frac{\nu}{n}\right)$$

If ν is a positive integer N, $S_N = S_{N+1} = \dots = 0$ and $F(z) = 0$

If $\nu = 0$, $F(z)$ is obviously 1. For other values of ν,

$$\lim_{n\to\infty} (P_n) = \frac{e^{-\nu\gamma}}{\Gamma(1-\nu)} \lim_{n\to\infty} e^{-\nu\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)}$$

(*Chap. XII, § 12.3.*)

So that if $\mathbf{R}(\nu) > 0$, $\lim P_n = 0$; if $\nu = i\beta$ (β real), P_n oscillates finitely; and if $\mathbf{R}(\nu) < 0$, P_n oscillates infinitely or diverges to $\pm\infty$.

10 7. Functions defined by Integrals. Consider the integral $\int_{z_0}^{z} w\,dz$, where the path of integration is the straight line joining z_0 to z when this straight line does not pass through a singularity of w. When it does pass through a singularity α, the path must be deflected by means of a small semicircle whose centre is α; and the path is made definite by choosing that semicircle whose description by a point z gives an increase of π to amp $(z - \alpha)$. We have already seen that if $G(z)$ is a function whose derivative is w, then

$$G(z) - G(z_0) = \int_{z_0}^{z} w\,dz$$

i e. we can write $\int_{z_0}^{z} w\,dz = F(z)$ where $F'(z) = w$, $F(z_0) = 0$ and the path of integration is the straight line joining z_0, z (modified if necessary).

Note. The semicircle used to avoid a singularity on the path is called an *indentation*. When dealing with the variation of a function when z describes the boundary of a given domain, and when α is a point of the boundary, we should usually draw a semicircle (or arc) that excludes the singularity from the domain

By Cauchy's Theorem the value of the integral is also $F(z)$ by any other path that can be deformed into the line joining z_0, z without crossing a singularity; and it may be called the principal value of the many-valued function defined by the integral for all possible paths. Let A, P be the points z_0, z and let $A_1(z_1)$ be a singularity of w. Also let us suppose (for the moment) that z_1 is the only singularity. (*Fig. 23.*)

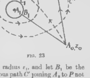


FIG. 23

Let C_1 be a small circle centre A_1 and radius ε_1, and let B_1 be the point of C_1 nearest to A_0. Take any continuous path C' joining A_0 to P not deformable into the straight line A_0P. (*Fig. 23.*) Then C' is equivalent to the *closed* path γ' consisting of $C' + \overrightarrow{PA_0}$ followed by $\overrightarrow{A_0P}$. The increase in amp $(z - z_1)$ when γ' is described is $2m_1\pi$ where m_1 is an integer ($\pm$). The integer is not zero, for then C' could be deformed into $\overrightarrow{A_0P}$. In

general, the value of w after a circuit of A_1 is changed to w_1 (when A_1 is a branch point of w); if A_1 is not a branch point, its value is unaltered. Thus for a singularity, in general,

$$\int_{C'} w\,dz = m_1\int_{C_1} w\,dz + \int_{z_0}^{B_1}(w - w_1)dz + \int_{z_0}^{z} w_1\,dz$$

but if the value is restored after m_1 circuits (or if A_1 is not a branch point),

$$\int_{C'} w\,dz = m_1\int_{C_1} w\,dz + F(z)$$

Similarly, if there are p singularities at A_s ($s = 1$ to p) and no two are on the same line through z_0, we obtain for the general path C'

$$\int_{C'} w\,dz = \sum_1^p\left\{m_s\int_{C_s} w_{s-1}\,dz + \int_{z_0}^{B_s}(w_{s-1} - w_s)dz + \int_{z_0}^{z} w_s\,dz\right\}$$

a definite order being chosen for the points A_s. When a point A lies on the line z_0A_s, it is necessary to indent the path z_0B_s in the usual way. In many cases, the integrals round C_s tend to definite limits I_s when $\epsilon_s \to 0$, and we can then write

$$\int_{C'} w\,dz = \sum_1^p\left\{m_sI_s + \int_{z_0}^{z_s}(w_{s-1} - w_s)dz + \int_{z_0}^{z} w_s\,dz\right\}.$$

In particular, if the only singularities enclosed are poles $A_1, A_2, \ldots, A_q$ (which are not branch points), we obtain

$$\int_{C'} w\,dz = F(z) + 2\pi i(m_1k_1 + m_2k_2 + \ldots + m_qk_q)$$

where k_r is the residue of w at A_r.

Examples. (i) Let $w = \int_{C'}\frac{dz}{1+z^2}$ where C' is any continuous path from O to z. The function defined by the relation $z = \tan w$ (viz. $\operatorname{Tan}^{-1} z$) satisfies the relation $dw/dz = 1/(1+z^2)$; and the above integral provides a suitable definition of $\operatorname{Tan}^{-1} z$. The principal value $\tan^{-1} z$ is defined to be $\int_0^z \frac{dz}{1+z^2}$ along the straight line joining O to z (suitably modified when $z = iy$ and $|y| > 1$). The function is not defined for $z = \pm i$, these being the only singularities.

Now
$$\frac{1}{1+z^2} = \frac{i}{2(z+i)} - \frac{i}{2(z-i)}.$$

If $z \neq iy$ ($|y| > 1$),

$$\tan^{-1} z = \frac{i}{2}\{\log(i+z) - \log(i-z)\}$$

since the value on the right $\to 0$ when $z \to 0$.

The residues at $z = \pm i$ are $\mp \frac{1}{2}i$ respectively. Therefore, if C' is a path from O to $P(z)$ (*Fig.* 24), which is such that the closed path formed by C' and $\overrightarrow{PO}$ encircles $\pm i$, m_1, and m_2 times respectively, then

$$\int_{C'}\frac{dz}{1+z^2}\,(= \operatorname{Tan}^{-1} z) = \tan^{-1} z + 2\pi i\cdot\frac{i}{2}(m_2 - m_1)$$
$$= \tan^{-1} z + N\pi$$

where N is an integer ($\pm$) or zero.

If APB (an angle between 0 and π inclusive) is denoted by θ,

$$\tan^{-1} z = \frac{i}{2}\left\{\log\frac{PB}{PA} \pm i(\theta - \pi + 2m\pi)\right\}$$

where m must be chosen so that $\tan^{-1}(0) = 0$, according as $\mathbf{R}(z) \gtrless 0$. But

$$\lim_{\epsilon \to 0} \theta = \pi$$

and therefore

$$\tan^{-1} z = \frac{i}{2}\log\frac{PB}{PA} \pm \left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

according as $\mathbf{R}(z) \gtrless 0$.

Also $\tan^{-1}(iy) = \frac{i}{2}\log\frac{1+y}{1-y}$ if $|y| < 1$.

If $y > 1$, $\tan^{-1}(iy)$, if the path is deflected into the region $\mathbf{R}(z) > 0$, must be

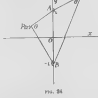


FIG. 24

given by the upper sign, i.e. is $\frac{i}{2}\log\frac{y+1}{y-1} + \frac{1}{2}\pi$, and $\tan^{-1}(-iy)$ for $y > 1$ is $\frac{i}{2}\log\frac{y-1}{y+1} - \frac{1}{2}\pi$ if the path is deflected into the Region $\mathbf{R}(z) < 0$.

This is verified by noting that $(y > 1)$,

$$\int_0^{iy}\frac{dz}{1+z^2} = iP\int_0^y\frac{dy}{1-y^2} + \lim_{\epsilon\to 0}\int_\gamma\frac{dz}{1+z^2} \text{ where } \gamma \text{ is the semicircle}$$

$$= \frac{i}{2}\log\frac{y-1}{y+1} + \tfrac{1}{2}\pi, \text{ since } \frac{1}{z^2+1} = \frac{-i}{2(z-i)} + g(z)$$

where $g(z)$ is analytic at $z = i$.

Again since $\int_0^{-z}\frac{dz}{1+z^2} = -\int_0^z\frac{dz}{1+z^2}$, we can simply define $\tan^{-1}(z)$ to be $-\tan^{-1}(-z)$ when $y < -1$.

Notes. (i) The whole z-plane is represented on the w-plane by the strip $-\pi/2 < u < \pi/2$, although the point $z = \infty$ is represented by $w = \pi/2$ and also by $w = -\pi/2$. The points A, B also may be regarded as being given by $c \pm i\infty$ respectively where $|c| < \pi/2$.

(ii) The conjugate functions for $\tan^{-1} z$ are in general given by $\theta =$ constant, $PA = kPB$, i.e. two orthogonal systems of coaxal circles, in which A, B are the critical points.

(iii) $\int_C \frac{dz}{\sqrt{(1-z^2)}}$ for any path starting from O with the initial value $+1$ for $\sqrt{(1-z^2)}$.

Assume for the moment that z is not real. Then the principal value $\sin^{-1} z$ is $\int_0^z \frac{dz}{\sqrt{(1-z^2)}}$ for the straight line Oz. The singularities ± 1 are branch points as well as infinities of the integrand.

Let C_1 denote the circle $|z-1| = \epsilon_1$ and C_2 the circle $|z+1| = \epsilon_2$. Near $z = 1$, $|(1-z^2)| > \epsilon_1(1-\epsilon_1)$ and therefore $\left|\int_{C_1} \frac{dz}{\sqrt{(1-z^2)}}\right| < \frac{2\pi\sqrt{\epsilon_1}}{\sqrt{(1-\epsilon_1)}}$, i.e. $\int_{C_1} \frac{dz}{\sqrt{(1-z^2)}} \to 0$ as $\epsilon_1 \to 0$; and similarly $\int_{C_2} \frac{dz}{\sqrt{(1-z^2)}} \to 0$ as $\epsilon_2 \to 0$.

A circuit (in either direction) round ± 1 changes $\sqrt{(1-z^2)}$ into $-\sqrt{(1-z^2)}$; therefore two consecutive circuits round either of these points may be ignored.

The value of the general integral due to one circuit of C_1 is therefore

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} + \int_1^0 \frac{dx}{-\sqrt{(1-x^2)}} + \int_0^z \frac{dz}{-\sqrt{(1-z^2)}} = \pi - \sin^{-1} z,$$

and the value for a single circuit round C_2 is $-\pi - \sin^{-1} z$, whilst the value for a circuit round C_1 followed by one round C_2 is $2\pi + \sin^{-1} z$; for a circuit round C_2 followed by one round C_1 the value is $-2\pi + \sin^{-1} z$. Since either of these double circuits may be repeated and then be followed by a single circuit round one of the points ± 1, we conclude that the general value of the integral $\mathrm{Sin}^{-1} z$ is

$$m\pi + (-1)^m \sin^{-1} z.$$

We have already shown that $\sin^{-1} z = -i \log \{iz + \sqrt{(1-z^2)}\}$. When $z = x$ (real) and $|x| < 1$, $\sin^{-1} z = \sin^{-1} x$, where $-\pi/2 < \sin^{-1} x < \pi/2$.

To give definiteness to $\sin^{-1} x$ when $|x| > 1$, we can choose the path from O to x that is deflected at $+1$, in such a way that the branch point is on the left of the path.

Thus amp $(1-z)$ increases by π and therefore $\sqrt{(1-z^2)}$ becomes $-i\sqrt{(x^2-1)}$ for the indentation at $+1$, i.e.

$$\sin^{-1} x\,(|x| > 1) = \frac{\pi}{2} - i\int_1^x \frac{dx}{\sqrt{(x^2-1)}} = \frac{\pi}{2} - i \log (x + \sqrt{(x^2-1)})$$
$$= \frac{\pi}{2} - i \cosh^{-1} x.$$

Then $x < -1$, $\sin^{-1} x = -\tfrac{1}{2}\pi + i \cosh^{-1} x$ (since $\sin^{-1}(-x) = -\sin^{-1} x$).

Note. The lines $u =$ constant, $v =$ constant have been shown to correspond to the system of confocal conics (foci ± 1). If the z-plane is cut along the real axis from $x = 1$ to $x = +\infty$ and $x = -1$ to $x = -\infty$ (the points of the cut being omitted from the z plane), there is a $1-1$ correspondence between the cut plane and the open region $-\tfrac{1}{2}\pi < u < \tfrac{1}{2}\pi$. The cut from 1 to $+\infty$ is represented by one boundary $u = \pi/2$, but it is represented twice on this line; when it is regarded as the limiting position of the upper edge of the cut, it is given by $v > 0$ and if of the lower edge it is given by $v < 0$.

10.71. Christoffel-Schwarz Transformations. Consider the transformation $w = \sin^{-1} z$ discussed in the example above; and suppose now we confine z to the *upper* half of the z-plane, the boundary of which is

$y = 0$ indented by small semicircles centres ± 1, the semicircles being drawn to *exclude* ± 1 from the region. Also let the radii of these circles tend to zero; then the value of w on the real axis has, in the limit, the following values

$$-\frac{\pi}{2} + i\cosh^{-1} x\,(-\infty < x < -1);\quad \sin^{-1} x\,(|x| < 1);$$
$$\frac{\pi}{2} + i\cosh^{-1} x\,(1 < x < \infty).$$

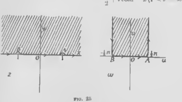


FIG. 25

As z describes the x-axis from $-\infty$ to $+\infty$, w describes the boundary of the semi-infinite rectangle determined by $u = \pm\pi/2$ $(v > 0)$ and $v = 0(|u| < \pi/2)$. (*Fig. 25.*) The upper half of the z-plane is on the *left* of the moving point; therefore it corresponds to the *interior* of this rectangle. There is a one-one correspondence between the regions and w is therefore called a *simple* (or *schlicht*) function for the domain $\mathbf{I}(z) > 0$.

This is a particular case of the more general transformation indicated by the relation

$$\frac{dw}{dz} = \frac{A}{(z - a_1)^{\lambda_1}(z - a_2)^{\lambda_2} \ldots (z - a_n)^{\lambda_n}} = f(z)$$

where A, a_r, λ_r are constants.

For purposes of illustration we shall consider the simplest type of the above transformation, viz. that in which the numbers a_r, λ_r are real, and determine the boundary in the w-plane that corresponds to the real axis indented by the upper halves of the circles C_r given by $|z - a_r| = \epsilon_r$ $(r = 1$ to $n)$. The origin of the w-plane may be chosen at any point since no constant of integration has been specified. In particular we may take

$$w = \int_0^z \frac{A\,dz}{(z - a_1)^{\lambda_1} \ldots (z - a_n)^{\lambda_n}}$$

so that $w = 0$ when $z = 0$ (provided $a_r \neq 0$ when w is not convergent there).

Since the effect of the multiplier A is merely to give a magnification

and rotation, the essential character of the transformation is not altered by taking A to be any particular number. The point $z = \infty$ is in general a singularity of the integrand, but since this point may be transformed into a point $x = c$ by the transformation $z = (z - c)^{-1}$ (c real and $\neq a_r$), we may without loss of generality assume that ∞ is not a singularity of $f(z)$. By choosing O at a suitable point (this being a simple translation of the z-plane) and taking a_r in the correct order we can write

$$0 < a_1 < a_2 \dots < a_n$$

and by a suitable choice of A we can take

$$f(z) = \frac{1}{(a_1 - z)^{\lambda_1} \dots (a_n - z)^{\lambda_n}}.$$

Let the initial value of z be 0 with initial value $|a_1{}^{\lambda_1} \dots a_n{}^{\lambda_n}|$ for $f(z)$ (i.e. $\operatorname{amp} f(z) = 0$).

A case of special simplicity arises when all the integrals round C_r tend to zero when $\varepsilon_r \to 0$, i.e. when w is finite for all z (including ∞).

On the circle $|z - a_r| = \varepsilon_r$, $|f(z)| < K/\varepsilon_r{}^{\lambda_r}$ where K is bounded, and therefore $\int_{C_r} f(z)dz \to 0$ when $\varepsilon_r \to 0$ if $\lambda_r < 1$. When $|z| \to \infty$, $|f(z)| = O(|z|^{-\Sigma\lambda_r})$ and therefore w is finite as $z \to \infty$ if $\Sigma\lambda_r > 1$.

If infinity is not a branch point, $\Sigma\lambda_r$ must be an integer. Its smallest value in that case is 2 and we shall see that in this case the real axis is transformed into the sides of a convex polygon and the region $I(z) > 0$ into its interior. Now $\operatorname{amp} f(z)$ is zero from $-\infty$ on the real axis to $x = a_1$, and its value at any point there is $\int_0^x \frac{dx}{(a_1 - x)^{\lambda_1} \dots (a_n - x)^{\lambda_n}}$. This increases from the negative real value

$$\int_0^{-\infty} \frac{dx}{(a_1 - x)^{\lambda_1} \dots (a_n - x)^{\lambda_n}} = -\int_0^{\infty} \frac{dt}{(a_1 + t)^{\lambda_1} \dots (a_n + t)^{\lambda_n}}$$

at $x = -\infty$ to zero at $x = 0$ and then to the positive real value $\int_0^{a_1} \frac{dx}{(a_1 - x)^{\lambda_1} \dots (a_n - x)^{\lambda_n}}$ at a_1. The description of C_1 causes a decrease in $\operatorname{amp} (a_1 - z)^{\lambda_1}$ of amount $\lambda_1\pi$ ($< \pi$); and as z describes a_1a_2, w describes the straight line from $w(a_1)$ to $w(a_2)$ which makes an angle $\lambda_1\pi$ with $v = 0$. The length of this line is

$$\int_{a_1}^{a_2} \frac{dx}{(x - a_1)^{\lambda_1} (a_2 - x)^{\lambda_2} \dots (a_n - x)^{\lambda_n}}.$$

Similarly when z describes the other segments $a_2a_3, \dots, a_{n-1}a_n$, w describes the sides of a polygon. (*Fig. 26.*) The polygon is convex since $\lambda_r < 1$. When z describes C_n, $\operatorname{amp} f(z)$ is $(\Sigma\lambda_r)\pi = 2\pi$, and therefore w moves parallel to the line $v = 0$, when z describes a_n to $A(+\infty)$. But ∞ is not a singularity of w, and the integral converges there. Therefore, by Cauchy's Theorem, the value of w at $A(+\infty)$ is the same as that at $A(-\infty)$. Thus as z describes a_n to $A(+\infty)$, w describes the

real axis $v = 0$ from a_n to A, thus completing the polygon. The real axis (indented) is transformed into the sides of this polygon (when the radii of the indentations tend to zero); and the interior of the polygon corresponds to the upper half of the z-plane. The function defined is simple for $I(z) > 0$.

FIG. 26

When the integrals round C_r do not converge to zero, suitable modifications may be made in the representation of w, when, for example, the integral tends to a limit. Thus if a_1 were a simple pole, and not a branch point, the point $w(a_1)$ is at infinity on the real axis of the w-plane. The segment corresponding to a_1a_2 is then parallel to the real axis at a distance $k\pi$ above it, where k is the residue of $f(z)$ at a_1.

Note. Since the real axis of z may be transformed by a bilinear substitution into the unit circle $|z| = 1$, the transformation above gives a representation of a polygon in a circle.

Examples. (i) $\dfrac{dw}{dz} = z^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}(1+z)^{-\frac{1}{2}} = f(z)$.

Take amp $f(z)$ to be zero for $0 < x < 1$ and let $w = 0$ when $z = 0$ (the integral being convergent there). The branch points are $0, \pm 1, \infty$, the integral converging at each.

As z describes the real axis (indented), w describes a quadrilateral $OACB$ whose vertices correspond to $0, 1, \infty, -1$ respectively. (*Fig.* 27.)

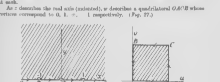


FIG. 27

The increases in amp dw at three vertices are each $\frac{1}{2}\pi$ and therefore the quadrilateral is a rectangle.

The length of OA is $\displaystyle\int_0^1 \frac{dx}{\sqrt{\{x(1-x^2)\}}}$; and the value of w at B is

$$\int_0^{-1} \frac{dz}{z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}(1+z)^{\frac{1}{2}}} = i\int_0^1 \frac{dt}{\sqrt{\{t(1+t)(1-t)\}}}$$

since $z = te^{i\pi}$, i.e. $OA = OB$, and the rectangle is a square. That all four sides are equal may be verified by the substitutions $x = 1/u$, $u = -v$ in the integral for OA.

Note. The length of the side is $\frac{1}{2}\displaystyle\int_0^1 u^{-\frac{3}{4}}(1-u)^{-\frac{1}{2}}\,du$ $(x = u^2)$. The value of this is $\{\Gamma(\frac{1}{4})\}^2/2\sqrt{(2\pi)}$ (*Chap. XII, § 12.24*) and therefore the substitution

$$w = \frac{2a\sqrt{(2\pi)}}{\{\Gamma(\frac{1}{4})\}^2}\int_0^z \frac{dz}{\sqrt{\{z(1-z^2)\}}}$$

transforms the real axis into the perimeter of a square of side a and the region $I(z) > 0$ into its interior.

(ii) $\dfrac{dw}{dz} = (\alpha - z)^{-\lambda}(\beta - z)^{-\mu}(\gamma - z)^{-\nu}$ $(\lambda + \mu + \nu = 2)$

Let $\lambda = 1 - A/\pi$, $\mu = 1 - B/\pi$, then $\nu = 1 - C/\pi$ where A, B, C are the angles of a triangle. Since the increase in amp $f(z)$ when z passes α is $\lambda\pi = \pi - A$, with a similar result for the point β, it follows that the real axis is transformed into a triangle ABC.

In particular, if we take α, β, γ to be 0, 1, ∞, the corresponding transformation is given by

$$\frac{dw}{dz} = \frac{1}{z^{\lambda}(1-z)^{\mu}}.$$

The length of the side opposite the angle C is $\displaystyle\int_0^1 \frac{dt}{t^{\lambda}(1-t)^{\mu}}$.

Note. This integral is easily evaluated in terms of Γ functions (*Chap. XII, § 12.24*), and from the properties of the Γ function it may be verified that the sides of the triangle are proportional to the sines of the opposite sides. The length of the side opposite A is $\displaystyle\int_1^{\infty} \frac{dt}{t^{\lambda}(t-1)^{\mu}}$ and the length of the side opposite B is $\displaystyle\int_0^{\infty} \frac{dt}{t^{\lambda}(1+t)^{\mu}}$. The side opposite C has length

$$B(1-\lambda,\, 1-\mu) = \frac{\sin C}{\pi}\,\Gamma\!\left(\frac{A}{\pi}\right)\Gamma\!\left(\frac{B}{\pi}\right)\Gamma\!\left(\frac{C}{\pi}\right)$$

and therefore $w = k\displaystyle\int_0^z \frac{dz}{z^{\lambda}(1-z)^{\mu}}$ transforms the real axis into a given triangle ABC if $\lambda = 1 - A/\pi$, $\mu = 1 - B/\pi$, $k = \dfrac{p}{\Gamma(A/\pi)\Gamma(B/\pi)\Gamma(C/\pi)}$, and p is the perimeter of the circumcircle of the triangle ABC.

In particular, $w = \dfrac{2\pi a}{\sqrt{3}\,\{\Gamma(\frac{1}{3})\}^3}\displaystyle\int_0^z \frac{dz}{(z - z^2)^{\frac{2}{3}}}$ transforms the real axis into an equilateral triangle of side a.

(iii) $\dfrac{dw}{dz} = \dfrac{(z-1)^{\frac{1}{2}}}{z(z+c)^{\frac{1}{2}}} = f(z)$, $(c > 0)$.

The singularities are 0, ∞, 1, $-c$ (the last two being branch points). The integral converges at 1, $-c$. Take $w = 0$ when $z = 1$, with amp $f(z) = 0$ for $x > 1$.

As x increases from 1 to $+\infty$, w is real and positive increasing from O at A to ∞ when $x \to +\infty$ (where A is $z = 1$) (*Fig.* 28.) Between O and A,

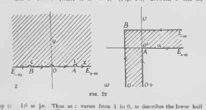


FIG. 28

amp $(z - 1)^{\frac{1}{2}}$ is $\frac{1}{2}\pi$. Thus as z varies from 1 to 0, w describes the lower half of its imaginary axis. The integral round the indentation at O increases w by πi (residue at $z = 0$) $= \pi i(i/\sqrt{c}) = -\pi/\sqrt{c}$. In the interval O to $-c$, amp z is now π and amp dw is $\frac{1}{2}\pi$, so that w moves upwards parallel to the imaginary axis to the point B for which $z = -c$. In the interval $x < -c$, amp $(z + c)^{\frac{1}{2}}$ increases by $\frac{1}{2}\pi$ and therefore amp w decreases by $\frac{1}{2}\pi$. w therefore moves parallel to the real axis $v = 0$ with increasing u, until finally when $x \to -\infty$, $w \to \infty$ as shown in *Fig.* 28.

The distance between $AO(+)$ and $BO(-)$ has been shown to be $\pi/\sqrt{c}$. The distance between $AE(+\infty)$ and $BE(-\infty)$ may be found by noting that

$$w_{E(-\infty)} - w_{E(+\infty)} = \lim \int_C f(z)dz$$

where C is a large semicircle, $|z| = R$, $I(z) > 0$, and $R \to \infty$. But when $|z|$ is large $f(z) = \frac{1}{z}G(z)$ where $G(z) = 1 + K$ where K is bounded and therefore

$$\lim_{R \to \infty} \int_C f(z)dz = \pi i$$

i.e. the distance between the parallel lines is π.

10.72. Analytic Continuation. Consider the function $f(z)$ defined by the power series $f(z) = z - z^2/2 + z^3/3 \ldots$

It is an analytic function within the circle $|z| = 1$. Without assuming the logarithmic function, we can easily prove the property

$$f(z) = f(z_0) + f\left(\frac{z - z_0}{1 + z_0}\right)$$

where $|z| < 1$ and $|z_0| < 1$, provided $\left|\frac{z - z_0}{1 + z_0}\right| < 1$.

For if $F(z, z_0) = f(z) - f(z_0)$ and $G(z, z_0) = \frac{z - z_0}{1 + z_0}$, then $\frac{\partial(F, G)}{\partial(z, z_0)} = 0$, since $f'(z)$ is obviously $\frac{1}{1 + z}$ ($|z| < 1$), i.e. $f(z) - f(z_0) = E\left(\frac{z - z_0}{1 + z_0}\right) = f\left(\frac{z - z_0}{1 + z_0}\right)$, since $f(z_0) = 0$ when $z_0 = 0$.

But the region determined by $|z - z_0| < |1 + z_0|$ is the interior of the circle centre z_0 and radius $|1 + z_0|$ and has a part outside the circle $|z| = 1$ (except when z_0 is real and negative). The relation, therefore, determines an analytic function for a region that is partly outside and which overlaps the original region $|z| < 1$. Now an analytic function in general has a unique expression as a power series for any point within a domain for which the function is defined. We may therefore regard the two expressions as different forms (both limited in scope) of the *same* analytic function, since they agree over their common domain. This process of *continuation* may be extended over a wider region of the plane and is only limited by the occurrence of singularities on the various circles of convergence. Thus in the case given above, $f(z)$ may be extended over any finite region of the plane that excludes the point -1. We may then say that a power series together with its continuations determines an analytic function.

Suppose that a function $f(z)$ is continued by means of the domains $D_1, D_2, \ldots, D_n$, where D_{r+1} overlaps D_r; and let $D_1, D_2', D_3', \ldots, D_n'$ indicate a second method of continuation where D_n' overlaps D_n. The value of $f(z)$ in this overlapping part is not necessarily the same; for the regions given by $D_1 + D_2 + \ldots + D_n$ and $D_1 + D_2' + \ldots + D_n'$ may be such that there lies between them another region across which $f(z)$ cannot be continued (i.e. where $f(z)$ has a singularity). The function is in this case many-valued; but if it is possible to continue $f(z)$ across the gap in a finite number of steps, then the values of $f(z)$ in D_n, D_n' must be identical. Thus to take a simple case to which the general case is reducible, let C_1, C_2, C_3 be three circles, each overlapping the other, but containing a gap between them. Suppose that a function $f_1(z)$ defined for C_1 is continued into C_2 giving a function $f_2(z)$ which is identical with $f_1(z)$ in C_1C_2; and similarly let $f_3(z)$ be the continuation of $f_1(z)$ into C_3 where $f_3(z) = f_1(z)$ in C_1C_3. Suppose also that $f_1(z)$ can be continued into a circle C_4 that contains the gap entirely within it, and that the value of $f_1(z)$ in C_4 is $f_4(z)$ where $f_1(z) = f_4(z)$ in C_1C_4. In the region $C_1C_2C_4$, $f_1 = f_2 = f_4$ and therefore $f_2 = f_4$ in C_2C_4; similarly $f_3 = f_4$ in C_3C_4, i.e. $f_2 = f_3 = f_4$ in $C_2C_3C_4$ and therefore $f_2 = f_3$ in C_2C_3.

The term analytic function, which is used to define a function for a certain limited domain, may be applied to describe the more general function obtained by its continuations through power series. The region of its existence is limited by its singularities, but for any point lying within this region (in the strict sense), there exists a power series for its representation. For this latter region, the function is given by its various power series together with their continuations, and is therefore sometimes described as *regular*. The word "*monogenic*" is sometimes used for analytic. The term *holomorphic* is used to describe a single-valued analytic function in a region where the function has no singularities, whilst *meromorphic* is used to describe the function when its only singularities are poles. The terms *uniform* and *multiform* are used for single-valued and many-valued respectively.

The following example illustrates the use of continuation in a simpler form.

Example. Consider the expansion of $(1 - z + z^2)^{-1} = f(z)$.

When $|z - z^2| < 1$, we have

$$f(z) = 1 + \sum_1^\infty z^n(1 - z)^n.$$

It is legitimate to rearrange in powers of z at least if $|z| + |z|^2 < 1$ for then the series written out at length is absolutely convergent; this inequality is satisfied if $|z| < \frac{1}{2}(\sqrt{5} - 1)$, i.e. $< 0{\cdot}6$ approx.

Thus $(1 - z + z^2)^{-1} = \sum_0^\infty u_n z^n$ for $|z| < {\cdot}6$ where

$$u_n = 1 - (n - 1) + \frac{(n - 2)(n - 3)}{1.2} - \frac{(n - 3)(n - 4)(n - 5)}{1.2.3} \ldots$$

there being $(\frac{1}{2}n + 1)$ terms in u_n if n is even and $\frac{1}{2}(n + 1)$ if n is odd. It is easily verified that $u_n + u_{n+2} = u_{n+1}$ and therefore $u_{n+1} + u_{n+3} = u_{n+2}$. Thus

$u_{n-1} - u_n$, so that since $a_0 = 1$, $u_1 = 1$, $u_2 = 0$ we find that u_n is either ± 1 or zero and therefore the radius of convergence of the series $\sum_{n}^{\infty} u_n z^n$ is actually 1. By the principle of continuation it follows that the expansion is valid for $|z| < 1$, since it is valid for the smaller region $|z| < \cdot 6$. The series is of course obtained immediately by expanding $(1 + z)/(1 + z^3)$: and in any case, a knowledge of the distance from $z = 0$ of the nearest singularity of $f(z)$ enables us to state beforehand that the radius of convergence must be $|-\omega| = |-\omega^2| = 1$.

Note. The continuation of the function given by a power series is related to the position of the singularities on the circle of convergence, and various tests have been devised for determining whether a given point is singular or not. It is important to note that the convergence or divergence of the series at a particular point does not by itself give any information as to whether that point is singular or not. Suppose, however, that the series $\sum_{n}^{\infty} a_n z^n$ has a unit radius of convergence. It can be shown that (i) if $a_n > 0$ or (ii) $\Sigma a_n \to \pm \infty$ (a_n being real), then $z = 1$ is singular; more generally, it has been established that if $\sum_{0}^{\infty} a_n z^n$ is divergent at $z = z_0$ on the unit circle, then z_0 is a singular point if $a_n \to 0$. (*Ref. Landau, Ergebnisse der Funktionentheorie, 18*)

10.8. Calculation of Real Definite Integrals by Contour Integration. By a suitable choice of integrand and contour it will be shown in the examples that follow how certain types of definite integrals may be evaluated. It will be found that the contours chosen consist of straight lines and arcs of circles; in the most important and interesting cases, the result is obtained by allowing some part of the contour to tend to infinity, and takes the form of an infinite integral. The establishing of the convergence of the integral is effected naturally in the course of the work.

10.81. Calculation of Residues. The residue theorem states that $\int_C f(z)dz = 2\pi i(A_1 + \ldots + A_s)$ where C is a closed contour within which $f(z)$ is analytic except at points $z_1, \ldots, z_s$ where there are poles of residues $A_1, \ldots, A_s$.

In many cases the integrand may be written in the form $F(z)/G(z)$ where $z = a$ is a root of $G(z) = 0$ but not a zero of $F(z)$.

(*a*) If the root is simple, then $G(z) = (z - a)H(z)$ where $H(a) \neq 0$ and therefore the residue at a is $F(a)/H(a) = F(a)/G'(a)$.

Examples. (i) The residue of $\dfrac{f(z)}{z^4 + 1}$ at any root α of the equation $z^4 = -1$ is $\dfrac{f(\alpha)}{4\alpha^3} = -\dfrac{\alpha f(\alpha)}{4}$.

(ii) The residue of $f(z)/(\sinh z)$ at $z = i\pi$ is $f(i\pi)/(\cosh i\pi) = -f(i\pi)$.

(*b*) If the root is multiple of order s, $G(z)$ is of the form $(z - a)^s H(z)$ $(H(a) \neq 0)$, and therefore the residue is the coefficient of $(z - a)^{s-1}$ in the expansion of $F(z)/H(z)$ in powers of $z - a$,

i.e. the residue is
$$\frac{1}{(s-1)!}\frac{d^{s-1}}{da^{s-1}}\left\{\frac{F(a)}{H(a)}\right\}.$$

It is usually better to write $z - a = \zeta$ in $F(z)/G(z)$ and expand in powers of ζ.

Examples. (i) Residue at α of $\dfrac{e^{az}}{(z-\alpha)^3(z-\beta)}$ is $\frac{1}{2}\dfrac{d^2}{d\alpha^2}\left(\dfrac{e^{a\alpha}}{\alpha-\beta}\right)$, i.e. is

$$\tfrac{1}{2}e^{a\alpha}\left\{\frac{a^2}{\alpha-\beta} - \frac{2a}{(\alpha-\beta)^2} + \frac{2}{(\alpha-\beta)^3}\right\}.$$

(ii) Find the residue at i of $\dfrac{z}{(z^2+1)^2(z+1)}$.

Take $z = i + \zeta$; then the residue is the coefficient of ζ in the expansion of $\dfrac{i+\zeta}{(2i+\zeta)^2(1+i+\zeta)}$, i.e. in $\dfrac{(i+\zeta)}{-4}\left(1-\dfrac{\zeta}{i}\right)\dfrac{1}{1+i}\left(1-\dfrac{\zeta}{1+i}\right)$. This gives $\frac{1}{4}$.

10.82. The Unit Circle $|z| = 1$.

I. The integral
$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$$
is equivalent to the contour integral $\int_C f(z)dz$ where

$$f(z) = -\frac{i}{z}F\left\{\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right\}$$

and C is the perimeter of the unit circle $|z| = 1$, since z on that circle may be taken as $\cos\theta + i\sin\theta$ $(0 \le \theta \le 2\pi)$. The integral may therefore be evaluated if $f(z)$ is analytic within and on C except at a finite number of poles within C.

Example. $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta\,d\theta}{5+4\cos\theta} = \frac{1}{2i}\int_C\frac{(z^4+1)dz}{z^2(2z+1)(z+2)}.$

The only poles within C are at $z = -\frac{1}{2}$ and $z = 0$.
Residue at $-\frac{1}{2}$ is $(1+\frac{1}{16}) \div (\frac{1}{4}.2.\frac{3}{2}) = \frac{17}{12}$.
Residue at $z = 0$ is the coefficient of z in the expansion of
$$(1+z^4)(1+2z)^{-1}(2+z)^{-1}$$
near $z = 0$, i.e. is $-5/4$.

Therefore the integral is $\pi\left(\dfrac{17}{12} - \dfrac{5}{4}\right) = \dfrac{\pi}{6}$.

II. Conversely, if $f(z)$ is a function analytic on and within C except at a finite number of poles, the integral $\int_C f(z)dz$ leads to a result of the form
$$\int_0^{2\pi}\{F(\theta) + iG(\theta)\}d\theta = A + iB \text{ when } z = e^{i\theta}.$$

Example. $\displaystyle\int_C\frac{e^{iaz}dz}{(3z+1)(z+3)} = \frac{\pi}{4}ie^{-ia/3}$, since $-\frac{1}{3}$ is the only pole inside.

Taking a real, we find, on putting $z = e^{i\theta}$, that
$$\int_0^{2\pi}\frac{e^{-a\sin\theta}\cos(a\cos\theta)d\theta}{5+3\cos\theta} = \frac{\pi}{2}\cos\tfrac{1}{3}a, \quad \int_0^{2\pi}\frac{e^{-a\sin\theta}\sin(a\cos\theta)d\theta}{5+3\cos\theta} = \frac{\pi}{2}\sin\tfrac{1}{3}a.$$

10.83. Infinite Semicircles. Let C be the contour determined by the upper half Γ of the circle $|z| = R$ and the diameter AB that closes it. (*Fig. 29.*) If $f(z)$ is chosen so that $\int_\Gamma f(z)dz$ tends to zero when $R \to \infty$, we may obtain the value of a definite integral of the type $\int_{-\infty}^{\infty} f(x)dx$. Suppose that $f(z)$ has no singularities on the real axis and that the only singularities within the finite semicircle are isolated (and therefore finite in number for a fixed R). Then $\int_\Gamma f(z)dz + \int_{-R}^{+R} f(x)dx = 2\pi i S_R$, where S_R is the sum of the residues of $f(z)$ within R.

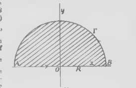


FIG. 29

Generally, the number of singularities will depend on R. If

$$\int_\Gamma f(z)dz \to 0 \text{ as } R \to \infty$$

then

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} 2\pi i S_R$$

if this limit exists.

The contour Γ should not of course pass through a singularity. Suppose that the singularities lie on the circles $|z| = r_n$, where

$$r_1 < r_2 < r_3 \ldots$$

(there being a finite number on each). Then there is no loss in generality if we take $R_n = \frac{1}{2}(r_n + r_{n+1})$, and write $\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} 2\pi i S_n$ where S_n is the sum of the residues of $f(z)$ within the semicircle determined by $|z| = R_n$, provided $\int_\Gamma f(z)dz \to 0$. It will be found that appropriate integrands for this contour are

$$\text{I. } e^{ipz}\phi(z) \qquad \text{II. } \{\log(z - \alpha)\}^p\phi(z), \ (p \text{ real})$$

I. Let $f(z) = e^{ipz}\phi(z)$

The following lemmas are sometimes useful in determining whether or not $\int_\Gamma f(z)dz \to 0$ as $R \to \infty$.

(*a*) If $|z\phi(z)| \to 0$ uniformly on Γ as $|z| \to \infty$, then $\int_\Gamma e^{ipz}\phi(z)dz$ tends to zero if $p > 0$.

(b) If $|\phi(z)| \to 0$ uniformly on Γ as $|z| \to \infty$, then $\int_\Gamma e^{ipz}\phi(z)dz$ tends to zero when $p > 0$.

These cases are not mutually exclusive.

We say that $|G(z)| \to k$ uniformly for the sector $\alpha_1 < \operatorname{amp} z < \alpha_2$ when $|z| \to \infty$, if, given ε (> 0), we can find R (> 0) such that

$$|G(z) - k| < \varepsilon$$

for all $|z| > R$ and all θ ($= \operatorname{amp} z$) in the interval $\alpha_1 < \theta < \alpha_2$.

(a) $\left|\int_\Gamma e^{ipz}\phi(z)dz\right| < \int_0^\pi e^{-pR\sin\theta}|z\phi(z)|d\theta$ $(z = Re^{i\theta}) < \pi\varepsilon$, since $p > 0$

and therefore $e^{-pR\sin\theta} < 1$ $(0 < \theta < \pi)$. Therefore $\int_1 f(z)dz \to 0$.

(b) $\left|\int_\Gamma e^{ipz}\phi(z)dz\right| < \varepsilon R\int_0^\pi e^{-pR\sin\theta}d\theta$ (where $|\phi(z)| < \varepsilon$). But

$$e^{-pR\sin\theta} < e^{-(2pR\theta)/\pi} \text{ in } 0 < \theta < \tfrac{1}{2}\pi \ (p > 0)$$

since $(\sin\theta)/\theta$ decreases steadily from 1 to $2/\pi$ in this interval

Therefore $\left|\int_\Gamma f(z)dz\right| < 2R\varepsilon\int_0^{\frac{1}{2}\pi} e^{-(2pR\theta)/\pi}d\theta = \dfrac{\pi\varepsilon}{p}(1 - e^{-pR}) < \dfrac{\pi\varepsilon}{p}$

i.e. $\int_1 f(z)dz \to 0$ as $R \to \infty$.

In particular let $\phi(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$, $Q(z)$ are polynomials of degrees m, n respectively, then $\int_1 f(z)dz \to 0$, when

$$(a)\ n \geq m + 2,\ p > 0 \quad (b)\ n \geq m + 1,\ p > 0.$$

Examples. (i) Consider $\int_C \dfrac{ze^{ipz}dz}{z^2 + a^2}$ $(a > 0)$.

By Lemma (b), $\int_\Gamma f(z)dz \to 0$ when $R \to \infty$ if $p > 0$.

The only singularity is at $z = ia$, and the residue there is

$$\frac{iae^{-pa}}{2ia} = \tfrac{1}{2}e^{-pa}.$$

Thus
$$\int_{-\infty}^{\infty} \frac{xe^{ipx}\,dx}{x^2 + a^2} = \pi i e^{-pa}$$

i.e. $\displaystyle\int_0^\infty \frac{x(e^{ipx} - e^{-ipx})dx}{x^2 + a^2} = \pi i e^{-pa}$ or $\displaystyle\int_0^\infty \frac{x\sin px}{x^2 + a^2}dx = \frac{\pi}{2}e^{-pa}$ $(p > 0,\ a > 0)$.

Note. If $p = 0$, the last integral is zero; if $p < 0$, the integral is $-\dfrac{\pi}{2}e^{pa}$ $(a > 0)$.

(ii) Let $f(z) = \dfrac{e^{ipz}}{(z^2 + a^2)(z^2 + b^2)}$ $(a > 0,\ b > 0,\ a \neq b)$.

By Lemma (a), $\int_\Gamma f(z)dz \to 0$ if $p > 0$.

Thus, as in example (i),

$$\int_0^\infty \frac{2\cos px\,dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i\left\{\frac{e^{-pa}}{2ia(b^2-a^2)} + \frac{e^{-pb}}{2ib(a^2-b^2)}\right\}$$

or

$$\int_0^\infty \frac{\cos px\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a^2-b^2)}(ae^{-pb}-be^{-pa})\ (p>0, a>0, b>0, a\neq b)$$

It is worth while pointing out here that this integral is a continuous function of a, b or p in the intervals $a>a_0>0$, $b>b_0>0$, $p>0$; and that it is legitimate to differentiate them under the integral sign with respect to a, b or p (within the above intervals), (for the proofs of these results, *see Chap. XI, §§ 11.54, 11.56*).

Thus (*a*) if $b\to a$, $\displaystyle\int_0^\infty \frac{\cos px\,dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}(1+ap)e^{-pa}\ (p>0, a>0)$

(*b*) Differentiation with regard to a gives

$$\int_0^\infty \frac{\cos px\,dx}{(x^2+a^2)^2(x^2+b^2)} = \frac{\pi}{4a^3b(a^2-b^2)^2}[-be^{-pa}\{pa(a^2-b^2)+3a^2-b^2\}+2a^3e^{-pb}]$$

(*c*) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2(x^2+b^2)} = \frac{\pi(2a+b)}{4a^3b(a+b)^2}$

(*d*) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2(x^2+b^2)^2} = \frac{\pi(a^2+3ab+b^2)}{4a^3b^3(a+b)^3}$

(*e*) $\displaystyle\int_0^\infty \frac{x\sin px\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi(e^{-pb}-e^{-pa})}{2(a^2-b^2)}$, &c.

It is necessary, however, that the resultant integral should be convergent (§ *11.56*). Thus the third derivative with regard to p gives

$$\int_0^\infty \frac{x^3\sin px\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a^2-b^2)}(a^2e^{-pa}-b^2e^{-pb}),$$

only for $p>0$, $a>0$, $b>0$, and the fourth derivative is not convergent.

(iii) Let $f(z) = \dfrac{z^2}{z^4+a^4}\ (a>0)$.

Here $\int_\Gamma f(z)dz\to 0$ by Lemma (*a*).

Therefore $\displaystyle 2\int_0^\infty \frac{x^2\,dx}{x^4+a^4} = 2\pi i$ (sum of residues at $a\alpha$, $a\alpha^3$) where

$$\alpha = (1+i)/\sqrt{2},\ \alpha^3 = (-1+i)/\sqrt{2}$$

or

$$S = \left[\frac{a^2\alpha^2}{4a^3\alpha^3} + \frac{a^2\alpha^6}{4a^3\alpha^9}\right] = \frac{\alpha^3+\alpha}{4a} = \frac{i\sqrt{2}}{4a}$$

so that

$$\int_0^\infty \frac{x^2\,dx}{(x^4+a^4)} = \frac{\pi\sqrt{2}}{4a}\ (a>0)$$

As in the previous example, it is legitimate to differentiate under the integral sign, any number of times, with regard to a (if $a>0$).

Thus $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^4+a^4)^2} = \frac{\pi\sqrt{2}}{16a^5}$ and $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^4+a^4)^3} = \frac{5\pi\sqrt{2}}{128a^9}$.

(iv) Let $f(z) = \dfrac{1}{(z^2+1)\cosh\pi z}$.

There are poles at i, $\frac{1}{2}i$, $\frac{3}{2}i$, ..., $(n-\frac{1}{2})i$, . . .

Take R, the radius of the semicircle, to be n. Then on $|z| = n$, $\cosh \pi z$ must have a lower bound $m > 0$ and therefore $|\int_\Gamma f(z)dz| < \dfrac{\pi n}{(n^2+1)m}$ which $\to 0$ as $n \to \infty$,

i.e.
$$2\int_0^\infty \frac{dx}{(x^2+1)\cosh \pi x} = 2\pi i\left\{-\frac{1}{2i} + \sum_1^n \frac{1}{\pi i}\,\frac{1}{(r-\frac12)^2-1}(-1)^r\right\}$$
$$= \pi + \sum_1^n (-1)^r\left\{\frac{1}{r-3/2} - \frac{1}{r+\frac12}\right\}$$
$$= -\pi + 4 + (-1)^{n-1}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right)$$

i.e.
$$\int_0^\infty \frac{dx}{(x^2+1)\cosh \pi x} = 2 - \tfrac12\pi.$$

11. Let $f(z) = \{\log(z-\alpha)\}^p\phi(z)$.

Take $\log(z-\alpha)$ to be the principal value and suppose that α is outside the semicircle, i.e. $I(\alpha) < 0$.

Assume that on $|z| = R$, $|\phi(z)| = O(R^{-\beta})$, $(\beta > 0)$. Then
$$|f(z)| = O\{(\log R)^p / R^\beta\}$$
and since the length of the semicircle is $2\pi R$, $\int_\Gamma f(z)dz \to 0$ if $\beta > 1$. In particular, if $\phi(z) = P(z)/Q(z)$ where P, Q are polynomials of degrees m, n, $\int_\Gamma f(z)dz \to 0$ if $n \geq m+2$.

Examples. (i) Let $f(z) = \dfrac{\{\log(z+i)\}^2}{z^2+1}$.

Then $\int_\Gamma f(z)dz \to 0$ and $\displaystyle\int_{-\infty}^\infty \frac{\{\log(x+i)\}^2}{x^2+1}dx = \pi(\log 2 + i\pi/2)^2$.

Now $\log(x+i) = \frac12\log(x^2+1) + i\left(\frac{\pi}{2} - \operatorname{arc\,tan} x\right)$.

Equation of the real parts gives
$$\frac14\int_{-\infty}^\infty \frac{\{\log(x^2+1)\}^2}{x^2+1}dx - \int_{-\infty}^\infty \frac{(\frac12\pi - \operatorname{arc\,tan} x)^2\,dx}{x^2+1} = \pi\{(\log 2)^2 - \tfrac14\pi^2\}$$

But $\displaystyle\int_{-\infty}^\infty \frac{(\frac12\pi - \operatorname{arc\,tan} x)^2\,dx}{x^2+1} = \int_{-\frac12\pi}^{\frac12\pi}\left(\frac{\pi}{2} - \theta\right)^2 d\theta = \tfrac13\pi^3$ and therefore, writing $\tan\theta$ for x, we find $\displaystyle\int_0^{\frac12\pi}(\log\cos\theta)^2\,d\theta = \tfrac12\pi\{(\log 2)^2 + \tfrac1{12}\pi^2\}$.

Equation of the imaginary parts gives
$$\int_{-\infty}^\infty \frac{\{\log(x^2+1)\}\{\frac12\pi - \operatorname{arc\,tan} x\}dx}{x^2+1} = \pi^2\log 2.$$

But $\displaystyle\int_{-\infty}^\infty \frac{\log(x^2+1)}{x^2+1}\operatorname{arc\,tan} x\,dx = 0$ by symmetry and therefore
$$\int_0^{\pi/2}(\log\cos\theta)\,d\theta = -\tfrac12\pi\log 2.$$

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{\log(x+ia)}{(x^2+b^2)^2}dx = 2\pi i$ (residue at $z = ib$) $(a > 0)$.

If $z = ib + \zeta$,

$$\text{integrand is } \frac{1}{4b^2\zeta^2}\left\{\log(a+b) + \frac{i\pi}{2} + \frac{\zeta}{i(b+a)} + \dots\right\}\left\{1 - \frac{\zeta}{ib} + \dots\right\}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{\left\{\frac{1}{2}\log(x^2+a^2) + i\left(\frac{1}{2}\pi - \arctan\frac{x}{a}\right)\right\}dx}{(x^2+b^2)^2} = \frac{\pi}{2b^3}\log(a+b) - \frac{\pi}{2b^2(a+b)} + \frac{\pi^2 i}{4b^3}.$$

Equation of real parts gives

$$\int_0^{\infty} \frac{\log(x^2+a^2)dx}{(x^2+b^2)^2} = \frac{\pi}{2b^3}\left\{\log(a+b) - \frac{b}{a+b}\right\}.$$

Equation of the imaginary parts gives simply

$$\int_0^{\infty} \frac{dx}{(x^2+b^2)^2} = \frac{\pi}{4b^3}.$$

10.831. Integrals from $c - i\infty$ to $c + i\infty$ (c real). Suppose that $f(z)$ has a finite number of poles in the finite part of the plane and that $f(z) \to 0$ as $|z| \to \infty$.

Consider the integral $\int_{\Gamma_1} e^{az}f(z)dz$ where a is real and Γ_1 is the *left* half of the circle $|z - c| = R$ (c real). The transformation $z = c + i\zeta$ changes Γ_1 into the *upper* half Γ_0 of the circle $|\zeta| = R$ and the integral into $\int_{\Gamma_0} e^{ai\zeta+ac}f(c+i\zeta)i\,d\zeta$ which tends to zero as $R \to \infty$ if $a > 0$.

Similarly the integral $\int_{\Gamma_2} e^{az}f(z)dz$ where Γ_2 is the *right* half of the circle $|z - c| = R$, is transformed by the substitution $z = c - i\zeta$ into $\int_{\Gamma_0} e^{-ai\zeta+ac}f(c-i\zeta)i\,d\zeta$, which tends to zero as $R \to \infty$ if $a < 0$.

Let $a > 0$; and integrate $\int_C e^{az}f(z)dz$ round the closed contour C consisting of the left half of the circle $|z - c| = R$ and the bounding diameter $x = c$. We obtain

$$\int_{c-iR}^{c+iR} e^{az}f(z)dz + \int_{\Gamma_1} e^{az}f(z)dz = 2\pi i S_1$$

where S_1 is the sum of the residues of $e^{az}f(z)$ within C. It is assumed that R, c are chosen so that C does not pass through a pole.

Let $R \to \infty$, then $\int_{c-i\infty}^{c+i\infty} e^{az}f(z)dz = 2\pi i S_1$ where S_1 is the sum of the residues of $e^{az}f(z)$ on the left of $x = c$.

Let $a < 0$; and integrate round the right half of $|z - c| = R$ and also along the bounding diameter $x = c$. Then

$$\int_{c-i\infty}^{c+i\infty} e^{az}f(z)dz = -2\pi i S_2$$

where S_2 is the sum of the residues of $e^{az}f(z)$ on the right of $x = c$.

If the number of poles is infinite, we must replace S_1 (or S_2) by the limit of the sum.

Note. If $|zf(z)| \to 0$ as $|z| \to \infty$, the integral round either semicircle tends to zero as $R \to \infty$ if $a = 0$; and therefore the integral is continuous at $a = 0$. This is verified by noting that $\int_C f(z)\,dz$ where C is the large circle $|z - c| = R$ is zero, i.e. $S_1 + S_2 = 0$.

Examples. (i) Let $f(z) = \dfrac{1}{(z-1)(z^2+1)}$, then $\int_{c-i\infty}^{c+i\infty} e^{az}f(z)dz$ is equal to

(a) $c > 1$; $\pi i(e^a - \cos a - \sin a)$ $(a > 0)$; 0 $(a < 0)$
(b) $0 < c < 1$; $-\pi i(\cos a + \sin a)$ $(a > 0)$, $-\pi i e^a$ $(a < 0)$
(c) $c < 0$; 0 $(a > 0)$, $-\pi i(e^a - \cos a - \sin a)$ $(a < 0)$.

(ii) Let $f(z) = \dfrac{1}{z-k}$ and $a = \log p$ $(p > 0)$. Then $\int_{c-i\infty}^{c+i\infty} \dfrac{p^z}{z-k}dz = 2\pi i p^k$ $(p > 1)$, 0 $(0 < p < 1)$, $(c > k)$.

10.832. Indented Semicircle. Let $F(z)$ have a pole (of order m) at $z = 0$. Indent the contour consisting of Γ, the upper half of $|z| = R$ and the diameter along $y = 0$, by the upper half γ of the small circle $|z| = \rho$. (*Fig. 30*) Suppose, as before, that $\int_\Gamma F(z)dz \to 0$ as $R \to \infty$;

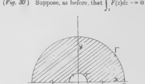


FIG. 30

and let us determine conditions for which $\int_\gamma F(z)dz$ tends to a limit K when $\rho \to 0$. When these conditions are satisfied, since

$$\int_\Gamma F(z)dz - \int_\gamma F(z)dz + \int_{DC} F(x)dx + \int_{AB} F(x)dx = 2\pi i S$$

where S is the sum of the residues within the contour, and A, B, C, D are respectively the points $\rho, R, -\rho, -R$, then

$$\int_0^\infty \{F(x) + F(-x)\}dx = 2\pi i S + K$$

where S is the sum of residues at all poles α for which $I(\alpha) > 0$.

Let the principal part of $F(z)$ at $z = 0$ be $G(z)$ where

$$G(z) = \sum_1^m A_r z^{-r}.$$

Then $\int_\gamma G(z)dz = \pi i A_1 + \sum_2^m \frac{A_r}{1-r}\{(-\rho)^{1-r} - \rho^{1-r}\}$ which tends to the limit $\pi i A_1$ only when $A_2 = A_4 = A_6 = \ldots = 0$. Thus

$$\int_\gamma F(z)dz \to \pi i A_1$$

if the principal part of $F(z)$ at $z = 0$ contains only *odd* powers of $1/z$, i.e. $\int_0^\infty \{F(x) + F(-x)\}dx = 2\pi i(S + \frac{1}{2}S_0)$ where S_0 is the residue at 0.

Examples. (i) Let $F(z) = e^{ipz}/z$.

$\int_\Gamma F(z)dz \to 0$ as $R \to \infty$ if $p > 0$. $F(z)$ is of the correct form at 0.

Therefore $\int_0^\infty \frac{2i \sin px}{x}dx = \pi i$ or $\int_0^\infty \frac{\sin px}{x}dx = \frac{1}{2}\pi\ (p > 0)$. Also the integral is equal to 0, $(p = 0)$ and $-\frac{1}{2}\pi\ (p < 0)$.

(ii) Let $F(z) = \frac{1}{z^3}(e^{iaz} + e^{ibz} + e^{icz} - e^{i(a+b+c)z})$ $(a,\ b,\ c$ real). Then $\int_\Gamma F(z)dz \to 0$ if $a, b, c \geq 0$. The principal part of $F(z)$ at $z = 0$ is $\frac{2}{z^3} + \frac{(ab + bc + ca)}{z}$ and is of the correct form

Therefore

$$\int_0^\infty \frac{\sin ax + \sin bx + \sin cx - \sin(a+b+c)x}{x^3}dx$$

$$= \frac{\pi}{2}(ab + bc + ca)\ (a, b, c \geq 0).$$

It is obvious from the form of $F(z)$ that we may also have a, b, c zero. Thus

$$\int_0^\infty \frac{\sin ax + \sin bx - \sin(a+b)x}{x^3}dx = \frac{\pi}{2}ab\ (a, b \geq 0).$$

The integral (which is convergent for all finite a, b, c) takes a different form when any of the numbers a, b, c is negative. The reader may easily verify that the value is

(i) $a, b \geq 0, c < 0$; $\frac{1}{2}\pi(a+c)(b+c)\ (a + b + c > 0)$;
$\frac{1}{2}\pi(a^2 + b^2 + ab + bc + ca)\ (a + b + c < 0)$

(ii) $a \geq 0,\ b, c < 0$: $\frac{1}{2}\pi(b^2 + c^2 + ab + bc + ca)\ (a + b + c > 0)$;
$-\frac{1}{2}\pi(a+b)(a+c)\ (a + b + c < 0)$

(iii) $a, b, c < 0$, $\quad -\frac{1}{2}\pi(bc + ca + ab)$.

Also $a > 0$, $b < 0$, $I = \frac{1}{2}\pi b(a + b)$ $(a + b > 0)$, $\quad \frac{1}{2}\pi a(a + b)$ $(a + b < 0)$.

$$a < 0,\ b < 0,\ I = -\tfrac{1}{2}\pi ab \text{ where } I = \int_0^\infty \frac{\sin ax + \sin bx - \sin (a + b)x}{x^3}\,dx.$$

Particular cases are

$$\int_0^\infty \frac{3 \sin x - \sin 3x}{x^3}\,dx = \frac{3\pi}{2};\quad \int_0^\infty \frac{(2 \sin 2x - \sin 4x)}{x^3}\,dx = 2\pi;$$

i.e.
$$\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{8};\quad \int_0^\infty \left(\frac{\sin x}{x}\right)^3 \cos x\,dx = \frac{\pi}{4}.$$

Note. We may also consider the case when the semicircle is indented at other points on the real axis (to avoid poles). The real integrals obtained will then be Principal Values. For example, by taking $F(z) = e^{ipz}/\{(z - c)(z^2 + a^2)\}$ it will easily be verified, if $p > 0$, that

$$P\int_0^\infty \frac{\cos px\,dx}{(x^2 - c^2)(x^2 + a^2)} = -\frac{\pi \sin pc}{2c(c^2 + a^2)} - \frac{\pi e^{-pa}}{2a(c^2 + a^2)} \text{ and}$$

$$P\int_0^\infty \frac{x \sin px\,dx}{(x^2 - c^2)(x^2 + a^2)} = \frac{\pi \cos pc}{2(c^2 + a^2)} - \frac{\pi e^{-pa}}{2(c^2 + a^2)}$$

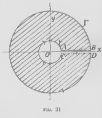


FIG. 31

10.84 *The Double Circle Contour.* Let Γ denote the circle $|z| = R$ and γ the circle $|z| = \rho$, where R is large and ρ is small. Let these be joined along the real axes from ρ to R. Take as a closed contour the line AB from ρ to R followed by Γ, then DC from R to ρ and finally γ (described clockwise). AB is coincident with CD but is shown distinct in *Fig. 31* for clearness.

Such a contour is suitable for the integration of a function that has a branch point at O, e.g.
$$F(z) = (\log z)^p f(z) \text{ or } z^{a-1}f(z).$$

In particular, let us suppose that $\int_\Gamma F(z)dz$ and $\int_\gamma F(z)dz$ both tend to zero when $R \to \infty$ and $\rho \to 0$ respectively. Let the value of $F(z)$ on AB be denoted by $F(x)$, and its value on CD (after a circuit of O) be $F_1(x)$. Then

$$\int_0^\infty \{F(x) - F_1(x)\}dx = 2\pi i S$$

where S is the sum of the residues of $F(z)$ at its singularities between γ and Γ (these singularities not being branch points). Also it is assumed

that $F(z)$ has no singularities on the real axis. We shall consider the typical cases

$$\text{I. } (\log z)^p \frac{P(z)}{Q(z)} \quad \text{II. } z^{a-1}\frac{P(z)}{Q(z)}, \ (p \text{ positive, } a \text{ real})$$

where in each case the principal values of $(\log z)^p$ and z^{a-1} are taken on AB, and where P, Q are polynomials of degrees m, n.

I. Let $F(z) = (\log z)^p \dfrac{P(z)}{Q(z)}$.

On Γ, $|F(z)| = O\left\{\dfrac{(\log R)^p}{R^{n-m}}\right\}$ and therefore

$$\left|\int_\Gamma F(z)dz\right| = O\left\{\frac{(\log R)^p}{R^{n-m-1}}\right\}, \text{ i.e. } \int_\Gamma F(z) \to 0 \text{ if } n > m + 1$$

On γ, $|F(z)| = O\{|\log \rho|^p\}$, ($z = 0$ not being a zero of $Q(z)$).

Therefore $\displaystyle\int_\gamma F(z)dz = O\{\rho|\log \rho|^p\}$ and therefore tends to zero as $\rho \to 0$. Thus, under these conditions,

$$\int_0^\infty \{(\log x)^p - (\log x + 2\pi i)^p\}\frac{P(x)}{Q(x)}dx = 2\pi i S, \text{ where } S$$

is the sum of the residues of $(\log z)^p \dfrac{P(z)}{Q(z)}$ at the zeros of $Q(z)$ (none of which are assumed to be real and positive).

By taking $p = 1, 2, 3, \ldots s$, we obtain s linear equations for the determination of

$$\int_0^\infty \frac{P(x)}{Q(x)}\,dx, \int_0^\infty \frac{P(x)}{Q(x)}\log x\,dx, \ldots, \int_0^\infty \frac{P(x)}{Q(x)}(\log x)^{s-1}\,dx.$$

Since, however, the real and imaginary parts may be equated, these equations are redundant.

Example. Find $\displaystyle\int_0^\infty \frac{(\log x)^2\,dx}{x^2+x+1}$.

Take $F(z) = \dfrac{(\log z)^3}{z^2+z+1}$; then $\int_\Gamma F(z)dz$ and $\int_\gamma F(z)dz$ both tend to 0. The poles are ω, $\omega^2 = \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi$.

Therefore $\displaystyle\int_0^\infty \frac{(\log x)^3 - (\log x + 2\pi i)^3}{x^2+x+1}\,dx = \frac{2\pi i}{2i\sin\frac{2}{3}\pi}\{(\log\omega)^3 - (\log\omega^2)^3\}$

i.e.
$$-3I_2 - 6\pi i I_1 + 4\pi^2 I_0 = \frac{-7}{i\sqrt{3}}\left(\frac{2i\pi}{3}\right)^3 = \frac{56\pi^3}{27\sqrt{3}}$$

where $I_n = \displaystyle\int_0^\infty \frac{(\log x)^n}{x^2+x+1}\,dx$.

Therefore $I_1 = \displaystyle\int_0^\infty \frac{\log x\,dx}{x^2+x+1} = 0$ (a result that may be verified by dividing

the range into the intervals (0, 1), (1, ∞) and writing $1/x$ for x in the second interval; similarly $I_n = 0$ if n is odd).

Also $I_3 = -\dfrac{56\pi^3}{81\sqrt{3}} + \dfrac{4\pi^2}{3}\cdot\dfrac{2}{\sqrt{3}}\left\{\arctan\left(\dfrac{2x+1}{\sqrt{3}}\right)\right\}_0^\infty = \dfrac{16\pi^3}{81\sqrt{3}}$.

II. Let $F(z) = z^{\alpha-1}\dfrac{P(z)}{Q(z)}$ (α real).

On Γ, $|F(z)| = O(R^{\alpha+m-n-1})$ and therefore $\int_\Gamma F(z)dz \to 0$ if $\alpha < n - m$.

On γ, $|F(z)| = O(\rho^{\alpha-1})$ and therefore $\int_\gamma F(z)dz \to 0$ if $\alpha > 0$.

Thus if $0 < \alpha < n - m$, we have

$$\int_0^\infty \{e^{(\alpha-1)\log x} - e^{(\alpha-1)(\log x + 2\pi i)}\}\frac{P(x)}{Q(x)}\,dx = 2\pi i S$$

i.e.
$$\int_0^\infty x^{\alpha-1}\frac{P(x)}{Q(x)}dx = -\frac{\pi e^{-\alpha\pi i}S}{\sin\alpha\pi}.$$

Examples. (i) $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx = -\frac{\pi e^{-\alpha\pi i}}{\sin\alpha\pi}(-1)^{\alpha-1}\ (0 < \alpha < 1)$

$$= -\frac{\pi e^{-\alpha\pi i}e^{(\alpha-1)\pi i}}{\sin\alpha\pi} = \frac{\pi}{\sin\alpha\pi}.$$

(ii) Find $\displaystyle\int_0^\infty \frac{x^{-p}\,dx}{1 + 2x\cos\lambda + x^2}$ where $-\pi < \lambda < \pi$.

For convergence $|p| < 1$. The poles are $-\cos\lambda \mp i\sin\lambda$ and therefore the correct amplitudes are $\pi \mp \lambda$ since these lie between 0, 2π

$$\int_0^\infty \frac{x^{-p}\,dx}{1 + 2x\cos\lambda + x^2} = -\frac{\pi e^{-\pi i(1-p)}}{\sin p\pi}S \text{ when } S = \frac{e^{-ip(\pi-\lambda)} - e^{-ip(\pi+\lambda)}}{2i\sin\lambda}$$

i.e. $\quad S = e^{-ip\pi}\dfrac{\sin p\lambda}{\sin\lambda}$ and $\displaystyle\int_0^\infty \frac{x^{-p}\,dx}{1 + 2x\cos\lambda + x^2} = \frac{\pi\sin p\lambda}{\sin p\pi\sin\lambda}.$

Thus $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{1 + x + x^2} = 2\int_0^\infty \frac{y^2\,dy}{1 + y^2 + y^4} = \frac{\pi}{\sqrt{3}},$

$$\int_0^\infty \frac{dx}{x^{\frac16}(1 + x^2)} = 6\int_0^\infty \frac{y^4\,dy}{1 + y^{12}} = \frac{\pi(\sqrt{3} - 1)}{\sqrt{2}}.$$

Note. When $Q(x)$ has zeros on the real axis, the contour may be indented to give principal values of real integrals.

10.85. Sector of a Circle (indented if necessary). Instead of the upper half of $|z| = R$ together with its diameter, we can take for C the boundary of the sector of this circle determined by $0 < \operatorname{amp} z < \alpha$ ($< \pi/2$). (*Fig.* 32 (i).) If $z = 0$ is a singularity, this sector may be indented by the corresponding arc γ of the small circle $|z| = \rho$. (*Fig.* 32 (ii).) Suppose that $\int_\Gamma F(z)dz \to 0$ as $R \to \infty$, and that $F(z)$ has at most a simple pole at $z = 0$ of residue A_0. Then the limit of $\int_\gamma F(z)dz$ is $i\alpha A_0$ (γ being determined counter-clockwise with reference to O).

On the radius OB, take $z = te^{i\alpha}$. Then when $R \to \infty$ and $\rho \to 0$ we have

$$\int_0^r F(x)dx - \int_0^r F(te^{i\alpha})e^{i\alpha}\,dt = 2\pi i S + i\alpha A_0$$

where S is the sum of the residues of $F(z)$ within the sector (it being

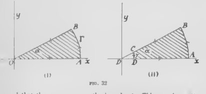


FIG. 32

assumed that there are none on the boundary). This may be written $\int_0^\infty F(te^{i\alpha})e^{i\alpha}\,dt = \int_0^\infty F(x)dx - 2\pi i S - i\alpha A_0$.

Examples. (i) Let $F(z) = e^{-z^2}$.

On Γ, $z = R(\cos\theta + i\sin\theta)$ and $|F(z)| = e^{-R^2\cos 2\theta}$ $(0 < \theta < \alpha)$.

Thus $\left|\int_\Gamma e^{-z^2}\,dz\right| < \int_\beta^{\frac{1}{2}\pi} \tfrac{1}{2}Re^{-R^2\sin\phi}\,d\phi$ where $\phi = \tfrac{1}{2}\pi - 2\theta$, $\beta = \tfrac{1}{2}\pi - 2\alpha$. But $\pi\sin\phi > 2\phi$ for $0 < \phi < \tfrac{1}{2}\pi$, i.e. $0 < \theta < \tfrac{1}{4}\pi$

i.e.
$$\left|\int_\Gamma e^{-z^2}\,dz\right| < \frac{\pi}{4R}\left[\exp\left\{-\frac{4R^2}{\pi}\left(\frac{\pi}{4} - \alpha\right)\right\} - \exp(-R^2)\right]$$

which $\to 0$ as $R \to \infty$ if $0 < \alpha < \tfrac{1}{4}\pi$.

There are no singularities within the sector, nor on it.

Therefore
$$\int_0^\infty e^{-t^2(\cos 2\alpha + i\sin 2\alpha)}e^{i\alpha}\,dt = \int_0^\infty e^{-x^2}\,dx.$$

It can be shown that $\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$. (*Chap. XII, § 12·24.*)

Therefore

$$\int_0^\infty e^{-x^2\cos 2\alpha}\{\cos(x^2\sin 2\alpha) - i\sin(x^2\sin 2\alpha)\}dx = \frac{\sqrt{\pi}}{2}(\cos\alpha - i\sin\alpha)$$

i.e.
$$\int_0^\infty e^{-x^2\cos 2\alpha}\cos(x^2\sin 2\alpha)dx = \frac{\sqrt{\pi}}{2}\cos\alpha\,;$$

$$\int_0^\infty e^{-x^2\cos 2\alpha}\sin(x^2\sin 2\alpha)\,dx = \frac{\sqrt{\pi}}{2}\sin\alpha\,(0 < \alpha < \pi/4).$$

These results are then obviously true for $0 > \alpha > -\pi/4$.

In particular $\displaystyle\int_0^\infty e^{-x^2/2}\cos\left(\frac{x^2\sqrt{3}}{2}\right)dx = \frac{\sqrt{3\pi}}{4}$; $\displaystyle\int_0^\infty e^{-\frac{1}{2}x^2}\sin\left(\frac{x^2\sqrt{3}}{2}\right)dx = \frac{\sqrt{\pi}}{4}$.

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}} \quad (\textit{Fresnel's Integrals.})$$

Also
$$\int_0^\infty e^{-ax^2}\sin(bx^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}\frac{\{(a^2+b^2)^{\frac{1}{2}}-a\}^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}} \quad (a > 0,\ b > 0).$$

(ii) Let $F(z) = e^{iz}/z$.

$\int_\Gamma F(z)dz \to 0$ for $0 < \alpha < \pi$, residue at $z = 0$ is 1.

Therefore
$$\int_0^\infty \{e^{ix(\cos\alpha + i\sin\alpha)} - e^{ix}\}\frac{dx}{x} = \alpha i$$

i.e.
$$\int_0^\infty e^{-x\sin\alpha}\sin(x\cos\alpha).\frac{dx}{x} - \int_0^\infty \frac{\sin x\,dx}{x} = -\alpha$$

since the integrals converge.

Thus
$$\int_0^\infty e^{-x\sin\alpha}\sin(x\cos\alpha).\frac{dx}{x} = \frac{\pi}{2} - \alpha \quad (0 \leq \alpha \leq \pi).$$

By a simple change of variable, we obtain also
$$\int_0^\infty e^{-ax}\sin bx\frac{dx}{x} = \frac{\pi}{2} - \text{arc tan}\left(\frac{a}{b}\right) \quad (a \geq 0,\ b > 0).$$

In particular
$$\int_0^\infty e^{-x}\sin x\frac{dx}{x} = \pi/4.$$

10.86. Rectangles (indented if necessary). The following examples show the use of rectangles when the integrand contains a periodic function.

Example. (i) Let $F(z) = \dfrac{e^{az}}{\cosh\pi z}$ and take the contour C to consist of the boundary of the rectangle determined by

$$x = \pm R,\ y = 0,\ y = 1 \quad (\textit{Fig. 33.})$$

The poles of $F(z)$ are $\pm i/2, \pm 3i/2, \ldots$, one of which is within C, the others outside.

Therefore
$$\int_C F(z)dz = 2\pi i.\frac{e^{\frac{1}{2}ai}}{\pi i} = 2e^{\frac{1}{2}ai}.$$

Along AB, $z = R + iy$ $(0 < y < 1)$ and $|F(z)| \leq \dfrac{2e^{aR}}{e^{\pi R} - e^{-\pi R}}$ (a real) and therefore $\int_{AB}F(z)dz \to 0$ if $a < \pi$ when $R \to \infty$. Similarly $\int_{CD}F(z)dz \to 0$ if $a > -\pi$. On BC, $z = x + i$ $(|x| < R)$ and therefore

$$\int_{BC}F(z)dz = -\int_{-R}^{R}\frac{e^{a(x+i)}\,dx}{(-\cosh\pi x)} = +e^{ai}\int_{-R}^{R}\frac{e^{ax}\,dx}{\cosh\pi x}.$$

Thus $\displaystyle\int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh\pi x}(1 + e^{ai}) = 2e^{\frac{1}{2}ai}$, giving $\displaystyle\int_{-\infty}^{\infty}\frac{e^{ax}\,dx}{\cosh\pi x} = \sec\tfrac{1}{2}a$ and

$$\int_0^\infty \frac{\cosh ax}{\cosh\pi x}dx = \tfrac{1}{2}\sec\tfrac{1}{2}a \quad (|a| < \pi).$$

(ii) Take $F(z) = \dfrac{e^{iaz}}{\sinh z}$ (a real), and C to be the boundary of the rectangle $ABCD$, $y = 0$, $y = \pi$, $x = \pm R$ indented at O, $i\pi$. (*Fig. 34.*)

The indentations consist of small semicircles γ_1 (of $|z| = \rho_1$), γ_2 (of $|z - i\pi| = \rho_2$) drawn into the rectangle.

Along $x = R$, $\left|\int_{AB} F(z)dz\right| \leq \int_0^\pi \dfrac{2e^{-ay}}{e^R - e^{-R}}\,dy$ which $\to 0$ when $R \to \infty$.

Similarly along $x = -R$, $\int_{CD} F(z)dz \to 0$.

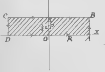


FIG. 33

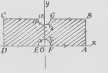


FIG. 34

Since the poles at O, $i\pi$ are simple we have as in previous examples $\int_{\gamma_1} F(z)dz \to -\pi i$ (γ_1 being described clockwise) when $\rho_1 \to 0$. The residue at $z = i\pi$ is $-e^{-a\pi}$ and therefore $\int_{\gamma_2} F(z)dz \to \pi i e^{-a\pi}$ when $\rho_2 \to 0$. There are no singularities within the indented rectangle, and therefore

$$P\int_{-\infty}^{\infty} \frac{e^{iax}\,dx}{\sinh x} - P\int_{-\infty}^{\infty} \frac{e^{ia(x+i\pi)}\,dx}{(-\sinh x)} = \pi i(1 + e^{-a\pi})$$

i.e.

$$P\int_{-\infty}^{\infty} \frac{e^{iax}}{\sinh x}\,dx = \frac{\pi i(e^{\frac{1}{2}a\pi} + e^{-\frac{1}{2}a\pi})}{(e^{\frac{1}{2}a\pi} - e^{-\frac{1}{2}a\pi})}$$

or $\int_0^\infty \dfrac{\sin ax}{\sinh x}\,dx = \dfrac{\pi}{2}\tanh\dfrac{a\pi}{2}$ (a real), the symbol P being now unnecessary.

(iii) Let $F(z) = \dfrac{e^{iaz}}{e^{2\pi z} - 1}$ (a real), for the boundary C given $x = 0$, $x = R$, $y = 0$, $y = 1$ indented at O, i. (*Fig. 35.*)

The indentations consist of small quadrants γ_1 (of $|z| = \rho_1$) and γ_2 (of $|z - i| = \rho_2$) drawn within the rectangle.

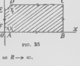


FIG. 35

On $x = R$, $|\int F(z)dz| < \int_0^1 \dfrac{e^{-ay}\,dy}{e^{2\pi R} - 1}$ which $\to 0$ as $R \to \infty$.

The points O, i are simple poles and therefore $\int_{\gamma_1} F(z)dz \to -\frac{1}{4}i$ when $\rho_1 \to 0$, the direction of the integral being clockwise. The residue at $z = i$ is $e^{-a}/2\pi$ and therefore $\int_{\gamma_2} F(z)dz \to -\frac{i}{4}e^{-a}$ when $\rho_2 \to 0$.

There are no singularities within C, and therefore the imaginary part of the integral $\int_C F(z)dz$ is zero.

Along the real axis $I\int F(z)dz = \int_{\rho_1}^{R} \frac{\sin ax}{e^{2\pi x}-1}\,dx$, and along the opposite side

$$I\int F(z)dz = -e^{-a}\int_{\rho_2}^{R} \frac{\sin ax}{e^{2\pi x}-1}\,dx.$$

Along the imaginary axis (of the rectangle)

$$I\int F(z)dz = -I\int_{\rho_1}^{1-\rho_2} \frac{ie^{-ay}\,dy}{e^{2\pi iy}-1} = \int_{\rho_1}^{1-\rho_2} \tfrac{1}{2}e^{-ay}\,dy = -\frac{1}{2a}\{e^{-a(1-\rho_2)} - e^{-a\rho_1}\}$$

Also $\lim I\int_{\gamma_1} F(z)dz = -\frac{1}{4}$ and $\lim I\int_{\gamma_2} F(z)dz = -\frac{1}{4}e^{-a}$.

Therefore when $\rho_1, \rho_2 \to 0$, $R \to \infty$ we find

$$\lim\left[\int_{\rho_1}^{R} \frac{\sin ax}{e^{2\pi x}-1}\,dx - e^{-a}\int_{\rho_2}^{R} \frac{\sin ax}{e^{2\pi x}-1} - \frac{1}{2a}\{e^{-a(1-\rho_2)} - e^{-a\rho_1}\}\right] = \tfrac{1}{4}e^{-a} + \tfrac{1}{4}$$

i.e.

$$\int_0^{\infty} \frac{\sin ax}{e^{2\pi x}-1}\,dx = \tfrac{1}{4}\frac{e^a+1}{e^a-1} - \frac{1}{2a} = \tfrac{1}{4}\coth\frac{a}{2} - \frac{1}{2a}.$$

Examples X

1. Explain why the following statement is not a *definition* of a complex number: 'A complex number is a number of the form $p + iq$ where p and q are real and i is a root of the equation $x^2 + 1 = 0$.'

Express the numbers given in *Examples 2–22* in the form $a + ib$ where a and b are real.

2. $(1-2i)^2$. **3.** $(3-i)^3$. **4.** $\dfrac{4-i}{(3+i)^2}$. **5.** $\dfrac{(3-2i)(1-i)}{(2+i)^2}$.

6. $\dfrac{(1+2i)^4}{(1-2i)^4} + \dfrac{(1-2i)^4}{(1+2i)^4}$. **7.** $\dfrac{(1+i)^3}{(1-i)^3} - \dfrac{(1-i)^3}{(1+i)^3}$. **8.** $\dfrac{\cos\theta + i\sin\theta}{\cos\phi + i\sin\phi}\cdot\dfrac{1-i}{1+i}$.

9. $\dfrac{(\cos\theta + i\sin\theta)^2}{(1+i\sqrt{3})^4}$. **10.** i^i. **11.** $(1+i)^i$. **12.** $\text{Log}\,(1+i)$.

13. $\tan(1+i\sqrt{3})$. **14.** $\sin(3i)$. **15.** $\cos(i-1)$. **16.** $\pi^{-2i\pi}$.

17. $\cot\dfrac{i\pi}{4}$. **18.** $\text{Sin}^{-1}(2)$. **19.** $\text{Cos}^{-1}\left(\dfrac{3i}{2}\right)$. **20.** $\text{Cosh}^{-1}(-1)$.

21. $\text{Tan}^{-1}(\cos\alpha + i\sin\alpha)$. **22.** $\text{Tanh}^{-1}(2)$.

Find the moduli and amplitudes of the numbers given in *Examples 23–30*.

23. $\dfrac{(1-i)^2}{(1+i)^4}$. **24.** $\dfrac{28}{(2-i\sqrt{3})(5-i\sqrt{3})}$. **25.** $\dfrac{(3+i)(1+i)}{(3-i)}$.

26. $1 + \cos\alpha + i\sin\alpha$. **27.** $\dfrac{1+\cos\alpha+i\sin\alpha}{1+i}$. **28.** $\dfrac{1}{1+(\cos\alpha+i\sin\alpha)^2}$.

29. $\dfrac{1 - e^{i\alpha}}{1 + e^{2i\beta}}$ 30. $\dfrac{(x + iy) - (p + iq)}{(x - iy) + (p - iq)}$ where x, y, p, q are real.

If the co-ordinates of z are x, y, find the co-ordinates of the numbers given in *Examples 31–5*.

31. z^2 32. $1/z^2$. 33. $(3 + 4z)/(2 - z)$ 34. $z^2 + 2z + 4$
35. $1 - 1/z$.

Interpret geometrically the relations given in *Examples 36–44*, where z is a variable number and z_r a fixed number.

36. $\mathbf{R}\left(\dfrac{z - z_1}{z_3 - z_2}\right) = 0$ 37. $\mathbf{I}\left\{\dfrac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}\right\} = 0$

38. $\operatorname{amp}\left(\dfrac{z - z_1}{z - z_2}\right) = \text{constant}$ 39. $\left|\dfrac{z - z_1}{z - z_2}\right| = \text{constant}$

40. $|z - 1| + |z + 1| = 2$ 41. $|z - i| - |z + i| = 2$
42. $|z - 1| - |z + 1| = 1$. 43. $|2z - i| + |2z + i| = 6$
44. $z_1(z_2 - z_3) + z_2(z_3 - z_1) + z_3(z_1 - z_2) = 0$

45. Find the co-ordinates of the third vertices of the two equilateral triangles that can be drawn on the line joining (x_1, y_1), (x_2, y_2) as base.

Prove the results given in *Examples 46–61*.

46. $\left(\dfrac{1 + \cos\alpha + i\sin\alpha}{1 + \cos\alpha - i\sin\alpha}\right)^n = \cos n\alpha + i\sin n\alpha$.

47. $\sum\limits_0^n(\cos\theta + i\sin\theta)^r = \dfrac{\sin\frac{1}{2}(n + 1)\theta}{\sin\frac{1}{2}\theta}(\cos\tfrac{1}{2}n\theta + i\sin\tfrac{1}{2}n\theta)$

48. $7\tan^2\alpha = \dfrac{1 + 35\tan^4\alpha}{3 + \tan^4\alpha}$ if $\alpha = \dfrac{\pi}{14}$

49. $\sec^4\left(\dfrac{\pi}{7}\right) + \sec^4\left(\dfrac{2\pi}{7}\right) + \sec^4\left(\dfrac{3\pi}{7}\right) = 416$

50. $\sec^2\left(\dfrac{2\pi}{9}\right) + \sec^2\left(\dfrac{4\pi}{9}\right) + \sec^2\left(\dfrac{8\pi}{9}\right) = 192$

51. $\sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} + \sin\dfrac{8\pi}{7} = \dfrac{\sqrt{7}}{2}$ 52. $\prod\limits_{r=1}^{7}\cos r\alpha = \dfrac{1}{128}$ if $\alpha = \pi/15$

53. $\tan^{-1}(73) = \dfrac{\pi}{4} + \tan^{-1}(\tfrac{1}{3}) + \tan^{-1}(\tfrac{1}{5}) + \tan^{-1}(\tfrac{1}{7}) + \tan^{-1}(\tfrac{1}{9})$

54. $\tan^{-1}(5) = \tan^{-1}(\tfrac{1}{3}) + \tan^{-1}(\tfrac{1}{7}) + \tan^{-1}(\tfrac{1}{8}) + \dfrac{\pi}{4}$

55. $2^{n-1}\cos^n\theta = \cos n\theta + {}^nC_1\cos(n - 2)\theta + \ldots + \tfrac{1}{2}\,{}^nC_{\frac{1}{2}n}$ or ${}^nC_{\frac{1}{2}(n-1)}\cos\theta$
56. $128\sin^5\theta\cos^3\theta = \sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta$.

57. $\dfrac{\sin n\theta}{\sin\theta} = (2\cos\theta)^{n-1} - {}^{n-2}C_1(2\cos\theta)^{n-3} + {}^{n-3}C_2(2\cos\theta)^{n-5}\ldots$

58. $x^8 - x^4 + 1 = (x^2 - 2x\cos\alpha + 1)(x^2 + 2x\cos\alpha + 1)$
$(x^2 - 2x\sin\alpha + 1)(x^2 + 2x\sin\alpha + 1)$ $(\alpha = 15^\circ)$

59. $\cos\dfrac{\pi}{n}\cos\dfrac{2\pi}{n}\ldots\cos\dfrac{(2n - 1)\pi}{n} = \dfrac{(-1)^n - 1}{2^{2n-1}}$ if n is a positive integer.

60. $\sin\dfrac{\pi}{4n}\sin\dfrac{3\pi}{4n}\ldots\sin\dfrac{(2n - 1)\pi}{4n} = 2^{\frac{1}{2}-n}$, if n is a positive integer

61. $(1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n = 2^{n+1}$ or -2^n

Express the functions given in *Examples 62–5* as linear combinations of the sines or cosines of multiples of θ.

62 $\sin^6\theta\cos^6\theta$ 63. $\sin^3\theta\cos^5\theta$ 64. $\sin^3\theta\cos^4\theta$ 65. $\sin^4\theta\cos^3\theta$

Solve completely the equations given in *Examples 66–84*.

66. $z^4 + i = 0$ **67.** $z^6 = 1 - i\sqrt{3}$ **68.** $(z-2)^2 + z^2 = 0$

69. $(z^2 - 1)^2 = 8z^2$ **70.** $z^4 + 10z^2 + 20z - 31 = 0$ **71.** $z^2 = i$

72. $z^4 + 1 = i\sqrt{3}$ **73.** $z^4 - 1 = i\sqrt{3}$ **74.** $z^3 - 1 = i$ **75.** $z^3 = \sqrt{3} + i$.

76. $2z^3 - 1 = i\sqrt{3}$ **77.** $z^{10} + 1 = 0$ **78.** $\cos 2z = 2$

79. $\sin z = i$ **80.** $\tan z = 1 + i$ **81.** $e^{2z+3} = i$ **82.** $\sinh z = 2i$

83. $2\cosh 3z = 1$ **84.** $\tanh 2z = -2$

85. Show that if n is an integer,

$$\frac{x^{\frac{1}{3}}}{3.(3n)!}\{(1 + x^{\frac{1}{3}})^{3n} + \omega(1 + \omega x^{\frac{1}{3}})^{3n} + \omega^2(1 + \omega^2 x^{\frac{1}{3}})^{3n}\}$$
$$= \frac{x}{2!(3n-2)!} + \frac{x^2}{5!(3n-5)!} + \dots + \frac{x^n}{(3n-1)!}$$

86. If $w = \dfrac{(z-1)^2(z^2+16)}{(z^2+4)^2}$, find the increase in amp w when z describes the circle $|z| = 3$ once in the counter-clockwise direction

87. If $w = az^2 + bz + c$ (a, b, c real), show that the circle $|z| = R$ becomes the limaçon $\rho = R(2aR\cos\theta + b)$ where $w - c - aR^2 = \rho e^{i\theta}$.

Determine for the equations given in *Examples 88–93*, the number of roots within each quadrant, and the number that are real or purely imaginary.

88. $z^3 + z^2 = 4$ **89.** $z^4 + 6z^2 + 10 = 0$ **90.** $z^3 + 10z^2 = 4$

91. $z^6 - 6z^5 - 10 = 0$ **92.** $z^2 + 3iz = 1 - i$ **93.** $z^4 + (1-i)z^3 = i$

94. Prove that all the roots of the equation $z^4 - 4z^3 - 16z^2 - 24z + 78 = 0$ lie between the circles $|z| = 7$, $|z| = 1.25$.

95. If $w^2 = (z^2 + 4)(z^2 - 4z + 8)$ and if z, with initial value $4\sqrt{2}$ for w at $(0, 0)$, describe the circle $x^2 + y^2 = hy$ once in the counter-clockwise direction, find the values of w (i) when z crosses the y-axis (ii) when z returns to 0, for the values 1, 3, 5 for h.

For *Examples 96–8*, show how to cut the z plane so that the branches of w are single-valued.

96. $w^4 = (z^2 + 1)^3(z^2 + 4)$ **97.** $w^3 = \dfrac{(z^2+1)^2}{(z-2)^2}$ **98.** $w^3 = \dfrac{z^2(z+1)^2}{(z^2+4)}$

99. Discuss the behaviour of the values of z^i when z tends to zero or to $\pm\infty$ along the real axis.

Prove the results given in *Examples 100–7*.

100. $\cos\alpha + x\cos 2\alpha + x^2\cos 3\alpha + \dots = \dfrac{\cos\alpha - x}{1 - 2x\cos\alpha + x^2}$ $(|x| < 1)$

101. $\sin\alpha - x\sin 2\alpha + x^2\sin 3\alpha \dots = \dfrac{\sin\alpha}{1 + 2x\cos\alpha + x^2}$ $(|x| < 1)$

102. $\sin\theta + \dfrac{1}{2^{\frac{1}{2}}}\sin 2\theta + \frac{1}{2}\sin 3\theta + \dfrac{1}{2^{\frac{3}{2}}}\sin 4\theta + \dots = \dfrac{2\sin\theta}{3 - 2\sqrt{2}\cos\theta}$

103. $\dfrac{\cos 2\theta}{2!\cos^2\theta} + \dfrac{\cos 3\theta}{3!\cos^3\theta} + \dots = e\cos(\tan\theta) - 2$

104. $1 - \dfrac{z^4}{4!} + \dfrac{z^8}{8!} - \dots = \cos\dfrac{z}{\sqrt{2}}\cosh\dfrac{z}{\sqrt{2}}$

105. $1 - \dfrac{\pi^4}{4.(4!)} + \dfrac{\pi^8}{4^2(8!)} - \dots = 0$ **106.** $\pi = (96)^{\frac{1}{2}}(35)^{\frac{1}{4}}(\sqrt{35} - \sqrt{33})^{\frac{1}{2}}$ approx

107. $1 + \dfrac{\pi^3}{3^2(3!)} + \dfrac{\pi^6}{3^4(6!)} + \dfrac{\pi^9}{3^6(9!)} + \dots = \frac{1}{3}e^{\pi/\sqrt{3}}$

Find the areas in the w-plane that correspond to the first quadrant of the circle $|z| = 1$, for the values of w given in *Examples 108–12*.

108. $\dfrac{1}{z}$ **109.** $z - 1$ **110.** $\dfrac{1}{z-1}$ **111.** $1 - \dfrac{1}{z}$ **112.** $\dfrac{z}{z-1}$

113. Find the bilinear transformation for which $z = 2, 3, 1$ corresponds to $w = 1, 2, 3$ respectively.

114. If $w = \dfrac{iz + 1 - i}{z + i}$, find the transform of $(x - 1)^2 + y^2 = 1$.

115. If $w = \dfrac{z + i}{z - i}$, show that the circle $(x - 1)^2 + y^2 = 1$ becomes the circle $u^2 + v^2 - 4v + 2u + 1 = 0$.

116. If $w = \dfrac{2z + i}{z - 2i}$, show that the circle $x^2 + y^2 = k$ is transformed into the circle $(4 - k)(u^2 + v^2) + 4(1 + k)u + 1 - 4k = 0$.

117. If $w = \dfrac{az + b}{cz + d}$ $(a, b, c, d$ real$)$, where $ad - bc = 1$, prove that the upper half of the z-plane is transformed into the upper half of the w-plane.

118. If $w = \dfrac{z}{z - 1}$, determine the regions of the w-plane that correspond to the eight regions of the z-plane bounded by $|z| = 1$ and the co-ordinate axes.

119. If $w = \dfrac{az + b}{cz + d}$, prove that the circle $|w| = 1$ becomes a straight line if $|a| = |c|$.

120. Show that if $w = e^{i\theta}\dfrac{z - \alpha}{z - \bar{\alpha}}$ where θ is real and $I(\alpha) > 0$, the half plane $I(z) > 0$ is transformed into the unit circle $|w| < 1$.

121. Find the general bilinear transformation of the half plane $R(z) > 0$ on the unit circle $|w| < 1$.

Obtain the level curves $|w| =$ constant for the functions w given in *Examples 122–4*.

122. e^z **123.** $z + a^2/z$ **124.** $\sin z$.

125. If $2w = z + \dfrac{1}{z}$, show that the circles $|z| = k$ correspond to confocal ellipses.

126. If $2w = az + b/z$, show that the circle $|z| = 1$ corresponds to an ellipse whose foci are given by $w^2 = ab$ and whose axes are $|a| \pm |b|$.

127. If $w = \dfrac{(z - ic)^2}{(z + ic)^2}$ $(c$ real$)$, show that the semi-circle determined by $|z| < c$, $R(z) > 0$, is transformed into the real w axis and the upper half of the w-plane.

128. If $w = \left(\dfrac{z - 1}{z + 1}\right)^4$, then the real axis $I(w) = 0$ is represented by three circles through $1, -1$ of centres $i, 0, -i$ and also by the real z-axis: show also that the eight regions bounded by these loci represent $I(w) > 0$ and $I(w) < 0$ alternately, the point $+ i\infty$ being in the region $I(w) > 0$.

129. If w is the principal value of z^i, prove that the ring-shaped region bounded by $|w| = e^{-\pi}$, $|w| = e^{\pi}$ corresponds to a similar ring-shaped region in the z-plane.

130. If $w = ic \cot \frac{1}{2}z$ $(c$ real$)$, determine the region in the w-plane that corresponds to the semi-infinite strip in the upper half of the z-plane bounded by $x = 0$, $x = \pi$, $y = 0$.

131. If $w = a + bz + cz^2$, prove that the x-axis corresponds to a parabola which reduces to a straight line if $I(b/c) = 0$.

132. If $w = 4/(z + 1)^2$, show that the circle $|z| = 1$ becomes a parabola.

133. Prove that the transformation $w = 2z/(1 + z^2)$ establishes a $1 - 1$ correspondence between the interior of the circle $|z| = 1$ and the whole of the w-plane cut along the real axis from 1 to ∞ and from -1 to $-\infty$.

134. A circle of radius a in the z-plane has its centre at the point $ia \sin \beta$. It is transformed to a curve in the w-plane by means of the relation $w = z + c^2/z$. Prove that this curve is a circular arc.

135. Show that if $w = az + bz^2$ (a, b real, positive and $a + b = 1$), the circle $|w| = 1$ lies outside the curve in the w-plane corresponding to $|z| = 1$ except at $w = 1$ where they touch.

136. Show that the substitution $w = (az + b)/(cz + d)$, where $ad \neq bc$, transforms the circle $|z| = 1$ into $|w| = 1$ if $a\bar{a} + b\bar{b} = c\bar{c} + d\bar{d}$ and $a\bar{b} = c\bar{d}$.

137. If $w = (z + k)/(z + 1)$ $(0 < k < 1)$, prove that the locus of points whose distance from the origin are unaltered by the transformation consists of two orthogonal circles.

138. The tangent at O to the circumcircle of OP_1P_2 meets P_1P_2 on Q. If z_1, z_2 correspond to P_1P_2 (O being the origin), and z corresponds to the midpoint of the chord of contact of the tangents from Q to the above circle, show that $\dfrac{1}{z} = \dfrac{1}{z_1} + \dfrac{1}{z_2}$.

139. If $w = \displaystyle\int_1^z \frac{z - 1}{z}\,dz$, show that as z describes the real axis from $-\infty$ to ∞, w describes its positive real axis from $+\infty$ to 0, then from 0 to $+\infty$ and then the line $v = -\pi$ from $+\infty$ to $-\infty$.

Determine for *Examples 140–3*, the regions in the w-plane that correspond to $I(z) > 0$.

140. $\dfrac{dw}{dz} = (1 - z^2)^{-\frac{1}{2}}$ **141.** $\dfrac{dw}{dz} = z^{-\frac{1}{2}}(z^2 - 1)^{-\frac{1}{2}}$

142. $\dfrac{dw}{dz} = z^{-\frac{1}{2}}(z^2 - 1)^{-\frac{1}{2}}$ **143.** $\dfrac{dw}{dz} = z^{-\frac{1}{2}}(z - 1)^{-\frac{1}{2}}(z + 1)^{-\frac{1}{2}}$

144. If $\dfrac{dw}{dz} = z^{-\frac{1}{2}}(z^2 - 1)^{-\frac{1}{2}}$, prove that $I(z) > 0$ is represented by the interior of a quadrilateral $ABCD$, whose vertices correspond respectively to $z = 0$, 1, ∞, -1, and lie on the circle whose diameter is AC; and that $AB = AD$ with $\angle BAD = \pi/3$.

145. Show that $w = C\displaystyle\int_0^z (1 - z^n)^{-\frac{2}{n}}\,dz$ represents the interior of the circle $|z| = 1$ on the interior of a regular polygon.

Prove by contour integration the results given in *Examples 146–203*.

146. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = \frac{\pi}{2}$ **147.** $\displaystyle\int_0^{2\pi} \frac{\cos 4\theta\, d\theta}{5 + 4\cos 2\theta} = \frac{\pi}{6}$

148. $\displaystyle\int_0^{2\pi} \frac{d\theta}{\cos\theta + \sin\theta - 2} = -\pi\sqrt{2}$

149. $\displaystyle\int_0^{2\pi} \cos^n\theta\, d\theta = 0$ if n is odd and $\displaystyle\int_0^{2\pi} \cos^n\theta\, d\theta = \frac{1.3\ldots(n-1)}{2.4\ldots n}2\pi$ if n is even.

150. $\displaystyle\int_0^{2\pi} \frac{\sin n\theta}{\sin\theta}\,d\theta = 2\pi$ or 0 according as n is odd or even.

151. $\displaystyle\int_0^{2\pi} \frac{d\theta}{9 + 16\sin^2\theta} = \frac{2\pi}{15}$

152. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a - b\cos\theta} = \frac{2\pi}{b^2}\{a - \sqrt{(a^2 - b^2)}\}$ $(a > |b| > 0)$

153. $\displaystyle\int_0^{2\pi} \frac{(4+\cos 2\theta)d\theta}{17+8\cos 2\theta} = \frac{\pi}{2}$. **154.** $\displaystyle\int_0^{2\pi} \frac{(4+5\cos\theta+\cos 2\theta)}{17+8\cos 2\theta}\,d\theta = \frac{\pi}{2}$

155. $\displaystyle\int_0^{2\pi} \frac{e^{\cos\theta}\cos(\theta+\sin\theta)d\theta}{5-4\cos\theta} = \frac{\pi\sqrt{e}}{3}$

156. $\displaystyle\int_0^{\infty} \frac{\cos px\,dx}{x^2+a^2} = \frac{\pi}{2a}e^{-pa}\ (p>0,\ a>0)$

157. $\displaystyle\int_0^{\infty} \frac{x\sin mx\,dx}{x^4+1} = \frac{\pi}{2}e^{-m/\sqrt{2}}\sin(m/\sqrt{2})\ (m>0)$

158. $\displaystyle\int_0^{\infty} \frac{\cos^2 x\,dx}{(x^2+1)^2} = \frac{\pi}{8}\left(1+\frac{3}{e^2}\right)$ **159.** $\displaystyle\int_0^{\infty} \frac{\cos(\frac{1}{2}\pi x)dx}{x^4+4} = \frac{\pi}{8}e^{-\frac{1}{2}\pi}$

160. $\displaystyle\int_0^{\infty} \frac{\cos(\pi x)dx}{x^4+x^2+1} = \frac{\pi}{2}e^{-\frac{1}{2}\pi\sqrt{3}}$

161. $\displaystyle\int_0^{\infty} \frac{x^3\sin px\,dx}{(x^4+a^4)} = \frac{\pi}{2}e^{-pa/\sqrt{2}}\cos(pa/\sqrt{2})\ (p>0,\ a>0)$

162. $\displaystyle\int_0^{\infty} \frac{(1-x^2)\cos(\frac{1}{2}\pi x)dx}{x^4+x^2+1} = \frac{\pi}{\sqrt{3}}e^{-\pi\sqrt{3}/4}$

163. $\displaystyle\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2-x+1}dx = \frac{2\pi}{\sqrt{3}}\log(2+\sqrt{3})$

164. $\displaystyle\int_0^{\infty} \frac{(x^2+1)\log(x^2+1)dx}{x^4+x^2+1} = \frac{\pi}{\sqrt{3}}\log(2+\sqrt{3})$

165. $\displaystyle\int_{-\infty}^{\infty} \frac{\tan^{-1}x}{x^2-x+1}dx = \frac{\pi^2}{6\sqrt{3}}$ **166.** $\displaystyle\int_0^{\infty} \frac{x\tan^{-1}x}{x^4+x^2+1}dx = \frac{\pi^2}{12\sqrt{3}}$

167. $\displaystyle\int_0^{\infty} \frac{x\tan^{-1}x}{(x^2+a^2)^2}dx = \frac{\pi}{4a(1+a)}\ (a>0)$. **168.** $\displaystyle\int_0^{\infty} \frac{\log(x^2+1)}{x^2+1}dx = \pi\log 2$

169. $\displaystyle\int_0^{\pi/2} \log\cos\theta\,d\theta = -\tfrac{1}{2}\pi\log 2$

170. $\displaystyle\int_0^{\infty} \frac{\cos mx-\cos nx}{x^2}dx = \frac{\pi}{2}(n-m)\ (n,\ m>0)$

171. $\displaystyle\int_0^{\infty} \frac{\sin(x^2)}{x}dx = \frac{\pi}{4}$ **172.** $\displaystyle\int_0^{\infty} \frac{\sin x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$

173. $\displaystyle\int_0^{\infty} \frac{\cos x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$ **174.** $\displaystyle\int_0^{\infty} \frac{x-\sin x}{x^3(x^2+1)}dx = \frac{\pi}{4e}(e-2)$

175. $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2(x^2+1)}dx = \frac{\pi(1+e^2)}{4e^2}$ **176.** $\displaystyle\int_0^{\infty} \frac{\sin^2 x\,dx}{x^2(x^2+1)^2} = \frac{\pi(5+e^2)}{8e^2}$

177. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x(2x^2-2\pi x+\pi^2)} = \frac{1}{\pi}(1+e^{-\frac{1}{2}\pi})$

178. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 2x\,dx}{(\pi-2x)(x^2+1)} = \frac{2\pi(1+e^2)}{e^2(\pi^2+4)}$

179. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\theta\, d\theta}{\tan\theta} = \int_0^\infty \frac{\tan^{-1} x\, dx}{x(1+x^2)} = \frac{\pi}{2}\log 2$ 180. $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^4}\, dx = \frac{\pi}{3}$

181. $\displaystyle\int_0^\infty \frac{\sin^5 x}{x^5}\, dx = \frac{115\pi}{384}$ 182. $\displaystyle\int_0^\infty \frac{\sin^6 x}{x^6}\, dx = \frac{11\pi}{40}$

183. $\displaystyle\int_0^\infty \frac{\sin x + \sin 2x - \sqrt{3}\sin x\sqrt{3}}{x^5}\, dx = \frac{\pi}{48}(17 - 8\sqrt{3})$

184. $\displaystyle\int_0^\infty \frac{\sin ax \sin bx \sin cx\, dx}{x^3} = \frac{\pi}{8}(2bc + 2ca + 2ab - a^2 - b^2 - c^2),$

$(a > b > c > 0,\ b + c > a)$ and $\frac{\pi}{2}bc\ (a > b > c > 0,\ b + c < a)$

185. $\displaystyle\int_0^\infty \frac{\sin ax \sin^2 bx}{x^3}\, dx = \frac{\pi a}{8}(4b - a)\ (0 < a < 2b),\ \frac{\pi b^2}{2}\ (a > 2b > 0).$

186. $\displaystyle\int_0^\infty \frac{\sin^3 ax}{x^3}\, dx = \frac{3\pi a^2}{8}\ (a > 0)$

187. $\displaystyle\int_0^\infty \frac{\sin^n x}{x^n}\, dx = \frac{\pi}{2^n (n-1)!}\left\{n^{n-1} - n(n-2)^{n-1} + \frac{n(n-1)}{1.2}(n-4)^{n-1} - \dots\right.$

where there are $\frac{1}{2}n$ terms (n even) or $\frac{1}{2}(n + 1)$ terms (n odd).

188. $\displaystyle P\int_0^\infty \frac{x^{a-1}\, dx}{1 - x} = \pi \cot a\pi\ (0 < a < 1)$

189. $\displaystyle\int_0^\infty \frac{x^a\, dx}{x^4 + 1} = \frac{\pi\sqrt{2}}{4\left(\cos\frac{\pi a}{4} + \sin\frac{\pi a}{4}\right)}\ (-1 < a < 3)$

190. $\displaystyle\int_0^\infty \frac{x^{2a}\, dx}{x^4 + x^2 + 1} = \frac{2\pi}{\sqrt{3}(1 + 2\cos 2a\pi)}\ (|a| < \tfrac{1}{2})$ 191. $\displaystyle\int_0^\infty \frac{(\log x)^2\, dx}{x^2 + 1} = \frac{\pi^3}{8}$

192. $\displaystyle\int_0^\infty \frac{x\, dx}{\sinh x} = \tfrac{1}{4}\pi^2$ 193. $\displaystyle\int_0^\infty \frac{x\cos mx\, dx}{\sinh x} = \frac{\pi^2 e^{m\pi}}{(1 + e^{m\pi})^2}\ (m \text{ real})$

194. $\displaystyle\int_{-\infty}^\infty \frac{e^{ax}\, dx}{e^x + 1} = \frac{\pi}{\sin a\pi}\ (0 < a < 1)$ 195. $\displaystyle\int_0^\infty \frac{\cos xt\, dx}{\cosh(x\sqrt{\frac{1}{2}\pi})} = \sqrt{\tfrac{1}{2}\pi}\,\frac{1}{\cosh(t\sqrt{\frac{1}{2}\pi})}$

196. $\displaystyle\int_0^\infty \frac{x^2\, dx}{\cosh x + \cos a} = \frac{a(\pi^2 - a^2)}{3\sin a}\ (|a| < \pi)$

197. $\displaystyle\int_0^\infty \frac{\sin 4x\, dx}{(\sin x)(x^2 + 1)} = \frac{\pi}{e^4}(1 + e^2)$

198. $\displaystyle\int_0^\infty \frac{\sin mx\, dx}{(\sin x)(x^2 + a^2)} = \frac{\pi}{a}\,\frac{e^{ma} - 1}{e^{2a} - 1}e^{-a(m-1)}\ (m \text{ even}),$

$\displaystyle\qquad = \frac{\pi}{a}\,\frac{e^{(m-1)a} - 1}{e^{2a} - 1}e^{-a(m-1)} + \frac{\pi}{2a}\ (m \text{ odd})$

199. $\displaystyle\int_0^\infty \frac{\operatorname{sech}^2 x\, dx}{4\pi^2 + x^2} = \frac{1}{12}$ 200. $\displaystyle\int_0^\infty \frac{dx}{(x^2 + 4\pi^2)\cosh\frac{1}{2}x} = \frac{4 - \pi}{4\pi}$

201. $\displaystyle\int_0^\infty \frac{dx}{(x^2 + 1)\cosh\frac{1}{2}\pi x} = 1 - \log 2$

202. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{x \sin 2x \, dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{4} \log (1 + a), \ (|a| < 1); \quad \frac{\pi}{4} \log \left(1 + \frac{1}{a}\right)$ $(|a| > 1)$ (a real)

203. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{1 - r \cos 2\theta}{1 - 2r \cos 2\theta + r^2} \log \tan \theta \, d\theta = \frac{\pi}{4} \log \frac{1 - r}{1 + r} \ (|r| < 1),$

$$-\frac{\pi}{4} \log \left(\frac{r - 1}{r + 1}\right) (|r| > 1)$$

204. If $f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ and has zeros α_r ($r = 1$ to n) not necessarily distinct; and if $u(t) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{e^{tz} dz}{f(z)}$ where C encloses all α_r, show that, if $D \equiv \dfrac{d}{dt}$, then

$f(D)u = 0$; $u(0) = u'(0) = u''(0) = \ldots = u^{(n-2)}(0) = 0$, $u^{(n-1)}(0) = 1/a_0$.

205. Find the solution of the equation $u''' + u'' - u' - u = 0$ that satisfies the initial conditions $u(0) = u'(0) = 0$, $u''(0) = k$.

206. Show that the multiple integral

$$\iint \ldots \int e^{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n} \, dx_1 \, dx_2 \ldots dx_n$$

extended over all positive and zero values of $x_1, x_2, \ldots, x_n$ that satisfy the relation $0 < x_1 + x_2 + \ldots < t$ (real numbers), is equal to the contour integral

$$\frac{1}{2\pi i}\int_C \frac{e^{tz} \, dz}{z(z - a_1) \ldots (z - a_n)}$$

where C encloses all the points a_r (real or complex).

Use *Example 206* to evaluate the integrals given in *Examples 207–10*.

207. $\iiint e^{x+y+2z} \, dx \, dy \, dz$ for $0 < x + y + z < 1$

208. $\iiiint e^{x+y+z+u} \, dx \, dy \, dz \, du$ for $0 < x + y + z + u < a$

209. $\iiint e^{x+2y+3z} \, dx \, dy \, dz$ for $0 < 2x + 3y + z < a$

210. $\iiint \ldots \int e^{x_1 + x_2 + \ldots + x_n} \, dx_1 \, dx_2 \ldots dx_n$

for $0 < x_1 + x_2 + \ldots + x_n < a$

211. Show that $\iint \ldots \int e^{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n} \, dx_1 \, dx_2 \ldots dx_{n-1}$ extended over all positive and zero values of $x_1, x_2, \ldots, x_{n-1}$ that satisfy the relation

$$0 < x_1 + x_2 + \ldots + x_{n-1} < a$$

where $x_n = a - x_1 - x_2 - \ldots - x_{n-1}$ is equal to the contour integral

$$\frac{1}{2\pi i}\int_C \frac{e^{az} \, dz}{(z - a_1)(z - a_2) \ldots (z - a_n)}$$

where C encloses all the points a_r.

212. Prove that $\displaystyle\int_{c - i\infty}^{c + i\infty} \frac{e^{az} \, dz}{\sin \pi z} = \frac{2ie^a}{1 + e^a}$ $(a > 0)$, $(0 < c < 1)$

213. Show that

(i) $\displaystyle\frac{1}{2\pi i}\int_{c - i\infty}^{c + i\infty} \frac{e^{tz}}{(z + n)^2} dz = te^{-nt} \ (t > 0), \ 0 \ (t < 0)$

(ii) $\displaystyle\frac{1}{2\pi i}\int_{c - i\infty}^{c + i\infty} \frac{e^{tz}}{z(z + n)^2} dz = \frac{1}{n^2}(1 - e^{-nt}) - \frac{t}{n} e^{-nt} \ (t > 0), \ 0 \ (t < 0)$

where in both cases $n > 0$, $c > 0$.

214. Prove the *Maximum Modulus Theorem.* If $f(z)$ is analytic in a domain D, and $|f(z)| \leq M$ on a simple closed curve C within D, then $|f(z)| < M$ for all points interior to C (except when $f(z)$ is constant).

215. If $f(z)$ is analytic and regular for $|z| < R$ and $|f(z)| < M$ for $|z| = R$, and $f(0) = 0$, show that $|f(z)| < M\,r/R$ for $|z| = r$ and $0 < r < R$. (*Schwarz's Lemma.*)

216. If $f(z)$ is analytic and regular for $r_1 < |z| < r_2$ and $\max |f(z)|$ on the circles $|z| = r_1, r_2, r_3$ are M_1, M_2, M_3 respectively, where $r_1 < r_2 < r_3$ show that

$$\begin{vmatrix} \log M_1 & \log r_1 & 1 \\ \log M_2 & \log r_2 & 1 \\ \log M_3 & \log r_3 & 1 \end{vmatrix} < 0$$

(*Hadamard's Three Circles Theorem*).

217. If $f(z)$ is analytic on and inside a closed contour C except at a number of poles within C, and does not vanish on C, show that $\dfrac{1}{2\pi i}\displaystyle\int_C \{f'(z)/f(z)\}dz = m - n$, where m is the number of poles and n the number of zeros within C, each zero (or pole) being reckoned according to its multiplicity.

218. If C is a closed level curve $|w| =$ constant $(\neq 0)$, where $w = f(z)$ is an analytic function, and $f'(z)$ does not vanish on C, show that $f(z)$ vanishes at least once within C.

219. An analytic function $f(z)$ and its derivative do not vanish on a given closed level curve C. If $f(z)$ has n zeros within C, show that $f'(z)$ has $(n - 1)$ zeros within C.

Solutions

2. $-3 \quad 4i$ **3.** $18 \quad 26i$ **4.** $0.26 \quad 0.32i$

5. $(0.68) \quad (0.76)i$ **6.** 1.6864 **7.** $2i$

8. $\sin(\theta - \phi) - i\cos(\theta - \phi)$ **9.** $\frac{1}{4}(\cos 2\theta \quad i\sin 2\theta)$ **10.** $e^{\ 2n\pi}$

11. $e^{(8n \ \ 1)\pi}\{\cos\log\sqrt{2} + i\sin\log\sqrt{2}\}$ **12.** $\frac{1}{2}\log 2 + i(2n + \frac{1}{4})\pi$

13. $\{(\tan y)\operatorname{sech}^2\sqrt{3} + i\sec^2 y\tanh\sqrt{3}\} \div \{1 + \tan^2 y\tanh^2\sqrt{3}\}\ (y = 1)$

14. $i\sinh 3$ **15.** $\cos y\cosh y + i\sin y\sinh y\ (y = 1)$ **16.** 1

17. $-i\coth\frac{1}{2}\pi$ **18.** $(2n + \frac{1}{2})\pi \quad i\log(2 \quad \sqrt{3})$

19. $2n\pi \pm (\frac{1}{2}\pi - i\log\frac{1}{2}(\sqrt{13} + 3))$ **20.** $(2n + 1)i\pi$

21. $(n + \frac{1}{2})\pi - \frac{1}{2}i\log\tan(\frac{1}{8}\pi - \frac{1}{4}\alpha)$ **22.** $\frac{1}{2}\log 3 + \frac{1}{2}i(2n + 1)\pi$

23. $\sqrt{2}/4, -\frac{1}{4}\pi$ **24.** $2, \frac{1}{2}\pi$ **25.** $\sqrt{2}$, arc tan 7

26. $2\cos\frac{1}{2}\alpha, \frac{1}{2}\alpha\ (-\pi < \alpha < \pi)$ **27.** $\sqrt{2}\cos\frac{1}{2}\alpha, \frac{1}{2}\alpha - \frac{1}{2}\pi\ (-\pi < \alpha < \pi)$

28. $\frac{1}{2}\sec\alpha, -\alpha\ (-\pi/2 < \alpha < \pi/2)$

29. $\cos\frac{1}{2}\alpha\sec\beta, \frac{1}{2}\alpha - \beta\ (-\pi < \alpha < \pi, \ -\frac{1}{2}\pi < \beta < \frac{1}{2}\pi)$

30. $\dfrac{\{(x - p)^2 + (y - q)^2\}^{\frac{1}{2}}}{\{(x + p)^2 + (y + q)^2\}^{\frac{1}{2}}}$, arc tan $\left(\dfrac{2(xy - pq)}{x^2 - y^2 - p^2 + q^2}\right)$

31. $x^3 - 3xy^2, 3x^2y - y^3$ **32.** $\dfrac{x^2 - y^2}{(x^2 + y^2)^2}, \dfrac{2xy}{(x^2 + y^2)^2}$

33. $\dfrac{6 + 5x - 4x^2 - 4y^2}{(x - 2)^2 + y^2}, \dfrac{11y}{(x - 2)^2 + y^2}$

34. $x^2 - y^2 + 2x + 4, 2y(x + 1)$ **35.** $\dfrac{x^2 + y^2 + x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}$

36. Locus of z is the line through z_3 perpendicular to the line joining z_1, z_2

37. The circumcircle of z_1, z_2, z_3.

38. Arc of a circle passing through z_1, z_2.

39. A circle for which z_1, z_2 are inverse points.

40. The straight segment joining $1, -1$.

41. The y-axis from $-i$ to $-i\infty$.

42. The half of the hyperbola for which $x < 0$, whose foci are ± 1, semi major axis $\frac{1}{2}$ and eccentricity 2.

43. Ellipse of foci $\pm\frac{1}{2}i$, semi axes $3/2, \sqrt{2}$, eccentricity $1/3$

44. The triangle z_1, z_2, z_3 is similar to the triangle z_4, z_5, z_6

45. $\frac{1}{2}\{(x_1 + x_2) \pm \sqrt{3}(y_1 - y_2)\}, \frac{1}{2}\{(y_1 + y_2) \mp \sqrt{3}(x_1 - x_2)\}$

46. Find $\tan 7\alpha$ in terms of $\tan\alpha$.

49. Use result $\tan 7\theta$ $7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta = 0$

50. The roots of $c^4 - 4c^3 - 12c^2 + 8c + 16 = 0$ are $\sec 2r\pi/9$ ($r = 1$ to 4). Use formula for $\cos 9\theta = 1$.

51. Express $\sin 2\pi/7 + \sin 4\pi/7 + \sin 6\pi/7$ as a product of sines

$$4\sin\pi/7\sin 2\pi/7\sin 3\pi/7$$

and use equation $\sin 7\theta = 0$.

52. *See Example 59.*

53. Use $(1+i)(3+i)(5+i)(7+i)(9+i) = 20(1+73i)$.

54. $\dfrac{(1+5i)}{(3+i)(1+i)} = \frac{1}{50}(7+i)(8+i)$

58. Put $z = e^{i\theta}$ and factorise $2\cos 4\theta - 1$

$$= 16(\cos\theta - \cos\alpha)(\cos\theta + \sin\alpha)(\cos\theta + \cos\alpha)(\cos\theta - \sin\alpha),\ \alpha = \pi/12$$

59. Consider $\sin n\theta = 0$. 60. Consider $\cos 2n\theta = 0$.

61. The expression is $2^n(\omega^n + \omega^{2n})$.

62. $2^{-13}(\cos 14\theta - 2\cos 12\theta - 5\cos 10\theta + 12\cos 8\theta + 9\cos 6\theta - 30\cos 4\theta - 5\cos 2\theta + 20)$

63. $2^{-11}(20\sin 2\theta - 5\sin 4\theta - 10\sin 6\theta + 4\sin 8\theta + 2\sin 10\theta - \sin 12\theta)$

64. $2^{-8}(\sin 9\theta - \sin 7\theta - 4\sin 5\theta + 4\sin 3\theta + 6\sin\theta)$

65. $2^{-10}(14\cos\theta - 6\cos 3\theta - 11\cos 5\theta - \cos 7\theta - 3\cos 9\theta - \cos 11\theta)$

66. $\pm\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right),\ \pm\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

67. $2^{\frac13}\left(\cos\frac{6r+1}{27}\pi - i\sin\frac{6r+1}{27}\pi\right)$ $r = 0$ to 8

68. $1 + i\cot\frac{2r+1}{12}\pi$ ($r = 0$ to 5)

69. $\pm i,\ \pm 7^{-\frac14}(\cos\frac12\alpha \pm i\sin\frac12\alpha)$, $\tan\alpha = \sqrt3/2$

70. $1,\ 2\alpha^4 - \alpha,\ 2\alpha^3 - \alpha^2,\ 2\alpha^2 - \alpha^3,\ 2\alpha - \alpha^4$ where $\alpha = \cos 2\pi/5 + i\sin 2\pi/5$

71. $-i,\ \frac12(\sqrt3 + i),\ \frac12(i - \sqrt3)$

72. $\pm 2^{-3/4}(\sqrt3 + i),\ \pm 2^{-3/4}(-1 + i\sqrt3)$

73. $\pm 2^{-5/4}\{\sqrt3 + 1 + i(\sqrt3 - 1)\},\ \pm 2^{-5/4}\{\sqrt3 - 1 - i(\sqrt3 + 1)\}$

74. $2^{-4/3}\{\sqrt3 + 1 + i(\sqrt3 - 1)\},\ 2^{-1/3}(-1 + i),$

$$2^{-4/3}\{\sqrt3 - 1 + i(\sqrt3 + 1)\}$$

75. $2^{1/3}(\cos\alpha + i\sin\alpha)$, $\alpha = 10°, 130°, 250°$

76. $\cos\alpha + i\sin\alpha$, $\alpha = 12°, 84°, 156°, 228°, 300°$

77. $\pm i,\ \pm\cos 18° \pm i\sin 18°,\ \pm\sin 36° \pm i\cos 36°$

78. $n\pi \pm \frac12 i\log(2 \pm \sqrt3)$

79. $(2n+1)\pi - i\log(\sqrt2 + 1),\ 2n\pi - i\log(\sqrt2 - 1)$

80. $n\pi - \frac12\tan^{-1}2 + \frac{i}{4}\log 5$ 81. $\pm\dfrac{i(4n+1)\pi}{4}$

82. $(2n \pm \frac12)\pi + i\log(2 \pm \sqrt3)$

83. $\frac12 i\pi(6n \pm 1)$ 84. $\frac14(2n+1)\pi - \frac14 i\log 3$ 86. 2π

88. One real and +, one in 2nd quadrant and one in the 3rd.

89. One within each quadrant.

90. One real and , and one in each quadrant.

91. One real and , one real and , and one in each quadrant.

92. One in the 2nd quadrant and two in the 4th.

93. One real (-1), one in each of the 1st, 2nd and 4th quadrants.

95. (i) $3^{\frac12}(65)^{\frac14}(\cos\frac12\alpha + i\sin\frac12\alpha)$, $\alpha = \tan^{-1}(\frac14)$, $5^{\frac12}(145)^{\frac14}(\cos\frac12\beta + i\sin\frac12\beta)$ $\beta = \tan^{-1}(12)$, $(21)^{\frac12}(689)^{\frac14}(\cos\frac12\gamma + i\sin\frac12\gamma)$, $\gamma = \tan^{-1}(\frac{8}{17})$. (ii) $4\sqrt2$, $4\sqrt2$, $4\sqrt2$.

96. Join $2i$ to $i\infty$, $2i$ to $-i\infty$ and i to $-i$.

97. Join ∞ to 2.

98. Join 0 to ∞ and 1 to $-\infty$.

99. Oscillates finitely in each case.

105. Put $z = \pi/\sqrt2$ in *Example 104*.

106. Equate to zero the first three terms of the series in *Example 105.*
108. Exterior of $|w| = 1$ in 4th quadrant.
109. First quadrant of $|w + 1| = 1$.
110. Exterior of $|w + \frac{1}{2}| = \frac{1}{2}$ in the 3rd quadrant.
111. Exterior of $|w - 1| = 1$, $u > 1$, $v < 0$.
112. Exterior of $|w - \frac{1}{2}| = \frac{1}{2}$, $u < \frac{1}{2}$, $v > 0$. **113.** $w = \dfrac{7z - 12}{3z - 5}$.
114. $u^2 + v^2 - 2u + 4v = 0$.
118. If A_1, A_2, A_3, A_4 are the four quadrants of $|z| = 1$ in the usual order, and A_5, A_6, A_7, A_8 are the exteriors corresponding to these, and if $A_1', \ldots, A_8'$ are the corresponding areas for $|w - \frac{1}{2}| = \frac{1}{2}$, then $A_1, A_2, \ldots, A_8$ become $A_7', A_3, A_8', A_6', A_5', A_4, A_1', A_2'$.
121. $w = e^{i\theta}\dfrac{z - z_0}{z + \bar{z}_0}$, where θ is real and $\mathbf{R}(z_0) > 0$.
122. Straight lines parallel to the y-axis.
123, 124. (*See Hardy, Pure Mathematics, XC, 28, 29.*)
125. Take $z = \cos\theta + i\sin\theta$. **130.** The 3rd quadrant.
134. If z is on the given circle, so also is c^2/z, and the chord joining these points always passes through the fixed point K whose polar with respect to the given circle is the x-axis. The mid-point of the chord is always on the circle described on the line joining K to $ia\sin\beta$ as diameter and describes the arc of this circle that lies within the given circle. Therefore $z + a^2/z$ also describes an arc.
135. $|w|^2 = (a + b)^2 - 4ab\sin^2\dfrac{\theta}{2}$ if $z = \cos\theta + i\sin\theta$. Therefore $|w| < 1$ except when $\theta = 0$, i.e. $z = 1 = w$.
138. If R is the point of contact of the other tangent from Q and S the mid-point of P_1P_2, the triangles P_1OS, ROP_2 are similar and therefore

$$z_1/\tfrac{1}{2}(z_1 + z_2) = 2z/z_2$$

140, 141. Equilateral triangles.
142. Isosceles right-angled triangle.
143. A triangle with angles 30°, 60°, 90°.
145. Let $z = \cos\theta + i\sin\theta$, where $2r\pi/n < \theta < 2(r + 1)\pi/n$, ($r = 0$ to $n - 1$). Increase in amp dz due to increment $\delta\theta$ is equal to $\delta\theta$. Increase in amp $(z - z_r)^{2/n}$ where $z_r = \exp(2ri\pi/n)$ is $\delta\theta/n$, i.e. increase in amp dw is zero. As z describes the arc between two consecutive vertices of the polygon determined by z_r, w describes a segment of a straight line. If the circle is indented at every z_r, excluding z_r from the interior, the increase in amp dw at every z_r is $2\pi/n$. Thus w describes a polygon of n sides, each angle of which is that of a regular polygon, and the polygon is obviously regular.
146–55. The unit circle $|z| = 1$.
156–7. The infinite semicircle Γ for which $\mathbf{I}(z) \geq 0$.
158. Integrate $(1 - e^{2iz})/(z^2 + 1)^2$ round Γ. **159–61.** Γ.
162. Integrate $e^{i\pi z}/(z^2 - z + 1)$ round Γ.
163. Integrate $\log(z + i)/(z^2 + 1)$ round Γ. **164.** Use *Example 163.*
165. Integrate $\log(1 - iz)/(z^2 - z + 1)$ round Γ.
166. Use *Example 165.*
167. Integrate $z\log(1 - iz)/(z^2 + a^2)^2$ round Γ. **169.** Use *Example 168.*
170. Γ indented at O. **171.** Take $u = x^2$. **172, 173.** Take $x = u^2$.
174–8. Γ indented at O or $\frac{1}{2}\pi$.
179. Integrate $\dfrac{\log(z + i)}{z(z^2 + 1)}$ round Γ indented at O.
180–2. *See Example 187.*
183. Integrate $(e^{iz} + e^{3iz} - \sqrt{3}e^{i\sqrt{3}z})/z^5$ round Γ indented at O.
184. Express $\sin ax \sin bx \sin cx$ as
$\frac{1}{4}\{\sin(a - b + c)x + \sin(b + c - a)x + \sin(a + b - c)x - \sin(a + b + c)x\}$.
185. Use *Example 184.* **186.** Use *Example 185.*

187. Integrate $f(z) = \left[e^{inz} - ne^{i(n-2)z} + \frac{n(n-1)}{1.2}e^{i(n-4)z} - \ldots\right]z^{-n}$, the last term within the bracket being $\frac{1}{2}(-1)^{\frac{1}{2}n} \cdot {}^{n}C_{\frac{1}{2}n}$ or $(-1)^{\frac{1}{2}(n-1)}\, {}^{n}C_{\frac{1}{2}(n-1)}\, e^{iz}$, round Γ indented at O. Note that $2^n i^n \sin^n x$ is equal to $f(x) + f(-x)$ when n is even and $f(x) - f(-x)$ when n is odd.

188–91. The double circle $|z| = \rho$, $|z| = R$ indented if necessary.

192–6. Rectangles.

197. Integrate $(e^{iz} + e^{2iz})/(z^2 + 1)$ round Γ.

198. Use result $\dfrac{\sin mx}{\sin x} = 2\cos(m-1)x + 2\cos(m-3)x + \ldots + 2\cos x$ (or 1) and *Example 156.*

199. Integrate $\{(z^2 + \pi^2)(1 + \cosh z)\}^{-1}$ round Γ or the infinite square $x = \pm N\pi$, $y = 0$, $y = 2N\pi$ (N integral $\to \infty$).

200, 201. Γ or an infinite square (*Example 199* solution).

202. Integrate $\dfrac{2z}{a - e^{-2iz}}$ round the rectangle $x = \pm\frac{1}{2}\pi$, $y = 0$, $y = R$.

203. Integrate $[\{(1 - r) + z^2(1 + r)\}\log z] \div [\{(1 - r)^2 + z^2(1 + r)^2\}(1 + z^2)]$ round Γ indented at O.

204. The integral may be differentiated with respect to t under the sign of integration; also $u(t)$, $u'(t)$, . . . are equal respectively to the coefficients of $1 : z$ in the expansions of their integrals for z near ∞. Thus $f(D)u(t) = \dfrac{1}{2\pi i}\displaystyle\int_C e^{zt}\,dz = 0$; and $u^{(r)}(0) = 0$ for $r = 0, 1, 2, \ldots, n - 2$, whilst $u^{(n-1)}(0) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{z^{n-1}\,dz}{f(z)} = \dfrac{1}{a_0}$.

205. By *Example 204*, $u = \dfrac{k}{2\pi i}\displaystyle\int_C \dfrac{e^{zt}\,dz}{(z + 1)(z^2 + 1)} = \frac{1}{2}k(e^{-t} - \cos t + \sin t)$.

206. *Ref. Proc. Edin. Math. Soc., Ser. 2, I(2) (1928).*

207. $\frac{1}{4}(e^2 - 2e - 1)$ **208.** $1 + e^a(\frac{1}{6}a^3 - \frac{1}{2}a^2 + a - 1)$

209. $-[\text{illegible}](2e^{3a} - 135e^{2a} + 108e^{a} - 35)$

210. $e^a\left\{\dfrac{a^{n-1}}{(n-1)!} - \dfrac{a^{n-2}}{(n-2)!} + \ldots + (-1)^{n-1}\right\} + (-1)^n$

211. *See Ref. Example 206, Solution.*

214. Consider the harmonic function $e^{Rf(z)}$.

215. Apply *Example 214* to $f(z)/z$.

216. Let $\Delta(z) = \begin{vmatrix} \log M_1 & \log r_1 & 1 \\ \log|f(z)| & \log|z| & 1 \\ \log M_3 & \log r_3 & 1 \end{vmatrix} = \lambda \log|z^\mu f(z)| + \nu$.

$\log|z^\mu f(z)|$ attains its maximum for $r_1 \leq |z| \leq r_3$ at some point of the boundary. But $\max \Delta(z) = 0$ for $|z| = r_1$ or $|z| = r_3$, and therefore for $|z| = r_2$,

$$\max \Delta(z) = \begin{vmatrix} \log M_1 & \log r_1 & 1 \\ \log M_2 & \log r_2 & 1 \\ \log M_3 & \log r_3 & 1 \end{vmatrix} < 0.$$

217. Near $z = a_r$, a zero of multiplicity m_r, $f'(z)/f(z)$ is of the form $m_r/(z - a_r)$ + analytic function, and near $z = b_s$, a pole of multiplicity n_s, $f'(z)/f(z)$ is of the form $-n_s/(z - b_s)$. The integral is therefore $\Sigma m_r - \Sigma n_s = m - n$.

218, 219. Let z describe the curve C from a point P_0, and let the length of the arc measured from P_0 be s. When z moves from P_0 to $z(s)$, let the angle turned through by the tangent be ψ $(0 \leq \psi \leq 2\pi)$. Then $\operatorname{amp}\dfrac{dz}{ds} = \psi$. The point corresponding to $z(s)$ is w where $w = \sigma e^{i\phi}$. Then $\operatorname{amp}\dfrac{dw}{ds} = \phi + \frac{1}{2}\pi$. But if $f'(z) \neq 0$

on C, $\frac{dw}{ds} = f'(z)\frac{dz}{ds}$ and therefore the increase in ϕ as z describes C is 2π + increase in amp $f'(z)$. Since $f'(z)$ is analytic in C (i.e. has no poles), the increase in amp $f'(z)$ is not negative (but may be zero). Thus the increase in ϕ, i.e. of amp w is at least 2π and therefore w has one zero at least within C. Also if there are n zeros of w within C the increase in amp w is $2n\pi$ and therefore the increase in amp $f'(z) = 2(n-1)\pi$ or there are $(n-1)$ zeros of $f'(z)$ within C.

CHAPTER XI

INFINITE SERIES, PRODUCTS AND INTEGRALS.

11. Convergence of Series. In considering the further properties of series and integrals we shall find it convenient at times to recapitulate the more important results obtained in earlier chapters.

A *necessary and sufficient* condition for the convergence of $\sum_1^\infty u_n$ is that corresponding to any ε (> 0) a suffix n_0 exists such that

$$\left|\sum_{n_0}^{m} u_n\right| < \varepsilon \text{ for all } m > n_0.$$

It is necessary but not sufficient for convergence that $\lim u_n$ should exist and have the value zero.

It is also necessary that $\lim nu_n$, *if it exists*, should be zero. For if $\lim nu_n = l$ ($\neq 0$), the terms are ultimately of the same sign (that of l) and are numerically greater than $\frac{1}{2n}|l|$. Such a series is divergent since $\sum_1^\infty \frac{1}{n}$ is divergent.

However,

(i) It is *not necessary* for convergence that $\lim nu_n$ should exist. For example, let $u_n = \frac{1}{n}$ when n is a perfect square and let $u_n = \frac{1}{n^2}$ when n is not a perfect square; then $\sum_1^\infty u_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{5^2} + \ldots + \frac{1}{8^2} + \frac{1}{3^2} + \ldots < 2\sum_1^\infty \frac{1}{n^2} \quad \sum_1^\infty \frac{1}{n^4}$.

Thus $\sum_1^\infty u_n$ converges but nu_n oscillates with limits 0, 1.

(ii) If u_n is a *decreasing* monotone ($\to 0$) and $\sum_1^\infty u_n$ converges, then $\lim nu_n$ *does* exist (and has the value zero). (*Pringsheim.*)

For $\sum_{n_0}^{m} u_n < \varepsilon$ and therefore $(m - n_0 + 1)u_m < \varepsilon$, all $m > n_0$.

Let $m \to \infty$, then since $(n_0 - 1)u_m \to 0$ so must $mu_m \to 0$.

(iii) It is *not sufficient* for convergence that $\lim nu_n$ should be zero. For example, $\sum \frac{1}{n \log n}$ diverges.

11.01. Tests for Convergence. (*Positive Terms.*) It has been shown in §§ 4.1–4.19 that the convergence (or divergence) of a series of *positive* terms may often be established by a comparison with the known series,

$$\text{(i) } \sum_1^\infty c^n (c > 0), \quad \text{(ii) } \sum_1^\infty \frac{1}{n^p}, \quad \text{(iii) } \sum_2^\infty \frac{1}{n(\log n)^p}.$$

The comparison is made by considering either (i) corresponding terms or (ii) corresponding ratios of successive terms. To effect these comparisons it is therefore usually sufficient to find *approximations* for (i) u_n or (ii) u_n/u_{n+1} when n is large.

Examples. (i) Let $u_n = \dfrac{n^{\log n}}{(\log n)^n}$ $(n \geq 2)$.

Let $A = n^{\log n}\, n^p$, $B = (\log n)^n$; then since $p \log n < n \log \log n$ and $(\log n)^2 < n \log \log n$ (n large, p independent of n), it follows that $\log A < \log B$ and $A < B$ (all p). Take $p = 2$ and we find that $u_n < \dfrac{1}{n^2}$ (n large). Thus the series converges.

(ii) Let $u_n = \dfrac{(2n+2)(2n+3)\ldots(3n+1)}{n!}\,\dfrac{x^{2n}}{2n+1}$ (x real).

Here
$$\frac{u_n}{u_{n+1}} = \frac{4}{27x^2}\left\{1 - \frac{3}{2n} + O\left(\frac{1}{n^2}\right)\right\}$$

The series therefore converges if $x^2 < 4/27$ and diverges otherwise (§ *4.18*).

11.02. The Cauchy-Maclaurin Integral Test. (*Positive Terms.*) Let $f(x)$ be a positive non-increasing function of x, defined for all $x > 1$. Let the integral $\int_1^x f(x)dx$ exist ($x > 1$) and denote $\int_1^n f(x)dx$ by I_n where n is a positive integer. Integration gives
$$0 < f(n) < I_n - I_{n-1} < f(n-1).$$
By addition,
$$f(2) + f(3) + \ldots + f(n) < I_n < f(1) + f(2) + \ldots + f(n-1)$$
i.e.
$$f(1) > S_n - I_n > f(n) > 0 \text{ (where } S_n = \sum_1^n f(n)).$$

Also $(S_n - I_n) - (S_{n-1} - I_{n-1}) = f(n) - \int_{n-1}^n f(x)dx < 0$, so that the sequence $S_n - I_n$ is a non-increasing monotone of positive numbers. Therefore $S_n - I_n$ tends to a limit between 0 and $f(1)$, and we deduce that the series $\sum_1^\infty f(n)$ converges or diverges with the *infinite* integral $\int_1^\infty f(x)dx$.

Notes. (i) We need only consider cases where $\lim_{x \to \infty} f(x) = 0$, since if this limit exists, it must be zero for a convergent integral; and in any case $\lim f(n)$ must be zero for a convergent series.

(ii) The series $\sum_m^\infty f(n)$ converges or diverges with the integral $\int_m^\infty f(x)\,dx$ and so the theorem is applicable when $f(x)$ is defined only for values $> m$ (m fixed), provided the other conditions are satisfied.

Examples. (i) The series $\sum_1^\infty \dfrac{1}{n^p}$ converges or diverges with $\int_1^\infty \dfrac{dx}{x^p}$. This integral converges if $p > 1$ and diverges if $p < 1$. The series therefore converges (diverges) when $p > 1$ ($p < 1$).

(ii) Let $f(n) = \dfrac{1}{n}$; then $\displaystyle\int_1^n \frac{dx}{x} = \log n$.

Thus $\displaystyle\lim_{n\to\infty}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n\right)$ exists and has a value γ between 0 and 1. This limit γ is called *Euler's Constant*, and its value is 0·5772 (approx.). (§ *12.12* (i).) The theorem shows also that

12·1(i)
$$\gamma = \sum_1^n\left(\frac{1}{r}\right) - \log(n+1) + \frac{\theta}{n+1} \quad (0 < \theta < 1)$$

Taking $n = 4$, we find that $0{\cdot}47 < \gamma < 0{\cdot}66$.

(iii) Show that $\displaystyle\sum_1^\infty \frac{1}{n(36n^2-1)} = 2\log 2 + \tfrac{3}{2}\log 3 - 3$.

Denote $\displaystyle\sum_1^n\left(\frac{1}{r}\right)$ by S_n; then since $\dfrac{1}{n(36n^2-1)} = -\dfrac{1}{n} + \dfrac{3}{6n-1} + \dfrac{3}{6n+1}$ it follows that

$$\sum_1^n \frac{1}{n(36n^2-1)} = 3S_{6n+3} - \tfrac{3}{2}S_{3n+1} - S_{2n+1} - \tfrac{1}{2}S_n - 3$$
$$= 3\log(6n+3) - \tfrac{3}{2}\log(3n+1) - \log(2n+1) - \tfrac{1}{2}\log n + \epsilon_n - 3$$

where $\epsilon_n \to 0$ as $n \to \infty$.

But

$$3\log(6n+3) - \tfrac{3}{2}\log(3n+1) - \log(2n+1) - \tfrac{1}{2}\log n \to 3\log 6 - \tfrac{3}{2}\log 3 - \log 2$$

and the result follows.

(iv) Show that $\displaystyle\lim_{n\to\infty}\left\{\frac{n}{n^2} + \frac{n}{1+n^2} + \ldots + \frac{n}{(n-1)^2+n^2}\right\} = \frac{\pi}{4}$.

By the theorem,

$$\frac{1}{m^2} + \frac{1}{m^2+1} + \ldots + \frac{1}{m^2+(n-1)^2} = \int_0^{n-1}\frac{dx}{m^2+x^2} + O\!\left(\frac{1}{m^2}\right)$$
$$= \frac{1}{m}\arctan\left(\frac{n-1}{m}\right) + O\!\left(\frac{1}{m^2}\right).$$

Therefore $\displaystyle\lim_{n\to\infty}\left\{\frac{n}{n^2} + \frac{n}{1+n^2} + \ldots + \frac{n}{(n-1)^2+n^2}\right\}$

$$= \lim_{n\to\infty}\left\{\arctan\left(\frac{n-1}{n}\right) + O\!\left(\frac{1}{n}\right)\right\} = \pi/4.$$

11.03 *Integral Test for a Double Series.* (*Positive Terms.*) A proof of a similar type will show that if $f(x, y)$ is a positive non-increasing monotone function of both variables, the double series $\Sigma\Sigma f(m, n)$ converges or diverges with the integral $\displaystyle\int_c^\infty\int_c^\infty f(x, y)\,dx\,dy$ taken over the rectangle $c \leq x < A$, $c \leq y < B$, when $A, B \to \infty$. Only functions $f(x, y)$ that tend to zero when $x, y \to \infty$ need be considered.

Example. Let $f(m, n) = [m^\alpha n^\beta + m^\gamma n^\delta]^{-1}$, $(\alpha, \beta, \gamma, \delta > 0)$.

If we allow for interchanges between m, n or between α, β and γ, δ the cases essentially different are

(i) $\alpha > 1$, $\beta > 1$, (ii) $\alpha < 1$, $\gamma < 1$, (iii) $\alpha > 1$, $\beta < 1$, $\gamma < 1$, $\delta > 1$.

(i) $\alpha > 1$, $\beta > 1$; $\displaystyle\int^\infty\!\!\int^\infty \frac{dx\,dy}{x^\alpha y^\beta + x^\gamma y^\delta} < \int^\infty\!\!\int^\infty \frac{dx\,dy}{x^\alpha y^\beta}$, i.e. $\displaystyle\left\{\int^\infty \frac{dx}{x^\alpha}\right\}\left\{\int^\infty \frac{dy}{y^\beta}\right\}$.

The double series therefore converges, since $\displaystyle\int^\infty \frac{dx}{x^\alpha}, \int^\infty \frac{dy}{y^\beta}$ converge.

(ii) $\alpha < 1$, $\gamma \geq 1$ (and suppose $\alpha \geq \gamma$). $\displaystyle\int_c^\infty\int_c^\infty \frac{dx\,dy}{x^\alpha y^\beta + x^\gamma y^\delta} > \int_c^\infty\int_c^\infty \frac{dx\,dy}{x^\alpha(y^\beta + y^\delta)}$.

The double series therefore *diverges*, since $\displaystyle\int_c^\infty \frac{dx}{x^\alpha}$ diverges.

(iii) $\alpha > 1$, $\beta < 1$, $\gamma < 1$, $\delta > 1$; then $\Delta = \alpha\delta - \beta\gamma > 0$. The straight line joining (α, β), (γ, δ) in a ξ-η plane is of the form $p\xi + q\eta = 1$, where $p = (\delta - \beta)/\Delta$, $q = (\alpha - \gamma)/\Delta$ $(p, q > 0)$.

If $X = x^\alpha y^\beta$, $Y = x^\gamma y^\delta$, then $\partial(x, y)/\partial(X, Y) = X^{\frac{\delta-\gamma}{\Delta}-1}\, Y^{\frac{\alpha-\beta}{\Delta}-1}/\Delta$ and

$$\iint \frac{dx\,dy}{x^\alpha y^\beta + x^\gamma y^\delta} = \frac{1}{\Delta}\iint \frac{X^{\frac{\delta-\gamma}{\Delta}-1}\, Y^{\frac{\alpha-\beta}{\Delta}-1}}{X + Y}\,dX\,dY.$$

Take $X + Y = u$ and $Y = uv$ and the integral becomes

$$\frac{1}{\Delta}\iint u^{p+q-2}\, v^{\frac{\alpha-\beta}{\Delta}-1}(1 - v)^{\frac{\delta-\gamma}{\Delta}-1}\,du\,dv.$$

Since $X = u(1 - v)$, we have $0 < v < 1$ for the rectangle $c < x < A$, $c < y < B$, and when $A, B \to \infty$, the extreme values of v tend to 0 and 1. But $(\alpha - \beta)/\Delta > 0$, $(\delta - \gamma)/\Delta > 0$ so that $\displaystyle\int_0^1 v^{\frac{(\alpha-\beta)}{\Delta}-1}(1 - v)^{\frac{\delta-\gamma}{\Delta}-1}\,dv$ converges, and therefore the double integral (and the double series) converges when $p + q < 1$ and diverges when $p + q > 1$.

Summarising: If the region in the $\xi - \eta$ plane for which $\xi \geq 1$, $\eta \geq 1$ is denoted by Ω, the double series converges if the line joining (α, β) to (γ, δ) has a part in Ω and otherwise diverges (see § 4.56).

Note. For a test involving the use of a single integral, *see Examples XI, 28.*

11.04. Convergence of Series in General. When the terms of a series Σu_n are not all of the same sign, the comparison tests cannot be directly applied (except to establish *absolute convergence*, § 4.21).

The best-known tests for convergence not necessarily absolute are called the *Abel-Dirichlet Tests*, which may be established by using the following lemma:

11.05. Abel's Lemma for Sequences. Let (i) $0 < v_n \leq v_{n-1}$ (all $n > 1$), (ii) $G_p = \operatorname{Max}\sum_1^r a_n$, $L_p = \operatorname{Min}\sum_1^r a_n$ where $r = 1, 2, 3, \ldots, p$; then $G_p v_1 \geq \sum_1^p a_n v_n \geq L_p v_1$.

For

$$\sum_1^p a_n v_n = s_1(v_1 - v_2) + s_2(v_2 - v_3) + \ldots + s_{p-1}(v_{p-1} - v_p) + s_p v_p$$

where $s_r = \sum_1^r a_n$; i.e. $G_p v_1 \geq \sum_1^p a_n v_n \geq L_p v_1$ since $v_{r-1} - v_r \geq 0$ (all r).

11.06 Dirichlet's Test for Convergence of Series. If (i) $\sum_1^\infty a_n$ oscillates *finitely* (or is convergent), (ii) $v_n \to 0$ steadily, then $\sum_1^\infty a_n v_n$ converges.

Let v_n *decrease* steadily to zero.

By the lemma, $\left|\sum_{m}^{m+p} a_n v_n\right| < K v_m$ where

$$K = \text{Max}\,|a_m + \ldots + a_{m+r}| \quad (r = 0 \text{ to } p).$$

But since $\sum_1^{\infty} a_n$ oscillates finitely (or is convergent), the sums

$$a_m, \quad \ldots \quad a_m + \ldots + a_{m+r}$$

must have an upper bound M (independent of m, p). Also an m exists such that $v_n < \varepsilon$ (all $n \geq m$) since $v_n \to 0$. Thus $\left|\sum_{m}^{m+p} a_n v_n\right| < M\varepsilon$ (all $p > 0$), i.e. the series converges. If v_n increases steadily to zero, it follows that $\sum_1^{\infty}(-a_n v_n)$ converges and therefore $\sum_1^{\infty} a_n v_n$ also.

11·07 Abel's Test for Convergence of Series. If (i) $\sum_1^{\infty} a_n$ is convergent, (ii) v_n is a bounded monotone, then $\sum_1^{\infty} a_n v_n$ is convergent.

For v_n tends to a finite limit l, and therefore the sequence $u_n = v_n - l$ tends steadily to zero.

But $\sum_1^{n} a_n v_n = \sum_1^{n} a_n u_n + l \sum_1^{n} a_n$ and therefore converges, for $\sum_1^{\infty} a_n u_n$ converges by Dirichlet's Test and $\sum_1^{\infty} a_n$ is convergent (given).

Examples. (i) $\displaystyle\sum_1^{\infty} \frac{\sin n\theta}{n^p}, \ \sum_1^{\infty} \frac{\cos n\theta}{n^p}.$

If $p > 1$, both series are *absolutely* convergent, since

$$\left|\frac{\sin n\theta}{n^p}\right| \leq \frac{1}{n^p} \text{ and } \left|\frac{\cos n\theta}{n^p}\right| \leq \frac{1}{n^p}$$

If $p < 0$, the series cannot converge since the nth terms do not tend to zero. Now $\sum_1^{n} \sin n\theta = \dfrac{\sin \frac{1}{2}n\theta}{\sin \frac{1}{2}\theta} \sin \frac{1}{2}(n+1)\theta$; $\sum_1^{n} \cos n\theta = \dfrac{\sin \frac{1}{2}n\theta}{\sin \frac{1}{2}\theta} \cos \frac{1}{2}(n+1)\theta$, $(\theta \neq 2m\pi)$ and therefore these latter series oscillate finitely $(\theta \neq 2m\pi)$.

If $p > 0$, both former series converge, by Dirichlet's Test, when $\theta \neq 2m\pi$.

If $p > 0$, $\theta = 2m\pi$, the sine series converges to zero, whilst the cosine series converges only when $p > 1$.

(ii) More generally, the series $\sum_1^{\infty} a_n \cos n\theta$, $\sum_1^{\infty} a_n \sin n\theta$ are convergent by Dirichlet's Test $(\theta \neq 2m\pi)$, if $a_n \to 0$. When $\theta = 2m\pi$, the second series converges to zero, and the first converges if $\sum_1^{\infty} a_n$ converges.

(iii) *Alternating Series* $v_1 - v_2 + v_3 - v_4 + \ldots$ $(v_n > 0)$. By Dirichlet's Test, this series converges if v_n tends steadily to zero. (*Leibniz's Rule*, § 4.23.)

11.08. The Convergence of $\sum_1^{\infty} a_n \cos n\theta$, $\sum_1^{\infty} a_n \sin n\theta$ $(a_n > 0)$. It is shown in *Example* (ii) above that these series converge when $a_n \to 0$ steadily.

The following rule is useful in practice for determining whether a_n is a sequence of this kind. If $a_n > 0$, and a_n/a_{n+1} is expressible in the form

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + \frac{\omega_n}{n^\lambda} \ (\lambda > 1,\ |\omega_n| < A, \text{ independent of } n)$$

then $a_n \to 0$ steadily if $\mu > 0$, and does not tend to zero if $\mu \leqslant 0$. (*Bromwich.*)

Let $\dfrac{a_n}{a_{n+1}} = 1 + \alpha_n$; then, (omitting, if necessary, a finite number of terms)

(i) if $\mu > 0$, $\sum\limits_1^\infty \alpha_n$ is a divergent series of positive terms.

(ii) if $\mu < 0$, $\sum\limits_1^\infty \alpha_n$ is a divergent series of negative terms.

(iii) if $\mu = 0$, $\sum\limits_1^\infty \alpha_n$ is absolutely convergent.

(i) $\mu > 0$; $a_1/a_n = \prod\limits_1^{n-1}(1 + \alpha_r) > 1 + \sum\limits_1^{n-1} \alpha_r$

(α_r may be assumed < 1, since $\lim \alpha_n = 0$)

Therefore a_n tends steadily to zero (a_{n+1} is obviously $< a_n$, n large).

(ii) $\mu < 0$; $a_1/a_n = \prod\limits_1^{n-1}(1 - \beta_r)$ (where $\beta_r = -\alpha_r$) $< \left[1 + \sum\limits_1^{n-1}\beta_r\right]^{-1}$

and therefore $a_n \to +\infty$, since $\Sigma\beta_r$ diverges.

(iii) $\mu = 0$; $\left|\dfrac{a_m}{a_n}\right| < \prod\limits_m^{n-1}(1 + |\alpha_r|) < \left[\prod\limits_m^{n-1}(1 - |\alpha_r|)\right]^{-1}$ ($|\alpha_r| < 1$).

But, given ε, an m exists for which $\sum\limits_m^{n-1}|\alpha_r| < \varepsilon$ (all $n > m$) since $\Sigma\alpha_r$ is absolutely convergent.

Thus $\left|\dfrac{a_m}{a_n}\right| < \left[1 - \sum\limits_m^{n-1}|\alpha_r|\right]^{-1} < \dfrac{1}{1 - \varepsilon}$ and therefore a_n cannot tend to zero.

Example. $1 - \dfrac{3}{\alpha} + \dfrac{3.4}{\alpha(\alpha + 1)} - \dfrac{3.4.5}{\alpha(\alpha + 1)(\alpha + 2)} + \ldots = \sum\limits_1^\infty(-1)^{n-1}a_n$.

Here $(a_n/a_{n+1}) = 1 + (\alpha - 3)/n + O(1/n^2)$.

The series converges only if $\alpha > 3$.

11.1. Uniform Convergence of a Sequence. A function $F(x)$ may be defined as $\lim\limits_{n\to\infty} f(x, n)$ for those values of x for which the limit exists. Suppose that $F(x)$ is so defined for all x in the interval $a < x < b$. Then for a fixed x in this interval, an integer n_0 exists such that, for any given ε (> 0), $|F(x) - f(x, n)| < \varepsilon$ for all $n > n_0$.

If, for definiteness, we take n_0 to be the least integer having the

required property, n_0 is a definite function of x; but, if it is possible to find an integer n_1, *independent* of x, for which the inequality is satisfied, the convergence of $f(x, n)$ to $F(x)$ in the interval $a < x < b$ is said to be *uniform*. This type of convergence is made clearer by drawing in the same figure the curves $y = f(x, n)$ for $n = 1, 2, 3, \ldots$ and also the curves $y = F(x) \pm \varepsilon$. The convergence is uniform if ultimately *all* the curves $y = f(x, n)$ lie between the curves $y = F(x) \pm \varepsilon$ for the *whole* interval (a, b). For simplicity in the following examples, we assume that $x > 0$.

Examples. (i) $f(x, n) = \dfrac{x^n}{1 + x^{2n}}$. (*Fig. 1.*)

Here $F(x) = 0\ (0 \leqslant x < 1)$; $F(1) = \frac{1}{2}$; $F(x) = 0\ (x > 1)$.
It is obvious from the figure that the convergence is not uniform in an

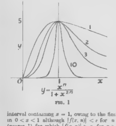


FIG. 1

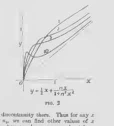


FIG. 2

interval containing $x = 1$, owing to the finite discontinuity there. Thus for any x in $0 < x < 1$ although $|f(x, n)| < \varepsilon$ for $n > n_0$, we can find other values of x (nearer 1) for which $|f(x, n)| > \varepsilon$, for $n > n_0$, however large n_0 may be taken. The sequence is uniformly convergent in each of the separate intervals

$$0 < x \leqslant c < 1, \quad 1 < c_1 \leqslant x \leqslant c_2.$$

In the former, for example, take n_0 so that $c^{n_0} < \dfrac{1}{2\varepsilon}\{1 - \sqrt{(1 - 4\varepsilon^2)}\}\ (\varepsilon < \frac{1}{2})$, but it is impossible to satisfy this inequality if $c = 1$.

(ii) $f(x, n) = \frac{1}{2}x\left(1 + \dfrac{2n}{1 + n^2x^2}\right)$. (*Fig. 2.*)

Here $F(x) = \frac{1}{2}x$ for *all* values of x, but $x = 0$ must be excluded from an interval of uniform convergence; for when $x = \dfrac{1}{n}$, $f(x, n) = \dfrac{1}{2n} + \frac{1}{2}\ (> \frac{1}{2})$; i.e. there are always points $x = \dfrac{1}{n}$ in the interval $0 < x < c$ for which $|F(x) - f(x, n)| > \frac{1}{2}$ however large n may be chosen. The sequence converges uniformly in the interval $0 < c_1 \leqslant x \leqslant c_2$, it being sufficient to choose n_0 to satisfy the inequality

$$n_0 > \frac{1}{2\varepsilon c_1}\{1 + \sqrt{(1 - 4\varepsilon^2)}\}\ (\varepsilon < \tfrac{1}{2}).$$

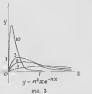
(iii) $f(x, n) = n^2xe^{-nx}$. (*Fig. 3.*)

Here $F(x) = 0$, all $x > 0$. But when $x = \frac{1}{n}$, $f(x, n) = n/e$ which tends to infinity. Thus $x = 0$ must be excluded from an interval of uniform convergence. It is uniformly convergent in $0 < c_1 \leq x \leq c_2$. It may be verified that if ξ is the greater root of the equation $\xi^2e^{-\xi} = c_1\varepsilon$ ($\varepsilon < 4/e^2c_1$), it is sufficient to take $n_0 = \xi/c_1$; if $\varepsilon > 4/e^2c_1$, any n_0 is sufficient.

(iv) $f(x, n) = \dfrac{x}{1 + n^2x^2}$. (*Fig. 4.*)

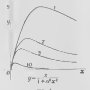
Here $F(x) = 0$ for all x and the convergence is uniform in any finite interval. Thus in $0 \leq x \leq c$, it is sufficient to take $n_0 = 1/2\varepsilon$, for $f(x, n) \geq 0$ and

$$f(x, n) = \frac{1}{2n} - \frac{(nx-1)^2}{2n(1 + n^2x^2)} \leq \frac{1}{2n} < \varepsilon.$$



$y = n^2xe^{-nx}$

FIG. 3



$y = \dfrac{x}{1+n^2x^2}$

FIG. 4

11.11. Properties of Uniformly Convergent Sequences.

1. If (i) $f(x, n) \to F(x)$ uniformly in $a \leq x \leq b$
and (ii) $f(x, n)$ is a continuous function of x in $a \leq x \leq b$, then $F(x)$ is a continuous function of x in $a < x < b$.

By (i), we can find n_0 such that $|F(x) - f(x, n)| < \varepsilon$ (all $n > n_0$) and for all x in (a, b).

By (ii) we can find δ (> 0) such that $|f(x, n_0) - f(x_1, n_0)| < \varepsilon$ for all x in $|x - x_1| < \delta$ where $a < x, x_1 < b$.

Therefore
$$|F(x) - F(x_1)| < |F(x) - f(x, n_0)| + |f(x, n_0) - f(x_1, n_0)| + |f(x_1, n_0) - F(x_1)| < 3\varepsilon$$
i.e. $F(x)$ is continuous at x_1, any point of (a, b).

In *Example* (i) above, $F(x)$ is not continuous at $x = 1$ and there is non uniform convergence there. In *Examples* (ii), (*iii*) $F(x)$ is continuous for all $x > 0$ although there is non-uniform convergence there.

Notes. (i) The necessary and sufficient condition that $f(x, n)$ should converge uniformly to $F(x)$ is that, given ε, we can find n_0, independent of x, such that $|f(x, n) - f(x, n_0)| < \varepsilon$ for all $n > n_0$.

(ii) The above examples show that uniform convergence is *sufficient* for the continuity of $F(x)$ when $f(x, n)$ is continuous, but that it is not necessary. It is, however, necessary and sufficient in the particular case when $f(x, n)$ is monotonic. Let $f(x, n)$ tend monotonically to $F(x)$ for every fixed x in (a, b), where f, F are continuous. Then $E(x, n) = |f(x, n) - F(x)|$ decreases steadily to zero for every

x and $E(x, n)$ is continuous. $E(x, n)$ attains its upper bound δ_n for one or more points $x_{n1}, x_{n2}, \ldots$ of the interval, and since $E(x, n)$ is a decreasing function, δ_n is monotonic, decreasing to a limit $\delta \geq 0$. The *whole* of the curve $y = E(x, m)$ lies in the interval $0 \leq y \leq \delta_n$ for all $m > n$. It is sufficient, therefore, for uniform convergence that $\delta = 0$. But if δ were not zero, and ξ were one of the limiting points of the set $x_{n1}, x_{n2}, \ldots$ ($n = 1, 2, 3, \ldots$), then in the neighbourhood of ξ, max $E(x, n)$ is *not less* than δ for all n. But since $E(x, n) \to 0$ for every x, and $E(x, n)$ is continuous, max $E(x, n)$ can be made as *small* as we please in the neighbourhood of ξ by taking n large enough. We thus arrive at a contradiction and δ must be zero. The *whole* of the curve $y = E(x, n)$ from and after some value n_0 of n, depending on ε, lies between the lines $y = 0$, $y = \varepsilon$, and therefore $E(x, n) \to 0$ uniformly, *i.e.* $f(x, n) \to F(x)$ uniformly.

(iii) It should be noted that in the proof of continuity we deal only with a particular point of the interval and that we may therefore write

$$\lim_{x \to x_0} \lim_{n \to \infty} f(x, n) = \lim_{n \to \infty} \lim_{x \to x_0} f(x, n)$$

when the convergence is uniform and the function is continuous.

11. If (i) $f(x, n) \to F(x)$ uniformly in $a < x < b$

and (ii) $f(x, n)$ is a continuous function of x in $a < x < b$ then

$$\lim_{n \to \infty} \int_{c_1}^{c_2} f(x, n)dx = \int_{c_1}^{c_2} \lim_{n \to \infty} f(x, n)dx, \text{ where } a < c_1 < c_2 < b.$$

By 1, $F(x)$ is continuous, also the integrals $\int_{c_1}^{c_2} f(x, n)$, $\int_{c_1}^{c_2} F(x)dx$ exist (and are continuous).

Using (i), we can find n_0 (independent of x) such that if

$$f(x, n) = F(x) + g(x, n)$$

then $|g(x, n)| < \varepsilon$, all $n > n_0$ and all x in the interval.

Thus $$\left|\int_{c_1}^{c_2} f(x, n)dx - \int_{c_1}^{c_2} F(x)dx\right| \leq \int_{c_1}^{c_2} |g(x, n)|dx < \varepsilon(c_2 - c_1).$$

Thus $$\lim_{n \to \infty} \int_{c_1}^{c_2} f(x, n)dx = \int_{c_1}^{c_2} F(x)dx.$$

Examples. (i) Let $f(x, n) = n^2 x e^{-nx}$ $(x \geq 0)$.

$$\int_0^c \lim f(x, n)dx = 0 \ (c > 0), \quad \lim \int_0^c f(x, n)dx = \lim \{1 - (1 + nc)e^{-nc}\} = 1.$$

The sequence is *not* uniformly convergent in $0 \leq x \leq c$.

However, $\int_{c_1}^{c_2} f(x, n)dx = (1 + nc_1)e^{-nc_1} - (1 + nc_2)e^{-nc_2}$ which $\to 0$ $(c_1, c_2 > 0)$ and the sequence is uniformly convergent in $0 < c_1 \leq x \leq c_2$.

(ii) Let $f(x, n) = \dfrac{x^{n-1}}{1 + x^n}$ $(x \geq 0)$.

$$\int_0^2 f(x, n)dx = \frac{1}{n}\log (1 + 2^n) \text{ which} \to \log 2 \text{ as } n \to \infty,$$

also $\int_0^2 \lim f(x, n)dx = \int_1^2 \frac{dx}{x} = \log 2$, since $F(x) = 0$ $(0 \leq x < 1)$, $F(1) = \frac{1}{2}$, $F(x) = \frac{1}{x}$ $(x > 1)$.

The sequence is non-uniformly convergent in $0 \leq x \leq 2$, but the results of the integration are not necessarily unequal.

(iii) Let $f(x, n) = \dfrac{x}{1 + n^2x^2}$ which converges uniformly to zero, all x. In this case $\lim \int_0^c \dfrac{x\,dx}{1 + n^2x^2} = \lim \dfrac{1}{2n^2} \log (1 + n^2c^2) = 0 = \int_0^c \lim f(x, n)dx.$

11.12. Uniformly Convergent Series. The infinite series $\sum\limits_1^\infty u_n(x)$ is said to converge uniformly to the Sum Function $S(x)$ in an interval $a < x < b$, if the sequence $S_n(x) = \sum\limits_1^n u_n(x)$ tends uniformly to $S(x)$ in that interval.

11.13. Tests for Uniform Convergence of Series.

I. *The M-Test.* (*Weierstrass.*) If (i) $|u_n(x)| < M_n$ $(a < x < b)$, where M_n is independent of x and (ii) $\sum\limits_1^\infty M_n$ is convergent, then $\sum\limits_1^\infty u_n(x)$ is uniformly convergent in $a < x < b$.

For, given ε, n_0 exists such that $\sum\limits_{n+1}^{n+p} M_r < \varepsilon$ (all $n > n_0$, and all positive integers p); and therefore

$$\left|\sum_{n+1}^{n+p} u_r(x)\right| \leqslant \sum_{n+1}^{n+p} |u_r(x)| < \sum_{n+1}^{n+p} M_r$$

and the value n_0 is obviously independent of x.

Notes. (i) The series is also *absolutely* convergent in $a < x < b$.
(ii) M_n may be replaced by $v_n(x)$ if $\Sigma v_n(x)$ is a *uniformly* convergent series of positive terms.

II. *The Dirichlet Test for Uniform Convergence of a Series.* (*Hardy.*) If (i) $\sum\limits_1^\infty a_n(x)$ oscillates finitely in $a < x < b$ in such a way that

$\overline{\lim} \left|\sum\limits_1^n a_r(x)\right| < K$ (independent of x) *or* if $\sum\limits_1^\infty a_n(x)$ converges,

and (ii) $v_n(x)$ is a non-increasing monotone (for every x in the interval) tending uniformly to zero, then $\sum\limits_1^\infty a_n(x)v_n(x)$ is uniformly convergent in $a < x < b$.

For, given ε, we can find n_0 (independent of x) such that $|v_n(x)| < \varepsilon$ for all $n > n_0$; and, by Abel's Lemma, $\left|\sum\limits_{n+1}^{n+p} a_n(x)v_n(x)\right| < \varepsilon K$ for all $n > n_0$, all x in the interval and positive integer values of p.

Thus $\sum\limits_1^\infty a_n(x)v_n(x)$ is uniformly convergent.

Notes. (i) If $\Sigma a_n(x)$ converges in $a < x < b$, then K obviously exists.
(ii) If $v_n(x)$ is continuous for all n, then if convergent (monotonically) to zero, its convergence is uniform (§ *11.11* (vi)).

111. *The Abel Test for Uniform Divergence of Series.* (*Hardy.*) If (i) $\sum_1^\infty a_n(x)$ is uniformly convergent in $a < x < b$, and (ii) $v_n(x)$ is a non-increasing monotone for every x in the interval such that $v_0(x) < K$ (independent of x), then $\sum_1^\infty a_n(x)v_n(x)$ is uniformly convergent in $a < x < b$.

For, given ε, we can find n_0 (independent of x) such that

$$\left|\sum_{n+1}^{n+p} a_n(x)\right| < \varepsilon \text{ for } n > n_0 \text{ and positive integer values of } p.$$

By Abel's Lemma,

$$\left|\sum_{n+1}^{n+p} a_n(x)v_n(x)\right| < \varepsilon v_{n+1}(x) < \varepsilon v_0(x) < \varepsilon K.$$

Thus $\sum_1^\infty a_n(x)v_n(x)$ converges uniformly in (a, b).

Notes. (i) If, in Abel's Test, $v_n(x)$ tends uniformly to its limit $v(x)$ ($\neq 0$) (which is therefore continuous when $v_n(x)$ is continuous), the result follows from Dirichlet's Test by putting $v_n(x) - v(x)$ for $v_n(x)$ in the latter.

(ii) By writing $-v_n(x)$ for $v_n(x)$ we obtain corresponding results for a non-decreasing monotone $v_n(x)$.

(iii) When x does not appear in $a_n(x)$, it is sufficient to state (a) in Abel's Test, that Σa_n should converge, (b) in Dirichlet's Test, that Σa_n should oscillate finitely (or be convergent).

Examples. (i) *Power Series* $\sum_0^\infty a_n x^n$. If R is the radius of convergence, the series is *uniformly* convergent when $|x| \leq R_1 < R$, by the M-Test; for we can take $M_n = |a_n| R_1^n$.

Now suppose that $\sum_0^\infty a_n R^n$ is convergent. Then $v_n(x) = (x/R)^n$ is a non-increasing monotone, *bounded* in $0 < x < R$ (although it is *not* uniformly convergent in $0 \leq x \leq R$). By Abel's Test $\sum_0^\infty a_n x^n \equiv \sum_0^\infty (a_n R^n)\left(\frac{x}{R}\right)^n$ is uniformly convergent in $0 \leq x \leq R$, since $\sum_0^\infty a_n R^n$ converges ($R \neq 0$).

Thus R belongs to the interval of uniform convergence; and similarly $-R$ belongs to it if $\sum_0^\infty a_n(-R)^n$ converges. Again, x^n is a continuous function; and therefore it follows that $F(x) = \sum_0^\infty a_n x^n$ is *continuous* in its interval; in particular $\lim_{x \to R} \sum_0^\infty a_n x^n = \sum_0^\infty a_n R^n$, if the latter series converges. (*Abel's Theorem.*)

(ii) *The series* (a) $\sum_1^\infty \frac{\sin nx}{n^p}$, (b) $\sum_1^\infty \frac{\cos nx}{n^p}$. By the M-Test, these series are uniformly (and absolutely) convergent for all x when $p > 1$. (Take $M_n = n^{-p}$.)

Now $\sum_1^n \sin nx = \frac{\cos \frac{1}{2}x - \cos(n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}$ ($x \neq 2m\pi$) and is equal to zero when $x = 2m\pi$

When $x \neq 2m\pi$, $\sum_1^p \sin nx$ oscillates between $-\frac{1}{2}\tan\frac{1}{4}x$ and $\frac{1}{2}\cot\frac{1}{4}x$ and these limits can be made as large as we please by taking x sufficiently near $(4p+2)\pi$ and $4p\pi$ respectively. Therefore, in applying the Dirichlet Test, these points must be excluded from an interval of uniform convergence. Taking, therefore, the intervals $2m\pi + \alpha \leq x \leq 2(m+1)\pi - \alpha$ $(0 < \alpha < \pi)$, and applying the Dirichlet Test, we deduce that the series is uniformly convergent if $p > 0$. Similarly $\sum_1^\infty \frac{\cos nx}{n^p}$ $(p > 0)$ is uniformly convergent in these intervals.

(iii) $\sum_1^\infty \frac{x^n}{n(1 + x + x^2 + \ldots + x^{n-1})}$

If $0 < x < 1$, $\sum_1^n x^r < n$ and therefore $n\sum_0^n x^r > (n+1)\sum_1^n x^r$.

Therefore the sequence $\frac{nx^n}{1 + x + \ldots + x^{n-1}}$ is a decreasing monotone for every x in $0 < x < 1$. When $x = 0$, the terms are all zero and when $x = 1$ the terms are all equal to unity.

Also for $0 < x < 1$, $\frac{nx^n}{1 + x + \ldots + x^{n-1}} = \frac{nx^n(1-x)}{1 - x^n} \to 0$.

Take $v_n(x) = \frac{nx^n}{1 + x + \ldots + x^{n-1}}$ and $a_n(x) = \frac{1}{n^2}$ and apply Abel's Test.

Then $v_n(x)$ is positive, non-increasing and is always ≤ 1 in $0 \leq x \leq 1$. The given series is therefore uniformly convergent in $0 \leq x \leq 1$ (although $v_n(x)$ does not here tend uniformly to a limit). The series diverges when $x > 1$.

11.14. Properties of Uniformly Convergent Series.

I. If (i) $S(x) = \sum_1^\infty u_n(x)$ is uniformly convergent in $a \leq x \leq b$, and (ii) $u_n(x)$ is continuous in $a \leq x \leq b$ (all n), then $S(x)$ is continuous in $a < x < b$ (§ *11.11*).

Note. Uniform convergence is a sufficient condition of continuity of $S(x)$. It is also a necessary condition when $u_n(x)$ is of constant sign (§ *11.11* (ii)).

II. If (i) $S(x) = \sum_1^\infty u_n(x)$ is uniformly convergent in $a \leq x \leq b$ and (ii) $u_n(x)$ is continuous in $a \leq x \leq b$ (all n), then

$$\int_{c_1}^{c_2}\left\{\sum_1^\infty u_n(x)\right\}dx = \sum_1^\infty\left\{\int_{c_1}^{c_2} u_n(x)dx\right\}$$

when $a < c_1 < c_2 < b$ (§ *11.11*).

III. If (i) $\sum_1^\infty u_n'(x)$ is uniformly convergent in $a \leq x \leq b$ and (ii) $u_n(x)$ is continuous in $a < x < b$ (all n), this being implied in (i), and (iii) $\sum_1^\infty u_n(x)$ is convergent in $a < x < b$, then $\frac{d}{dx}\{\sum_1^\infty u_n(x)\} = \sum_1^\infty u_n'(x)$ for any value in the interval.

For by II, $\int_{c_1}^{x}\{\sum_1^\infty u_n'(x)\}dx = \sum_1^\infty (u_n(x) - u_n(c_1))$ $(a < c_1 < x \leq b)$.

i.e. $$\int_{c_1}^{x}\{\sum_1^\infty u_n'(x)\}dx = \sum_1^\infty u_n(x) - \sum_1^\infty u_n(c_1), \text{ (using (iii))}$$

or $$\frac{d}{dx}\{\sum_1^\infty u_n(x)\} = \sum_1^\infty u_n'(x).$$

Examples. (i) Expand $\log\{1 + \sqrt{(1 - x)}\}$ and prove that

$$(a)\ \log 2 = \frac{1}{2}\cdot\frac{1}{2} + \frac{1.3}{2.4}\cdot\frac{1}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{6} + \cdots$$

$$(b)\ \log\left(\frac{1 + \sqrt{2}}{2}\right) = \frac{1}{2}\cdot\frac{1}{2} - \frac{1.3}{2.4}\cdot\frac{1}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{6} - \cdots$$

If $u = \log\{1 + \sqrt{(1 - x)}\}$, $u'(x) = -\dfrac{1 - \sqrt{(1 - x)}}{2x\sqrt{(1 - x)}}$ $(x \neq 0)$, and $\frac{1}{4}$ $(x = 0)$

i.e. $u'(x) = -\dfrac{1}{4} - \dfrac{1.3}{2.4}\cdot\dfrac{x}{2} - \dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^2}{3} \cdots$ $(|x| < 1)$, the series being uniformly convergent for $|x| \leq 1 - \epsilon < 1$.

Integration gives $u(x) - u(0) = -\dfrac{1}{2}\dfrac{x}{2} - \dfrac{1.3}{2.4}\cdot\dfrac{x^2}{4} - \dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^3}{6} \cdots$ $(|x| < 1)$.

When $x = 1$, if the general term is $-a_n$, then $a_n > 0$ and $\dfrac{a_n}{a_{n+1}} = 1 + \dfrac{3}{2n} + O\left(\dfrac{1}{n^2}\right)$ and therefore the series converges (§ 4.19).

Thus (a) $\log 2 = \sum_0^\infty \dfrac{1.3.5 \ldots (2n + 1)}{2.4.6 \ldots (2n + 2)}\cdot\dfrac{1}{(2n + 2)}$.

When $x = -1$, the series converges by Leibniz's Rule.

Thus (b) $\log\left(\dfrac{1 + \sqrt{2}}{2}\right) = \sum_0^\infty (-1)^n\cdot\dfrac{1.3.5 \ldots (2n + 1)}{2.4.6 \ldots (2n + 2)}\cdot\dfrac{1}{2n + 2}$.

It follows also that $\dfrac{1}{2} + \dfrac{1.3.5}{2.4.6}\cdot\dfrac{1}{3} + \dfrac{1.3.5.7.9}{2.4.6.8.10}\cdot\dfrac{1}{5} + \cdots = \log(1 + \sqrt{2})$.

Note. An infinite series may sometimes be integrated term by term when it is not uniformly convergent. The most important class of such series are those that are described as *boundedly convergent*. A series is said to be boundedly convergent for the interval $a \leq x \leq b$ if it converges at all points of the interval and if the sum of n terms has a finite upper bound M, independent of n and x. It is easy to see, for example, that if $S_n(x)$ converges uniformly to $S(x)$ except at a finite number of points in $a \leq x \leq b$ then term-by-term integration is legitimate if the convergence is bounded. Let there be one such point c in the interval. Then $\int_{c-\delta}^{c+\delta} S_n(x)dx \to 0$ as $\delta \to 0$ for all n (since $|S_n(x)| < M$).

i.e. $$\lim_{n\to\infty}\int_{c-\delta}^{c+\delta} S_n(x)dx \to 0 \text{ when } \delta \to 0$$

Thus $\lim\limits_{n\to\infty}\int_a^b S_n(x)dx$ exists, being defined as $\lim\limits_{\delta\to 0}\lim\limits_{n\to\infty}\left\{\int_a^{c-\delta} + \int_{c+\delta}^b\right\}S_n(x)dx$.

But $\lim\limits_{n\to\infty}\left\{\int_a^{c-\delta} + \int_{c+\delta}^b\right\}S_n(x)dx = \left\{\int_a^{c-\delta} + \int_{c+\delta}^b\right\}S(x)dx$, by uniform convergence. The expression on the right when $\delta \to 0$ defines $\int_a^b S(x)dx$, since $|S(x)| \leq M$.

Example. Let $S_n(x) = \dfrac{1}{1 + n^2x^2}$. Then $S_n(x)$ is not uniformly convergent at $x = 0$. Here $S(x) = 0$ $(x \neq 0)$ and $S(x) = 1$ $(x = 0)$. In this case $S_n(x)$ is boundedly convergent since $S_n(x) \leq 1$ all x, n and $S(x) \leq 1$. Thus $\int_0^1 S(x)dx = 0$ and $\lim\limits_{n \to \infty} \int_0^1 S_n(x)dx$ which is equal to $\lim \left(\frac{1}{n} \tan^{-1} n\right)$ is zero also.

11.15. Uniform Convergence of Sequences of Complex Numbers. A sequence $S(n, z)$ is said to tend uniformly to a limit $S(z)$ (where $z = x + iy$), in a given domain D, if, given ε, we can find an integer n_0 independent of z such that

$$|S(n, z) - S(z)| < \varepsilon \text{ for all } n > n_0 \text{ and all } z \text{ in } D.$$

Notes. (i) The domain may consist of the points of a continuous curve C, in which case the sequence is said to converge uniformly along C.

(ii) The necessary and sufficient condition that $S(n, z)$ should tend uniformly to $S(z)$ is that, given ε, we can find n_0 (independent of z in D) such that

$$|S(n, z) - S(n_0, z)| < \varepsilon$$

for all $n > n_0$ and all z in D.

(iii) The only functions of the complex variable that we shall consider here are analytic functions; and it is to be expected that the properties of uniformly convergent sequences of analytic functions will be simpler than the corresponding properties of general functions of the real variable.

11.16. Properties of Uniformly Convergent Sequences of Analytic Functions. Let (i) $S(n, z)$ be analytic in D

(ii) $S(n, z) \to S(z)$ uniformly in every region interior to D.

Then (*a*) $\int_{z_1}^{z_2} S(n, z)dz \to \int_{z_1}^{z_2} S(z)dz$, where the path of integration is a simple curve lying within D.

(*b*) $S(z)$ is an analytic function within D.

(*c*) $S^{(r)}(z)$ is an analytic function for all r and its value is

$$\lim_{n \to \infty} \int_{z_1}^{z_2} S^{(r)}(n, z)dz \quad (\textit{Weierstrass}).$$

(*a*) Choose n_0 so that $|S(n, z) - S(z)| < \varepsilon$ for all $n > n_0$ and all z within D. Then

$$\left|\int_{z_1}^{z_2} S(z)dz - \int_{z_1}^{z_2} S(n, z)dz\right| < \int_{z_1}^{z_2} |S(n, z) - S(z)|ds$$

where ds is the element of arc of the path of integration. But

$$\int_{z_1}^{z_2} |S(n, z) - S(z)|ds < \varepsilon l$$

where l is the length of the path.

Thus
$$\int_{z_1}^{z_2} S(n, z)dz \to \int_{z_1}^{z_2} S(z)dz.$$

(*b*) Let z_0 be a point within D and C a simple closed curve within D and containing z_0 within it.

Then $\quad S(z_0) = \lim S(n, z_0) = \lim \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{S(n, z)dz}{(z - z_0)}$

$$= \frac{1}{2\pi i}\int_C \frac{S(z)dz}{z - z_0}$$

by (a), since $S(n, z)/(z - z_0)$ is uniformly convergent to $S(z)/(z - z_0)$.

Therefore

$$\frac{S(z_0 + \delta z_0) - S(z_0)}{\delta z_0} = \frac{1}{2\pi i}\int_C S(z)\left\{\frac{1}{(z - z_0)^2} + \frac{\delta z_0}{(z - z_0)^2(z - z_0 - \delta z_0)}\right\}dz$$

But $\left|\dfrac{S(z)}{(z - z_0)^2(z - z_0 - \delta z_0)}\right|$ is bounded on C; and therefore $S'(z_0)$ exists and is equal to $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{S(z)dz}{(z - z_0)^2}$.

But $\displaystyle\int_C \frac{S(z)dz}{(z - z_0)^2} = \int_C \frac{\lim S(n, z)dz}{(z - z_0)^2} = \lim \int_C \frac{S(n, z)dz}{(z - z_0)^2} = \lim S'(n, z_0)$.

We have proved therefore that $S(z)$ is analytic within D and has the value $\lim S'(n, z)$.

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By a similar proof, using the method of *Chapter X, § 104*, for obtaining derivatives of an analytic function in terms of integrals, we may deduce that $S^{(r)}(z)$ exists and has the value $\lim S^{(r)}(n, z)$. §10 41

Notes. (i) It may, of course, be proved that $S(z)$ is continuous in D if $S(n, z)$ is continuous. Thus if $S(n, z)$ is analytic, $S(z)$ is continuous at least.

(ii) The convergence of $S^{(r)}(n, z)$ to $S^{(r)}(z)$ is obviously uniform.

(iii) It is sufficient that $S(n, z)$ should tend uniformly to $S(z)$ along C and $S(n, z)$ should be analytic along C and within it, in order that $S(z)$ should be analytic within C.

11.17. Infinite Series of Complex Variables. From the previous paragraph we deduce that if the series $\sum\limits_1^\infty u_n(z)$ is uniformly convergent in D, the sum $S(z)$ is continuous when $u_n(z)$ is continuous and analytic when $u_n(z)$ is analytic. The integral of $S(z)$ along a simple path within D may be effected term-by-term; and when $u_n(z)$ is analytic, the derivatives are obtained by differentiating the series term-by-term.

11.171. Tests for Uniform Convergence of Series of Complex Variables. The most useful test in practice is the *M-test*:

If (i) $\sum\limits_1^\infty M_n$ is a convergent series of positive constants, and (ii) $|u_n(z)| < M_n$ for all points in D, then $\sum\limits_1^\infty u_n(z)$ is uniformly (and absolutely) convergent in D.

The proof is similar to that for the real variable.

Note. Tests for convergence (ordinary or uniform) of the Abel-Dirichlet type suitable for complex terms have been given by Bromwich (*Infinite Series*, 80).

11.18. Power Series in the Complex Variable. The series $\sum\limits_0^\infty a_n z^n$ is uniformly (and absolutely) convergent for $|z| < R$, where R is the radius of convergence; and the region of uniform convergence contains those points for which $\sum\limits_0^\infty a_n z^n$ converges when $|z| = R$, provided the mode in which z approaches the boundary is of an appropriate type (§ *10.4*).

11.19. Convergence on the Circle of Convergence. If $R \neq 0$, we may without loss of generality assume that $R = 1$, since the substitution $\zeta = z/R$ gives a power series in ζ whose radius is unity.

I. Let a_n be real, and take $z = \cos\theta + i\sin\theta$. The resulting series $\sum\limits_0^\infty a_n(\cos n\theta + i\sin n\theta)$ is convergent, by Dirichlet's Test, if $a_n \to 0$ (except possibly when θ is a multiple of 2π).

II. Let a_n be complex; since $|a_n/a_{n+1}| \to 1$, we can multiply z by a suitable factor of the form $e^{i\alpha}$ to ensure that (a_n/a_{n+1}) takes the form $1 + \phi(n)$ where $\phi_n \to 0$. In many cases it will be found that a_n/a_{n+1} can be written $1 + (\mu/n) + (\omega_n/n^\lambda)$ where $|\omega_n|$ is bounded and $\lambda > 1$.

Then (i) If $\mathbf{R}(\mu) > 1$, we have already shown that there is *absolute convergence*.

(ii) If $\mathbf{R}(\mu) \leq 0$, the series cannot converge since the general term does not tend to zero. (§ *11.08*)

(iii) If $0 < \mathbf{R}(\mu) \leq 1$, it may be shown that the series converges (not absolutely) except for $z = 1$ (*Weierstrass, Ges. Werke, I, 185*).

Examples. (i) *The Binomial Expansion.* $\sum\limits_0^\infty (-1)^n \dfrac{\nu(\nu-1)\ldots(\nu-n+1)}{n!} z^n$.

When convergent, its value is $(1 - z)^\nu$, and its radius of convergence is 1. It is *absolutely* convergent (i) for $|z| < 1$, (ii) $|z| = 1$, $\mathbf{R}(\nu) > 0$. In this case

$$\frac{a_n}{a_{n+1}} = \frac{n+1}{n-\nu} = 1 + \frac{1+\nu}{n} + O\left(\frac{1}{n^2}\right).$$

It is not convergent for $|z| = 1$, when $\mathbf{R}(\nu) \leq -1$, and by the previous paragraph, the series converges (not absolutely) for $-1 < \mathbf{R}(\nu) \leq 0$, except when $z = 1$.

(ii) *The Geometric Series.* $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$, when $|z| < 1$ and the series does not converge at any point of the circle.

Let $z = re^{i\theta}$ $(0 \leq r < 1, -\pi < \theta \leq \pi)$; then

$$\frac{1}{1 + r(\cos\theta + i\sin\theta)} = \sum_0^\infty (-1)^n r^n(\cos n\theta - i\sin n\theta)$$

Thus
$$\frac{1 + r\cos\theta}{1 + 2r\cos\theta + r^2} = 1 - r\cos\theta + r^2\cos 2\theta \ldots$$

$$\frac{r\sin\theta}{1 + 2r\cos\theta + r^2} = r\sin\theta - r^2\sin 2\theta + \ldots$$

These equations are true for *all* θ and for $0 < |r| < 1$, but it is convenient to restrict r, θ as above.

Writing $\theta + \pi$ for θ we obtain

$$\frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} = 1 + r\cos\theta + r^2\cos 2\theta + \ldots$$

$$\frac{r\sin\theta}{1 - 2r\cos\theta + r^2} = r\sin\theta + r^2\sin 2\theta + \ldots$$

Simple deductions from these are

$$\frac{1 - r^2}{1 - 2r\cos\theta + r^2} = 1 + 2\sum_1^\infty r^n\cos n\theta; \quad \frac{1 - r^2}{1 + 2r\cos\theta + r^2} = 1 + 2\sum_1^\infty(-1)^n r^n\cos n\theta;$$

$$\frac{\cos\theta - r}{1 - 2r\cos\theta + r^2} = \sum_1^\infty r^{n-1}\cos n\theta; \quad \frac{\cos\theta + r}{1 + 2r\cos\theta + r^2} = \sum_1^\infty(-1)^{n-1} r^{n-1}\cos n\theta.$$

$\int_0^\pi \frac{\cos n\theta\, d\theta}{1 - 2r\cos\theta + r^2} = \frac{\pi r^n}{1 - r^2}$ (n integral, $0 < r < 1$), the series being uniformly convergent for $r < 1$ and all θ by the M-test.

(ii) *The Logarithmic Series.* $\int_0^z \frac{dz}{1 + z} = \log(1 + z)$, where the path of integration is the line joining O to z, $(z \neq -1)$.

The infinite series for $(1 + z)^{-1}$ converges for $|z| < 1$. If $P(re^{i\theta})$ is a point inside this circle, we have $\log(1 + z) = \log A_1P + i\phi$, where A_1 is the point -1, and ϕ (between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$) is the angle that $\overrightarrow{A_1P}$ makes with OX. (*Fig. 5.*)

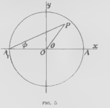

FIG. 5

Thus $\frac{1}{2}\log(1 + 2r\cos\theta + r^2) + i\arctan\left(\frac{r\sin\theta}{1 + r\cos\theta}\right)$

$$= z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \ldots$$

$$= \sum_1^\infty(-1)^{n-1}\frac{r^n}{n}\cos n\theta + i\sum_1^\infty(-1)^{n-1}\frac{r^n}{n}\sin n\theta \; (0 < r < 1).$$

By Dirichlet's Test, both series on the right are convergent for $r = 1$ (except the cosine series for $\theta = \pi$). The above equation, apart from the exceptional case, is true for $r = 1$.

Thus
$$\sum_1^\infty (-1)^{n-1}\frac{r^n}{n}\cos n\theta = \tfrac{1}{2}\log(1 + 2r\cos\theta + r^2).$$
$$\sum_1^\infty (-1)^{n-1}\frac{r^n}{n}\sin n\theta = \arctan\frac{r\sin\theta}{1 + r\cos\theta}$$
for $0 < r < 1$ and all θ and
$$\sum_1^\infty (-1)^{n-1}\frac{\cos n\theta}{n} = \tfrac{1}{2}\log(4\cos^2\tfrac{1}{2}\theta)\ (-\pi < \theta < \pi).$$
$$\sum_1^\infty (-1)^{n-1}\frac{\sin n\theta}{n} = \tfrac{1}{2}\theta\ (-\pi < \theta < \pi);\ \sum_1^\infty (-1)^{n-1}\frac{\sin n\theta}{n} = 0\ (\theta = \pm\pi).$$
The same series is represented for $-\pi < \theta < \pi$ by the part of the line through $(\pi, \frac{1}{2}\pi)$, $(-\pi, -\frac{1}{2}\pi)$ that lies in the interval. It is finitely discontinuous at the ends, and its value for other values of θ is obtained from its obvious periodicity in 2π.

Writing $\pi - \theta$ for θ we find
$$\sum_1^\infty r^n\cos n\theta = -\tfrac{1}{2}\log(1 - 2r\cos\theta + r^2);$$
$$\sum_1^\infty \frac{r^n}{n}\sin n\theta = \arctan\frac{r\sin\theta}{1 - r\cos\theta}\ (0 < r < 1,\ \text{all } \theta).$$
$$\sum_1^\infty \frac{\cos n\theta}{n} = -\tfrac{1}{2}\log(4\sin^2\tfrac{1}{2}\theta).$$
$$\sum_1^\infty \frac{\sin n\theta}{n} = \tfrac{1}{2}(\pi - \theta)\ (0 < \theta < 2\pi).$$

Notes. (i) The series $C(\theta) = \sum_1^\infty \frac{\cos n\theta}{n}$, $S(\theta) = \sum_1^\infty \frac{\sin n\theta}{n}$ are, by Dirichlet's Test, uniformly convergent in an interval that excludes the points $2n\pi$.

(ii) If $S(n, \theta) = \sum_1^n \frac{\sin n\theta}{n}$, it is of some interest to consider the limiting form of the curve $y = S(n, x)$ as n tends to infinity and x tends to zero. The double limit has different values according to the way in which the parameters n, x tend to infinity and zero respectively; and we have shown above that in particular $\lim_{n\to\infty}\lim_{x\to 0} S(n, x) = 0$ whilst $\lim_{x\to 0}\lim_{n\to\infty} S(n, x) = \frac{1}{2}\pi$.

$$S(n, x) = \int_0^x \Big(\sum_1^n \cos n\theta\Big)d\theta = \int_0^x \frac{\sin(n + \frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}\,d\theta = -\tfrac{1}{2}x + \int_0^x \frac{\sin(n - \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}\,d\theta,$$
and we have shown that $\lim_{n\to\infty}\int_0^x \frac{\sin(n + \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}\,d\theta = \frac{1}{2}\pi\ (x \neq 0),\ 0\ (x = 0)$.

Consider the function $T(n, x) = \int_0^x \frac{\sin(n + \frac{1}{2})\theta}{\theta}\,d\theta$.

If $x \neq 0$, $T(n, x) = \int_0^u \frac{\sin\phi}{\phi}\,d\phi$, $\phi = (n + \frac{1}{2})\theta$, $u = (n + \frac{1}{2})x$, and therefore $T(n, x) \to \pi/2\ (x \neq 0)$ whilst $T(n, x) \to 0\ (x = 0)$.

Thus $S(n, x) = -\frac{1}{2}x + T(n, x) + \int_0^x \sin(n + \frac{1}{2})\theta\left\{\frac{1}{2\sin\frac{1}{2}\theta} - \frac{1}{\theta}\right\}d\theta$ and we infer that $U(n, x) = \int_0^x \sin(n + \frac{1}{2})\theta\left\{\frac{1}{2\sin\frac{1}{2}\theta} - \frac{1}{\theta}\right\}d\theta \to 0$ for $0 < x < 2\pi$, a result that may be verified directly by integration by parts.

Now when $x \to 0$, the maximum value of $T(n, x)$ for a given n occurs when $u = (n + \frac{1}{2})x \to \pi$ (since $T'(u) = \sin u/u$) i.e. the maximum value of $T(n, x)$ is $\int_0^\pi \frac{\sin \phi}{\phi}\, d\phi$ ($= 1.85$ approx.). The convergence of $U(n, x)$ to zero is easily shown to be uniform.

Thus the limit of the curve $y = S(n, x)$ consists of a set of segments through $n\pi$ equal and parallel to the segment joining $(0, \frac{1}{2}\pi)$ to $(2\pi, -\frac{1}{2}\pi)$ together with a set of segments through $(2n\pi, 0)$ parallel to the segment of the y-axis between $y = +1.85$ and $y = -1.85$ approx. These segments parallel to the y axis project *above* the line $y = \frac{1}{2}\pi$ (and below $y = -\frac{1}{2}\pi$) by an amount 0.28 approx. (*The Gibbs Phenomenon.*)

It should be noted that we have proved that the series is *boundedly* convergent (and that therefore it is legitimate to integrate term by term over any interval in order to obtain the integral of the sum.)

(iv) *The Series obtained by Integrations of* $\sum_1^\infty \frac{\sin n\theta}{n}$. Integration from θ to π gives

$$(\cos\theta + 1) + \frac{1}{2^2}(\cos 2\theta - 1) + \ldots = \frac{1}{4}\pi^2 - \frac{1}{2}\pi\theta + \frac{1}{4}\theta^2.$$

Thus

$$\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{3^2}\cos 3\theta + \ldots = \frac{1}{4}\pi^2 - \frac{1}{2}\pi\theta + \frac{1}{4}\theta^2 - \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\right)$$

the series on the right being convergent (absolutely).

The series on the left is uniformly convergent for all θ.

Putting $\theta = 0$, we find that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$ from which we deduce that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6} \text{ and } 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$

and therefore

$$\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{3^2}\cos 3\theta + \ldots = \frac{\pi^2}{6} - \frac{1}{4}\theta(2\pi - \theta)\ (0 \leq \theta \leq 2\pi).$$

Integration of the last result from 0 to θ gives

$$\sin\theta + \frac{1}{2^3}\sin 2\theta + \frac{1}{3^3}\sin 3\theta \ldots = \frac{\theta(\theta - \pi)(\theta - 2\pi)}{12}\ (0 \leq \theta \leq 2\pi)$$

and in particular (for $\theta = \frac{1}{2}\pi$) $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \ldots = \frac{\pi^3}{32}$.

A further integration gives

$$\sum_1^\infty \frac{\cos n\theta}{n^4} = -\frac{\theta^4}{48} + \frac{\pi\theta^3}{12} - \frac{\pi^2\theta^2}{12} + \sum_1^\infty \frac{1}{n^4}.$$

$\theta = \pi$ gives $-\frac{7}{8}\sum_1^\infty \frac{1}{n^4} = -\frac{\pi^4}{48} + \sum_1^\infty \frac{1}{n^4}$ from which we deduce $1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}$,

$$1 - \frac{1}{2^4} + \frac{1}{3^4} - \ldots = \frac{7\pi^4}{720},\ 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}.$$

Also $\theta = \pi/4$ leads to the result

$$1 - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \frac{1}{11^4} \ldots = \frac{11\sqrt{2}\pi^4}{1536}.$$

(v) *The Series for* arc tan z. $\arctan z = \int_0^z \frac{dz}{1 + z^2} = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 \ldots |z| < 1$.

When $|z| = 1$, take $z = \cos\theta + i\sin\theta$: and therefore

$$\arctan(\cos\theta + i\sin\theta) = \sum_0^\infty (-1)^n \frac{\cos(2n + 1)\theta}{2n + 1} + i\sum_0^\infty (-1)^n \frac{\sin(2n + 1)\theta}{2n + 1}$$

except when $\theta = \pm \frac{1}{2}\pi$; and the convergence of these series of cosines and sines is uniform in $-\frac{1}{2}\pi + \alpha \leq \theta \leq \frac{1}{2}\pi - \alpha$ $(\alpha > 0)$.

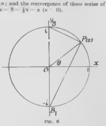

FIG. 6

Now arc tan z (on the unit circle)

$$\tfrac{1}{2}i \log\frac{i - z}{i + z}$$

$$= \tfrac{1}{2}i\left\{\log\left(\frac{PB}{B_1P}\right) + i\frac{\pi}{2}\right\} \quad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

where B_1, P, B are the points $-i$, z, i and $\mathbf{R}(z) > 0$.

Also $\dfrac{PB}{B_1P} = \tan\frac{1}{2}\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta}$

This gives the results

$$\left.\begin{aligned} &\cos\theta - \tfrac{1}{3}\cos 3\theta + \tfrac{1}{5}\cos 5\theta \ldots = \tfrac{1}{4}\pi \\ &\sin\theta - \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta \ldots = \tfrac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \end{aligned}\right\} - \tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi.$$

The sum of the sine series may also be written

$$\tfrac{1}{4}\log\left(\frac{1 + \sin\theta}{1 - \sin\theta}\right) \text{ or } \tfrac{1}{4}\log(\sec\theta + \tan\theta)^2.$$

When $\theta \to \pm\pi/2$, the cosine series tends to zero and the sine series to $\pm\infty$.

Putting $\theta - \pi/2$ for θ in these series we obtain also

$$\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \ldots = \frac{\pi}{4}; \quad \cos\theta + \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta \ldots = \tfrac{1}{2}\log\cot\frac{\theta}{2}$$
$$(0 < \theta < \pi).$$

Integration in the interval $0 < \alpha < \theta < \pi - \alpha$ gives

$$\cos\theta + \frac{1}{3^2}\cos 3\theta + \frac{1}{5^2}\cos 5\theta \ldots = C - \tfrac{1}{4}\pi\theta$$

but since the series on the left is uniformly convergent at $\theta = 0$, we obtain

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8} = C \text{ so that } \sum_0^\infty \frac{\cos(2n + 1)\theta}{(2n + 1)^2} = \frac{\pi}{8}(\pi - 2\theta)\ (0 \leq \theta \leq \pi).$$

A further integration gives:

$$\sin\theta + \frac{1}{3^3}\sin 3\theta + \frac{1}{5^3}\sin 5\theta \ldots = \frac{\pi\theta}{8}(\pi - \theta)\ (0 \leq \theta \leq \pi).$$

Notes. (i) For the interval $-\pi < \theta < 0$, $\sum\limits_0^\infty \dfrac{\sin(2n+1)\theta}{2n+1} = -\tfrac{1}{4}\pi$ and for $-\pi < \theta < 0$, we have $\sum\limits_0^\infty \dfrac{\cos(2n+1)\theta}{(2n+1)^2} = \dfrac{\pi}{8}(\pi + 2\theta)$; $\sum\limits_0^\infty \dfrac{\sin(2n+1)\theta}{(2n+1)^3} = -\dfrac{\pi\theta}{8}(\pi + \theta)$.

(ii) Integration of the series for $\frac{1}{2} \log \cot \theta/2$ will give

$$\sin\theta + \frac{1}{3^2}\sin 3\theta + \frac{1}{5^2}\sin 5\theta + \ldots = \tfrac{1}{2}\theta \log \cot \tfrac{1}{2}\theta + \tfrac{1}{2}\int_0^\theta \frac{\theta\, d\theta}{\sin\theta} \; (0 < \theta < \pi).$$

(iii) The various formulae in this example may of course be deduced directly from those in the previous example. Thus if we take

$$S(\theta) = \sum_1^\infty \frac{\sin n\theta}{n}, \text{ then } S(\theta) = \tfrac{1}{2}(\pi - \theta)\; (0<\theta<2\pi);\; S(0) = 0 = S(2\pi).$$

$$T(\theta) = \sum_1^\infty \frac{\sin 2n\theta}{n}, \text{ then } T(\theta) = \tfrac{1}{2}(\pi - 2\theta)\; (0<\theta<\pi);\; \tfrac{1}{2}(3\pi - 2\theta)\; (\pi<\theta<2\pi)$$

and $T(0) = T(\pi) = T(2\pi) = 0$,

so that

$$\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta \ldots = S(\theta) - \tfrac{1}{2}T(\theta) = \frac{\pi}{4}\; (0 < \theta < \pi), \quad -\frac{\pi}{4}\; (\pi < \theta < 2\pi)$$

and the sum is zero for $\theta = 0, \pi, 2\pi$.

11.2. Infinite Products. If the sequence

$$P_n = (1 + u_1)(1 + u_2) \ldots (1 + u_n)$$

tends to a limit $P\ (\neq 0)$, when n tends to infinity, we write $P = \prod\limits_1^\infty (1 + u_n)$ and call the expression on the right an infinite product. If $P = 0$, the product is said to *diverge to zero*, thus preserving the correspondance between the infinite product and the infinite series $\sum\limits_1^\infty \log|1 + u_n|$.

Since $\log|P_n| = \sum\limits_1^n \log|1 + u_n|$, it is necessary (but not sufficient) for convergence that $\lim u_n$ should be zero. Consequently, in investigating conditions for convergence it is sufficient to assume that all the terms $(1 + u_n)$ are *positive*. For definiteness also, we shall assume that $|u_n| < \frac{1}{2}$ so that $\frac{1}{2} < 1 + u_n < \frac{3}{2}$. In applications, however, there may be a *finite* number of terms (when there is convergence) that do not lie between these limits and there may be a finite number of negative terms. It will always be assumed also that no term u_n is -1; if, in an actual case, there are a finite number of terms u_n equal to -1, the product is said to converge to zero, when the product of the remaining terms converges.

By the mean value theorem

$$\log(1 + u_n) = u_n - \frac{u_n^2}{2(1 + \theta u_n)^2}\; (0 < \theta < 1).$$

Therefore

$$0 < u_n - \log(1 + u_n) < 2u_n^2 \;(\text{all } n, \text{ since } 1 + \theta u_n > 1 - |u_n|)$$

i.e.

$$0 < \sum_{m+1}^{m+p} u_n - \sum_{m+1}^{m+p} \log(1 + u_n) < 2\sum_{m+1}^{m+p} u_n^2.$$

If then $\sum_1^\infty u_n^2$ is *convergent*, the infinite product

(i) *converges*, when $\sum_1^\infty u_n$ converges;

(ii) *diverges to* $+\infty$, when $\sum_1^\infty u_n$ diverges to $+\infty$;

(iii) *diverges to zero*, when $\sum_1^\infty u_n$ diverges to $-\infty$;

(iv) *oscillates*, when $\sum_1^\infty u_n$ oscillates.

Again, since $1 + \theta u_n < 1 + |u_n| < \frac{3}{2}$, then $u_n - \log(1 + u_n) > \frac{2}{9}u_n^2$, so that if $\sum_1^\infty u_n^2$ diverges, the infinite product diverges to zero, when (i) $\sum_1^\infty u_n$ converges or (ii) $\sum_1^\infty u_n$ diverges to $-\infty$ or (iii) $\sum_1^\infty u_n$ oscillates in such a way that its upper limit is *not* $+\infty$.

No information is given from these inequalities when $\sum_1^\infty u_n^2$ diverges and $\sum_1^\infty u_n$ diverges to $+\infty$ or has $+\infty$ as its upper limit. In such cases, the infinite product may or may not converge. (*See Examples* (ii), (iii) *below*.)

There must, however, be divergence when Σu_n is divergent and u_n is of constant sign (*see* next paragraph).

It may be seen from the above that in general a sufficient (but unnecessary) condition for convergence is the convergence of both $\sum_1^\infty u_n$ and $\sum_1^\infty u_n^2$.

Examples. (i) $(1 + \frac{1}{2}x)(1 - \frac{1}{3}x^2)(1 + \frac{1}{4}x^3) \dots$

Here $\sum_1^\infty u_n = \frac{1}{2}x - \frac{1}{3}x^2 + \dots$; $\sum_1^\infty u_n^2 = \frac{x^2}{2^2} + \frac{x^4}{3^2} + \frac{x^6}{4^2} + \dots$

$\sum_1^\infty u_n$ converges if $-1 \leq x < 1$; $\sum_1^\infty u_n^2$ converges if $|x| \leq 1$.

Therefore the product converges when $-1 < x < 1$.

When $x = -1$, $\sum_1^\infty u_n$ diverges to $-\infty$ and $\sum_1^\infty u_n^2$ converges. The product diverges to zero.

When $|x| > 1$, u_n does not tend to zero, and the product is not convergent.

(ii) $\prod_1^\infty\left(1 + \frac{1}{\sqrt{n}}\right)$.

Here $\sum_1^\infty u_n$, $\sum_1^\infty u_n^2$ diverge, but $\prod_1^\infty\left(1 + \frac{1}{\sqrt{n}}\right) > 1 + \sum_1^\infty \frac{1}{\sqrt{n}}$ and therefore the product diverges to $+\infty$.

(iii) $\left(1 + \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{2} - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{3}}\right)\left(1 + \frac{1}{3} - \frac{1}{\sqrt{3}}\right) \dots$

i.e. $$\prod_2^\infty(1 + u_\nu) \text{ where } u_{2n} = \frac{1}{\sqrt{n}},\ u_{2n} = \frac{1}{n} - \frac{1}{\sqrt{n}}.$$

Here $\sum_2^{2n} u_\nu = \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$; $\sum_2^{2n+1} u_\nu = \frac{1}{\sqrt{(n+1)}} + \sum_2^{2n} u_\nu$, so that $\sum_2^\infty u_\nu$ diverges to $+\infty$.

Also $\sum_2^{2n} u_r^2 = \sum_2^n\left(\frac{1}{r^2} - \frac{2}{r^{5/2}} + \frac{2}{r}\right)$; $\sum_2^{2n+1} u_r^2 = \frac{1}{n+1} + \sum_2^{2n} u_r^2$, so that $\sum_2^\infty u_r^2$ also diverges to $+\infty$.

But $\prod_2^{2n+1}(1+u_r) = \left(1 - \frac{1}{\sqrt{(n+1)}}\right)\prod_2^{2n}(1+u_r)$; $\prod_2^{2n}(1+u_r) = \prod_2^n\left(1 + \frac{1}{r^{3/2}}\right)$; so that the infinite product converges since $\sum_2^\infty \frac{1}{r^{3/2}}$, $\sum_2^\infty \frac{1}{r^2}$ both converge.

11.21. The Case when u_n is of Constant Sign. Suppose $\alpha_n > 0$; then if $\sum_1^\infty \alpha_n$ is convergent, so also is $\sum_1^\infty \alpha_n^2$ for $\alpha_n < 1$ (ultimately) and α_n^2 is then less than α_n.

The convergence of $\sum_1^\infty \alpha_n$ is therefore *sufficient* for the convergence of the product $\prod_1^\infty(1+\alpha_n)$.

Also since $\prod_1^n(1+\alpha_n) > 1 + \sum_1^n \alpha_n$, the divergence of $\sum_1^\infty \alpha_n$ implies the divergence of the product.

Thus the convergence of $\sum_1^\infty \alpha_n$ is *necessary* for the convergence of the product.

Again, $0 < \prod_1^\infty(1-\alpha_n^2) < 1$ and therefore the convergence of $\prod_1^\infty(1+\alpha_n)$ implies that of $\prod_1^\infty(1-\alpha_n)$; whilst the divergence of $\prod_1^\infty(1+\alpha_n)$ (to $+\infty$) implies the divergence of $\prod_1^\infty(1-\alpha_n)$ to zero.

Summarizing. A necessary and sufficient condition for the convergence of $\prod_1^\infty(1+\alpha_n)$ and $\prod_1^\infty(1-\alpha_n)$ ($\alpha_n > 0$, all n), is the convergence of $\sum_1^\infty \alpha_n$.

11.22. Absolute Convergence of an Infinite Product. The product $\prod_1^\infty(1+u_n)$ is said to be *absolutely convergent* when the series $\sum_1^\infty \log(1+u_n)$ is absolutely convergent (assuming as before that $(1+u_n) > 0$).

If $0 < \alpha < 1$, $|\log(1-\alpha)| = \log\left(\frac{1}{1-\alpha}\right) > \log(1+\alpha)$.

Therefore, $\log(1+|u_n|)$ which equals $|\log(1+u_n)|$ when $u_n > 0$ is less than $|\log(1+u_n)|$ when $u_n < 0$.

i.e. $\sum_1^\infty \log(1+|u_n|)$ converges when $\sum_1^\infty|\log(1+u_n)|$ converges.

If, then, $\sum_1^\infty \log(1+u_n)$ is absolutely convergent, the infinite product $\prod_1^\infty(1+|u_n|)$ is convergent, a necessary (and sufficient) condition for which is the convergence of $\sum_1^\infty|u_n|$.

Now suppose that $\sum_1^\infty |u_n|$ converges; then $\sum_1^\infty u_n$ and $\sum_1^\infty u_n^2$ both converge since $u_n^2 - |u_n^2| < |u_n|$, i.e. $\prod_1^n (1 + u_n)$ tends to a limit P and $\prod_1^n |1 + u_n|$ tends to the limit $|P|$. Thus $\sum_1^\infty |\log(1 + u_n)| \to \log |P|$

Summarizing.—A necessary and sufficient condition for the *absolute* convergence of $\prod_1^\infty (1 + u_n)$ is the *absolute* convergence of $\sum_1^\infty u_n$. Also, since the sum of the series $\sum_1^\infty \log (1 + u_n)$, when absolutely convergent, is independent of the order of the terms, it follows that the value of an absolutely convergent infinite product is unaltered by a derangement of the factors.

11.23. Uniform Convergence of an Infinite Product. The sequence $P_n(x) = \prod_1^n (1 + u_n(x))$ is said to tend *uniformly* to the function $P(x) = \prod_1^\infty (1 + u_n(x))$, if, given ε (> 0), there exists an integer m, independent of x, for which

$$\left| \frac{P_{m+p}(x)}{P_m(x)} - 1 \right| < \varepsilon \quad \text{(all integers } p\text{)}.$$

It follows from this definition that if (i) $\sum_1^\infty M_n$ is a convergent series of positive constants and (ii) $|u_n(x)| < M_n$ for all x in the interval $a < x < b$, then the product converges uniformly in this interval. For the products $\prod_1^\infty (1 \pm M_n)$ converge and therefore an integer m exists for which

$$1 - \varepsilon < \prod_{m+1}^{m+p} (1 - M_n) < \prod_{m+1}^{m+p} (1 + M_n) < 1 + \varepsilon.$$

But $\prod_{m+1}^{m+p} (1 + u_n(x))$ lies between $\prod_{m+1}^{m+p} (1 - M_n)$ and $\prod_{m+1}^{m+p} (1 + M_n)$ and therefore $\left| \frac{P_{m+p}(x)}{P_m(x)} - 1 \right| < \varepsilon$ for all integers p, the choice of m being obviously independent of x.

If, in addition, $u_n(x)$ is continuous in $a < x < b$, the product is also continuous in this interval: for the series $\sum_1^\infty \log (1 + u_n(x))$ is uniformly convergent.

11.24. The Logarithmic Derivative of an Infinite Product. If $P(x) = \prod_1^\infty (1 + u_n(x))$ is continuous, so also is $\log P(x)$ $(1 + u_n(x) > 0)$. This possesses a derivative given by

$$\frac{P'(x)}{P(x)} = \sum_1^\infty \frac{u_n'(x)}{1 + u_n(x)}$$

when the series on the right is a uniformly convergent series of continuous

functions. For the application of this result, we must then assume at least that $u_n(x)$ possesses a continuous derivative.

Examples. (i) $\prod\limits_1^\infty (1 + nx^n)$

The sequence $nx^n \to 0$ only when $|x| < 1$. When $|x| < 1$, $\sum\limits_1^\infty nx^n$ is absolutely convergent.

Therefore $\prod\limits_1^\infty (1 + nx^n)$ is convergent (absolutely and uniformly) in the interval

$$-1 < x_1 \leq x \leq x_2 < 1.$$

(ii) $\prod\limits_1^\infty \left(\dfrac{x^2 + x^{2n}}{1 + x^{2n}}\right) = \prod\limits_1^\infty \left(1 + \dfrac{x^2 - 1}{1 + x^{2n}}\right)$.

When $x = \pm 1$, the product converges to 1. The sequence $(1 + x^{2n})^{-1}$ converges to zero only when $|x| > 1$. Also $\sum\limits_1^\infty (1 + x^{2n})^{-1} < \sum\limits_1^\infty x^{-2n}$, which converges when $|x| > 1$. Thus if c_1 is any number > 1, we can take $M_n = (c_2^2 - 1)/(1 + c_1^{2n})$ where c_2 is any number $> c_1$. Then $0 < u_n(x) < \dfrac{c_2^2 - 1}{1 + c_1^{2n}}$ when $c_1 \leq |x| \leq c_2$.

The product converges *absolutely* when $|x| > 1$, and it converges uniformly for

$$1 < c_1 \leq x \leq c_2.$$

Actually $P_n(x)$ is easily seen to be $\dfrac{2x^{2n}}{1 + x^{2n}}$, from which we deduce that $P_n(x) \to 2$ when $|x| > 1$ and $P_n(x) \to 1$ when $|x| = 1$, whilst $P_n(x)$ diverges to zero when $|x| < 1$.

(iii) To show that $\prod\limits_1^\infty (1 + x^{2n}) \prod\limits_1^\infty (1 + x^{2n-1}) \prod\limits_1^\infty (1 - x^{2n-1}) = 1$ if $|x| < 1$. The products are all absolutely convergent when $|x| < 1$, since $\sum\limits_1^\infty x^{2n-1}$, $\sum\limits_1^\infty x^{2n}$ are absolutely convergent. The factors of the product may be taken in any order; thus

$\prod\limits_1^\infty (1 + x^{2n}) \prod\limits_1^\infty (1 + x^{2n-1}) = \prod\limits_1^\infty (1 + x^n)$; $\prod\limits_1^\infty (1 - x^{2n}) \prod\limits_1^\infty (1 - x^{2n-1}) = \prod\limits_1^\infty (1 - x^n)$

i.e. $\prod\limits_1^\infty (1 + x^{2n}) \prod\limits_1^\infty (1 + x^{2n-1}) \prod\limits_1^\infty (1 - x^{2n}) \prod\limits_1^\infty (1 - x^{2n-1}) = \prod\limits_1^\infty (1 - x^{2n})$.

The result follows since $\prod\limits_1^\infty (1 - x^{2n})$ is not zero.

(iv) The product $(1 + \frac{1}{2})(1 - \frac{1}{3})(1 + \frac{1}{4})(1 - \frac{1}{5}) \ldots$ is convergent (not absolutely) since $\sum\limits_2^\infty (-1)^n \dfrac{1}{n}$ converges (not absolutely) and $\sum\limits_2^\infty \dfrac{1}{n^2}$ is convergent. Its value is $\lim\limits_{n \to \infty} \left(\dfrac{3}{2} \cdot \dfrac{2}{3} \cdots \dfrac{2n}{2n + 1}\right)$ or $\lim\limits_{n \to \infty} \left(\dfrac{3}{2} \cdot \dfrac{2}{3} \cdots \dfrac{2n + 3}{2n + 2}\right) = 1$.

A rearrangement of its factors will in general alter its value. For example, consider $(1 + \frac{1}{2})(1 + \frac{1}{4})(1 - \frac{1}{3})(1 + \frac{1}{6})(1 + \frac{1}{8})(1 - \frac{1}{5}) \ldots$

If the product of n factors is P_n, then

$$\lim_{n \to \infty} P_{3n+1} = \lim_{n \to \infty} P_{3n+2} = \lim_{n \to \infty} P_{3n} = \lim_{n \to \infty} P_n, \text{ if } \lim_{n \to \infty} P_{3n} \text{ exists.}$$

But $$P_{3n} \to \lim_{n \to \infty} \frac{\prod\limits_1^n \left(1 - \dfrac{1}{16n^2}\right)}{\prod\limits_1^n \left(1 - \dfrac{1}{4n^2}\right)} = \frac{\pi}{2 \sin \frac{\pi}{2}} \cdot \frac{4 \sin \frac{\pi}{4}}{\pi} = \sqrt{2} \quad (\S\ 11.34).$$

(v) Let $F(x) = \prod_1^\infty \left(1 + \frac{x}{n}\right)e^{-x/n}$.

$$\left(1 + \frac{x}{n}\right)e^{-x/n} = 1 + \frac{x^2}{2n^2}(1 + \theta), \text{ where } \theta = O\left(\frac{x}{n}\right).$$

If x_1 is any finite number (> 0), we can choose n_0 sufficiently large to ensure that $|\theta| < \frac{1}{2}$ for $n > n_0$.

Then $\frac{x^2}{2n^2}(1 + \theta) < \frac{3x^2}{4n^2}$ for all $|x| \le x_1$ and $n > n_0$. Therefore, since $\sum_1^\infty \frac{1}{n^2}$ is convergent, the infinite product is uniformly and absolutely convergent for all finite x.

The series obtained by differentiation is $\sum_1^\infty \left(\frac{1}{x + n} - \frac{1}{n}\right)$ (x not being a negative integer), and since $nx/(x + n) \to x$ when $n \to \infty$, the sequence $nx/(x + n)$ is bounded throughout the intervals $-1 + k \le x \le x_1$, $-m + k \le x \le l < -m + 1$ (m being a positive integer). Also $nx/(x + n)$ is ultimately of constant sign (that of x). Thus a constant K can be found such that $\left|\frac{x}{n(x + n)}\right| < \frac{K}{n^2}$, showing that the series $\sum_1^\infty \left(\frac{1}{x + n} - \frac{1}{n}\right)$ is uniformly (and absolutely) convergent in these intervals.

We may therefore write $\frac{F'(x)}{F(x)} = \sum_1^\infty \left(\frac{1}{x + n} - \frac{1}{n}\right)$ and similarly obtain the result $\frac{d^2}{dx^2}\{\log |F(x)|\} = -\sum_1^\infty \frac{1}{(x + n)^2}$, for the same intervals.

11.25. Infinite Products of Complex Numbers. When u_n is complex, it will be found sufficient for the cases likely to arise to apply the test of absolute convergence, viz. the convergence of $\sum_1^\infty |u_n|$.

Also, if $\prod_1^\infty (1 + u_n(z))$ has the value $P(z)$ for a given domain of the complex variable z, the convergence is *uniform* if $|u_n(z)| \le M_n$ in that domain, $\sum_1^\infty M_n$ being a convergent series of real positive constants.

Note. It does not follow, of course, that $\sum_1^\infty \log(1 + u_n) = \log P$ when $\prod_1^\infty (1 + u_n) = P$. It is easy to see that it is equal, however, to some definite value of Log P. Let α be the principal value of amp P and α_n the principal value of amp P_n. Then since $P_n/P \to 1$, from and after some definite value of n the principal value of P_n must tend to α; so that if the amplitude of P_n is $2k\pi + \alpha$, $\log P_n \to \log P + 2k\pi$.

11.3. Expansions of Analytic Functions. *Taylor's and Laurent's.* We shall now obtain one or two of the more important expansions of analytic functions. Two of these have already been established—the *Taylor* and the *Laurent* expansions, the latter being inclusive of the former (§§ 10.42, 10.47).

Laurent's Expansion. If a is an isolated singularity of $f(z)$, an analytic function, then

$$f(z) = \ldots \frac{b_n}{(z-a)^n} + \ldots + \frac{b_1}{(z-a)} + a_0 + a_1(z-a) + \ldots + a_n(z-a)^n + \ldots$$

the expansion being valid for all points within the circle $|z-a| = R$ (except at $z = a$), where R is the distance from a to the nearest singularity. The coefficients are given by

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}, \quad b_n = \frac{1}{2\pi i}\int_C (z-a)^{n-1}f(z)dz,$$

C being any circle $|z-a| = R_0\ (< R)$.

When $f(z)$ is analytic at a, the coefficients b_r are zero and we have *Taylor's Expansion* in which we may write $a_n = \dfrac{f^{(n)}(a)}{n!}$.

Example. Obtain the expansion of $e^{xz-y/z}$ in powers of z.

Here $e^{xz-y/z} = \sum_0^\infty a_n z^n + \sum_1^\infty b_n z^{-n}$, where

$$a_n = \frac{1}{2\pi i}\int_C \frac{e^{xz-y/z}dz}{z^{n+1}}, \quad b_n = \frac{1}{2\pi i}\int_C e^{xz-y/z}z^{n-1}dz.$$

Take C to be the circle $|z| = 1$ and write $z = e^{i\theta}$; then

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(x-y)\cos\theta}\cos\{(x+y)\sin\theta - n\theta\}d\theta \quad (x, y \text{ real})$$

since $\int_0^{2\pi} e^{(x-y)\cos\theta}\sin\{(x+y)\sin\theta - n\theta\}d\theta = 0$. (Put $2\pi-\theta$ for θ.)
Similarly (or by putting $-n$ for n), we obtain

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(x-y)\cos\theta}\cos\{(x+y)\sin\theta + n\theta\}d\theta$$

In particular $e^{\frac{1}{2}\xi(t-1/t)} = \sum_{-\infty}^{+\infty} J_n(\xi)t^n$ where $J_n(\xi)$ (*the Bessel Function of integral order n*) is given by

$$J_n(\xi) = \frac{1}{2\pi}\int_0^{2\pi}\cos(\xi\sin\theta - n\theta)d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(\xi\sin\theta - n\theta)d\theta$$

(since $\cos\{\xi\sin(2\pi-\theta) - n(2\pi-\theta)\} = \cos(\xi\sin\theta - n\theta)$.)

Also $$J_{-n}(\xi) = \frac{1}{\pi}\int_0^{\pi}\cos(\xi\sin\theta + n\theta)d\theta = (-1)^n J_n(\xi)$$

(since $\cos\{\xi\sin(\pi-\theta) + n(\pi-\theta)\} = (-1)^n\cos(\xi\sin\theta - n\theta)$.)

But $xz - y/z = \frac{1}{2}\xi(t-1/t)$ if $\xi = 2\sqrt{(xy)}$, $t = z\sqrt{(x/y)}$, with an appropriate choice of square roots,

i.e. $$e^{xz-y/z} = \sum_{-\infty}^{\infty} J_n\{2\sqrt{(xy)}\}(x/y)^{\frac{1}{2}n}z^n$$

and $$\frac{1}{\pi}\int_0^{\pi} e^{(x-y)\cos\theta}\cos\{(x+y)\sin\theta - n\theta\}d\theta = (x/y)^{\frac{1}{2}n}J_n\{2\sqrt{(xy)}\}$$

or $$\int_0^{\pi} e^{\lambda\cos\theta}\cos(\mu\sin\theta - n\theta)d\theta = \pi\left(\frac{\mu+\lambda}{\mu-\lambda}\right)^{\frac{1}{2}n}J_n\{\sqrt{(\mu^2-\lambda^2)}\}.$$

11.31. The Darboux Expansion. Let $f(z)$ be analytic on the straight line joining a to $a+h$ and let $\phi(t)$ be a polynomial of degree n.

Denote the integral $\int_0^1 \phi^{(n-r)}(t) f^{(r+1)}(a+th)dt$ by I_r where $z = a + th$ and t is real.

Integration by parts gives

$$hI_r = \phi^{(n-r)}(1)f^{(r)}(a+h) - \phi^{(n-r)}(0)f^{(r)}(a) - I_{r-1} \; (r = 1 \text{ to } n)$$
$$hI_0 = \phi^{(n)}(0)\{f(a+h) - f(a)\}, \text{ since } \phi^{(n)}(t) = \text{constant} = \phi^{(n)}(0).$$

i.e. $\phi^{(n)}(0)\{f(a+h) - f(a)\}$

$$= \sum_{m=1}^{n} (-1)^{m-1}h^m\{\phi^{(n-m)}(1)f^{(m)}(a+h) - \phi^{(n-m)}(0)f^{(m)}(a)\} + R_n$$

where $$R_n = (-1)^n h^{n+1}\int_0^1 \phi(t)\, f^{(n+1)}(a+th)dt.$$

Example Let $\phi(t) = (t-1)^n$.

Then $\phi(1) = \phi'(1) = \ldots = \phi^{(n-1)}(1) = 0$, $\phi^{(n)}(1) = n!$

$$\phi(0) = (-1)^n;\ \phi'(0) = n(-1)^{n-1};\ \ldots\ldots\ \phi^{r}(0) = \frac{n!}{(n-r)!}(-1)^{n-r}.$$
$$\ldots\ldots\ \phi^{(n)}(0) = n!$$

So that

$$n!\{f(a+h) - f(a)\} = \sum_1^n h^m.\frac{n!}{m!}f^{(m)}(a) + (-1)^n h^{n+1}\int_0^1 (t-1)^n f^{(n+1)}(a+th)dt$$

or

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + \frac{h^{n+1}}{n!}\int_0^1 (1-t)^n f^{(n+1)}(a+th)dt$$

Taylor's Series with a Remainder.

11.32. Lagrange's Expansion. (Rouché's Theorem.) Let $\phi(z)$ be analytic in a region enclosing the point a; then the equation

$$F(z) = z - a - t\phi(z) = 0 \; (\phi(a) \neq 0),$$

may be expected to have a root ζ which is a function of t tending to the value a when t tends to zero. *Lagrange's Expansion* gives an expression for $f(\zeta)$ as a power series in t. To obtain the region of validity of this series, we shall use a lemma known as Rouché's Theorem.

Rouché's Theorem. If $f(z)$, $h(z)$ are both analytic within and on a closed contour C and $|h(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + h(z)$ have the same number of zeros within C.

Since $|f(z)| > |h(z)|$ (which is ≥ 0), it follows that $f(z)$ and $f(z) + h(z)$ cannot vanish on C.

Now let $w(z)$ be any function that is analytic within and on C but does not vanish on C, and consider $\int_C \frac{w'(z)}{w(z)}dz = I$. For every zero of $w(z)$ (of multiplicity k), the integrand has a pole of residue k, and there are no other singularities. Therefore $I = 2\pi i m$ where m is the number of zeros within C, each zero being reckoned according to its multiplicity. But I is also equal to the change in any particular branch of $\log w$ as z describes C, i.e. is equal to $i\theta$ where θ is the increase in amp w when z describes C. Therefore the increase in amp w is equal to $2m\pi$ where m is the number of

zeros of w within C. Conversely, we infer that if the increase of amp w is $2m\pi$, there are m zeros within C.

Let $w(z) = 1 + \rho(z)$; then as z describes C, $w(z)$ describes a closed curve in the w-plane, for which $w = 0$ is an exterior point of $|\rho| < 1$ on C. Thus the increase in amp w is zero.

If the number of zeros of $f(z) + h(z)$ within C is equal to m, then

$$2m\pi = \text{increase in amp } f(z)(1 + \rho(z)) \text{ where } \rho(z) = \frac{h(z)}{f(z)}$$

$$= \text{increase in amp } f(z) \text{ if } |\rho| < 1$$

i.e. the number of zeros of $f(z)$ within C is also m.

Take $f(z) = z - a$ and $h(z) = -t\phi(z)$, then the number of zeros of $F(z) = f(z) + h(z) = z - a - t\phi(z)$ is equal to the number of zeros of $z - a$, i.e. *one*, when C is a closed contour surrounding a and

$$\left|\frac{t\phi(z)}{z - a}\right| < 1 \text{ on } C.$$

Let C be the circle $z = a + \rho e^{i\theta}$, and assume that $|t\phi(z)| < \rho$ on this circle.

Let $g(z)$ be any function that is analytic within and on C. Then

$$\frac{1}{2\pi i}\int_C \frac{g(z)dz}{z - a - t\phi(z)} = \frac{g(\zeta)}{1 - t\phi'(\zeta)}$$

where ζ is the zero of $z - a - t\phi(z)$ within C.

The integral on the left is

$$\frac{1}{2\pi i}\int_C \frac{g(z)}{(z - a)}\left\{1 + \frac{t\phi}{z - a} + \ldots + \frac{t^n\phi^n}{(z - a)^n} + \frac{t^{n+1}\phi^{n+1}}{(z - a)^n(z - a - t\phi)}\right\}dz$$

$$= g(a) + \frac{t}{1!}\frac{d}{da}\{g(a)\phi(a)\} + \frac{t^2}{2!}\frac{d^2}{da^2}\{g(a)\phi^2(a)\} + \ldots$$

$$+ \frac{t^n}{n!}\frac{d^n}{da^n}\{g(a)\phi^n(a)\} + R_n$$

where $$R_n = \frac{t^{n+1}}{2\pi i}\int_C \frac{g(z)\{\phi(z)\}^{n+1}\,dz}{(z - a)^{n+1}(z - a - t\phi(z))}.$$

Let M_1, M_2 be the upper bounds of $|g(z)|$ and $|t\phi(z)|$ on C, respectively.

Then $|R_n| \leq \dfrac{M_1 M_2^{n+1}}{\rho^n(\rho - M_2)}$ which $\to 0$ as $n \to \infty$, since $M_2 < \rho$.

Thus $$\frac{g(\zeta)}{1 - t\phi'(\zeta)} = g(a) + \sum_1^\infty \frac{t^n}{n!}\frac{d^n}{da^n}\{g(a)(\phi(a))^n\}.$$

Now let $g(\zeta) = (1 - t\phi'(\zeta))f(\zeta)$, so that $f(\zeta)$ is also analytic since $1 - t\phi'(\zeta) \neq 0$ (at a simple zero);

then $$f(\zeta) = f(a)(1 - t\phi'(a)) + \sum_1^\infty \frac{t^n}{n!}\frac{d^n}{da^n}\{f(a)(1 - t\phi'(a))(\phi(a))^n\}$$

$$= f(a) + \sum_1^\infty \frac{t^n}{n!}\frac{d^{n-1}}{da^{n-1}}\{f'(a)(\phi(a))^n\}.$$

Suppose, for example, that $\phi(z)$ is a polynomial in z (or an entire function), and let $M(\rho) = \text{Max}\,|\phi(z)|$ on C. Then the formula is valid if $|t| < \rho/M(\rho)$, for then $|t\phi(z)| < \rho$.

Let μ be Max $\dfrac{\rho}{M(\rho)}$ as ρ increases; then the expansion is valid when $|t| < \mu$.

Now $\lim\limits_{\rho \to 0} \dfrac{\rho}{M(\rho)} = 0$ (since $\phi(a) \neq 0$) and $\lim\limits_{\rho \to \infty} \dfrac{\rho}{M(\rho)} = 0$ (except in the trivial case when $\phi(z)$ is linear). Thus $\dfrac{\rho}{M(\rho)}$ must increase to some maximum as ρ increases from zero.

Examples. (i) Let $z = 1 + tz^2$ and find the series for z^a where z is the root that tends to 1 when t tends to zero. The maximum of $|z|^2$ on $|z - 1| = \rho$ is $(1 + \rho)^2$ and the maximum value of $\rho/(1 + \rho)^2$ is $\frac{1}{4}$ (when $\rho = 1$). Thus the radius of convergence of the series in t is $\frac{1}{4}$.

Lagrange's formula gives

$$z^a = a^a + \sum_1^\infty \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}}(aa^{a-1}a^{2n})\ (a = 1)$$

$$= 1 + at + \frac{a(a+3)}{2!}t^2 + \frac{a(a+4)(a+5)}{3!}t^3 + \dots$$

This, of course, is the same as $\left(\dfrac{1 + \sqrt{(1 - 4t)}}{2}\right)^a$; and it should be noted that $z = 0$, the only (possible) singularity of z^a is on the circle $|z - 1| = 1$.

(ii) Let $F(z) = z - a - \frac{1}{2}t(z^2 - 1) = 0$ $(a \neq 1)$

Then $\qquad F'(\zeta) = 1 - t\zeta$ (where ζ is the root that $\to a$ when $t \to 0$)

and $$\frac{1}{1 - t\zeta} = 1 + \sum_1^\infty \frac{t^n}{n!} \frac{d^n}{da^n}\left\{\frac{(a^2 - 1)^n}{2^n}\right\}.$$

But $$t\zeta = 1 - \sqrt{(1 - 2at + t^2)}$$

and therefore $$\frac{1}{\sqrt{(1 - 2at + t^2)}} = 1 + \sum_1^\infty \frac{t^n}{2^n . n!} \frac{d^n}{da^n}(a^2 - 1)^n.$$

The coefficient of t^n in this expansion is called the *Legendre Polynomial* of degree n, and is usually written $P_n(a)$.

The radius of convergence is in this case obviously the distance from 0 to the nearest singularity of the function on the left, i.e. the radius of convergence is $a - \sqrt{(a^2 - 1)}$ $(a > 1)$, 1, $(a \leq 1)$ (a real). This may be verified by calculating Max$\left\{\dfrac{\rho}{M(\rho)}\right\}$ as in the previous example. The results are

(i) $a > 1$; $\quad M(\rho) = \frac{1}{2}(a + \rho - 1)(a + \rho + 1)$, $\mu = a - \sqrt{(a^2 - 1)}$ when $\rho = \sqrt{(a^2 - 1)}$.

(ii) $a < 1$; $\quad M(\rho) = \dfrac{\rho^2 + 1 - a^2}{2\sqrt{(1 - a^2)}}$, $\mu = 1$ when $\rho = \sqrt{(1 - a^2)}$.

11.33. Expansion of Meromorphic Functions in a Series of Rational Functions. (*Mittag-Leffler.*) Let $f(z)$ be analytic at all points in the finite part of the plane except at a number of isolated simple poles a_r $(r = 1, 2, 3, \dots)$ which form a simple sequence. Let the suffixes be arranged so that $|a_r| \leq |a_{r+1}|$. Suppose also that it is possible to choose a simple sequence R_m $(\neq \text{any } |a_r|)$ such that $R_m \to \infty$ as $m \to \infty$.

Let Max $|f(z)|$ on the circle $|z| = R_m$ be a bounded function for all m and as $m \to \infty$; i.e. let $|f(z)| < M$ a finite constant for all m and for all z on these circles.

Let z be a point within the circle $|z| = R_m$, which is not a pole and let $a_1, a_2, \ldots a_k$ be the poles within this circle.

Then
$$\frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)d\zeta}{\zeta - z} = f(z) + \sum_1^k \frac{b_r}{a_r - z}$$

where b_r is the residue of $f(z)$ at a_r and C_m is the circle $|z| = R_m$.

But
$$\frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)d\zeta}{\zeta} + \frac{z}{2\pi i}\int_{C_m} \frac{f(\zeta)d\zeta}{\zeta(\zeta - z)}$$
$$= f(0) + \sum_1^k \frac{b_r}{a_r} + E$$

where $|E| \leq \dfrac{|z|}{2\pi}\displaystyle\int_{C_m} \frac{M\,ds}{R_m(R_m - |z|)} < \frac{M|z|}{R_m - |z|}$ (ds being the element of arc on C_m),

i.e. $E \to 0$ as $m \to \infty$ and therefore $f(z) = f(0) + \sum\limits_1^\infty b_r\left(\dfrac{1}{z - a_r} + \dfrac{1}{a_r}\right)$.

Notes (i) If for all points in the region for which $|z| \leq c$, $R_m - |z|$ has a lower bound (> 0), then m can be chosen so that $\dfrac{M|z|}{R_m - |z|} < \epsilon$ for all z in the region. The convergence of the series is therefore uniform in any finite region if the poles are excluded from this region by means of small circles with the poles as centres.

(ii) If the condition $|f(z)| < M$ be replaced by the condition $|f(z)| < MR_m{}^p$ ($p > 0$) on the circle $|z| = R_m$, we may obtain by similar reasoning the expansion
$$f(z) = f(0) + zf'(0) + \ldots + \frac{z^p f^{(p)}(0)}{p!} + \sum_1^\infty b_r\left(\frac{1}{z - a_r} + \frac{1}{a_r} + \frac{z}{a_r{}^2} + \ldots + \frac{z^p}{a_r{}^{p+1}}\right).$$

Examples (i) Let $f(z) = \operatorname{cosec} z - \dfrac{1}{z}$ ($z \neq 0$); $f(0) = \lim\limits_{z \to 0}\left(\operatorname{cosec} z - \dfrac{1}{z}\right) = 0$.

The poles of $f(z)$ are $\pm n\pi$ ($n = 1, 2, 3, \ldots$) and the residue at $n\pi$ is $(-1)^n$.

Let $R_m = (m + \frac{1}{2})\pi$.

On $|z| = R_m$, $|\sin z|^2 = \frac{1}{2}(\cosh 2y - \cos 2x)$ an even function of x, y. If $x = R_m \cos\theta$, $y = R_m \sin\theta$, it is sufficient to consider $0 \leq \theta \leq \frac{1}{2}\pi$. Now $\dfrac{d}{d\theta}|\sin z|^2 = (x\sinh 2y - y\sin 2x) > 0$ since $\sinh 2y \geq 2y$, $\sin 2x \leq 2x$. Thus $|\sin z|^2$ (and therefore $|\sin z|$) increases steadily as θ increases from 0 to $\frac{1}{2}\pi$. Its least value is unity and therefore $\left|\dfrac{1}{\sin z} - \dfrac{1}{z}\right| < 1 + \dfrac{1}{R_m} \leq 1 + \dfrac{2}{3\pi}$. Therefore
$$\frac{1}{\sin z} - \frac{1}{z} = \sum_{-\infty}^{\infty}{}' (-1)^n\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right).$$

The series is absolutely convergent except at $z = 0, \pm n\pi$, since
$$\left|\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)\right| = \left|-\frac{1}{n\pi}\left(1 + \frac{z}{n\pi} + \frac{z^2}{n\pi(n\pi - z)}\right) + \frac{1}{n\pi}\right| = O\left(\frac{1}{n^2}\right).$$

Grouping the terms, we find

$$\operatorname{cosec} z = \frac{1}{z} + \sum_{1}^{\infty}(-1)^n\left\{\frac{1}{z - n\pi} + \frac{1}{n\pi}\right\} + (-1)^n\left\{\frac{1}{z + n\pi} - \frac{1}{n\pi}\right\}$$

$$= \frac{1}{z} - \frac{2z}{z^2 - \pi^2} + \frac{2z}{z^2 - 4\pi^2} - \frac{2z}{z^2 - 9\pi^2} + \cdots$$

(ii) Let $f(z) = \dfrac{e^{az}}{e^z - 1} - \dfrac{1}{z}$ (a real), $f(0) = \lim\limits_{z \to 0}\left(\dfrac{e^{az}}{e^z - 1} - \dfrac{1}{z}\right) = a - \dfrac{1}{2}$.

The poles are $\pm 2n\pi i$, and the residue at $z = 2n\pi i$ is

$$e^{2an\pi i} = \cos 2an\pi + i \sin 2an\pi.$$

The circles of the theorem may obviously be replaced by a sequence of squares given by $x = \pm R_m$, $y = \pm R_m$.

In this case take $R_m = (2m + 1)\pi$.

On $x = R_m$, $|f(z)| < \dfrac{e^{aR_m}}{e^{R_m} - 1}$, which $\to 0$ as $R_m \to \infty$ if $a < 1$ and $\to 1$ if $a = 1$.

On $x = -R_m$, $|f(z)| < \dfrac{e^{-aR_m}}{1 - e^{-R_m}}$, which $\to 0$ as $R_m \to \infty$ if $a > 0$ and $\to 1$ if $a = 0$.

On $y = \pm R_m$, $|f(z)| = \dfrac{e^{ax}}{e^x + 1}$, and it is a simple exercise to show that this function lies between 0 and 1 for all x if $0 < a < 1$.

Thus
$$\frac{e^{az}}{e^z - 1} = \frac{1}{z} + \left(a - \frac{1}{2}\right) + \sum_{-\infty}^{\infty}(\cos 2an\pi + i \sin 2an\pi)\left(\frac{1}{z - 2n\pi i} + \frac{1}{2n\pi i}\right)$$

$$= \frac{1}{z} + \left(a - \frac{1}{2}\right) + \sum_{1}^{\infty}\left\{\frac{2z \cos 2an\pi - 4n\pi \sin 2an\pi}{z^2 + 4n^2\pi^2} + \frac{\sin 2an\pi}{n\pi}\right\}.$$

Now $\sum\limits_{1}^{\infty}\dfrac{\sin n\theta}{n} = \frac{1}{2}(\pi - \theta)$ $(0 < \theta < 2\pi)$, and 0 $(\theta = 0, 2\pi)$ (§ *11.19* (iii)) so that

$\sum\limits_{1}^{\infty}\dfrac{\sin 2an\pi}{n\pi} = \frac{1}{2} - a$ $(0 < a < 1)$, and 0 $(a = 0, 1)$.

Thus
$$\frac{e^{az}}{e^z - 1} = \frac{1}{z} + \sum_{1}^{\infty}\frac{2z \cos 2an\pi - 4n\pi \sin 2an\pi}{z^2 + 4n^2\pi^2} \quad (0 < a < 1)$$

from f 12

and
$$\frac{e^z}{e^z - 1} = \frac{1}{z} + \frac{1}{2} + \sum_{1}^{\infty}\frac{2z}{z^2 + 4n^2\pi^2}; \quad \frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{1}^{\infty}\frac{2z}{z^2 + 4n^2\pi^2},$$

the last two results being also obvious deductions from each other. Other deductions are

$$\coth z = 1 + \frac{2}{e^{2z} - 1} = \frac{1}{z} + \sum_{1}^{\infty}\frac{2z}{z^2 + n^2\pi^2}; \quad \cot z = i \coth iz = \frac{1}{z} + \sum_{1}^{\infty}\left(\frac{1}{z + n\pi} + \frac{1}{z - n\pi}\right)$$

and $\operatorname{cosec}^2 z = \sum\limits_{-\infty}^{\infty}\dfrac{1}{(z - n\pi)^2}$ (by differentiation).

11.34. Analytic Functions expressed as Infinite Products. Let $f(z)$ be analytic for all finite z and possess simple zeros at $z = a_r$ $(r = 1, 2, 3, \ldots)$ where $|a_r| < |a_{r+1}|$ and $a_r \to \infty$ as $r \to \infty$.

Then $f'(z)/f(z)$ is analytic for all finite z except at the points $z = a_r$ which are simple poles of residue unity.

If the conditions of the theorem given in the previous paragraph are satisfied by $f'(z)/f(z)$, then

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_1^\infty \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right) \quad (a_1 \neq 0).$$

Owing to the uniform convergence of the series in a region that excludes the poles, we may integrate from 0 to z along a simple path that does not pass through a singularity.

Thus $\quad \operatorname{Log} f(z) - \operatorname{Log} f(0) = z\dfrac{f'(0)}{f(0)} + \displaystyle\sum_1^\infty \left(\operatorname{Log}\left(1 - \frac{z}{a_n}\right) + \frac{z}{a_n}\right)$

where the values of the Logarithms depend on the path chosen.

Taking exponentials, we have

$$f(z) = f(0) \exp\left(z\frac{f'(0)}{f(0)}\right) \prod_1^\infty \left\{\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}\right\}$$

and the infinite product is necessarily convergent.

Example. Take $f(z) = \cot z - \dfrac{1}{z}$ which has been shown above to be equal to $\displaystyle\sum_{-\infty}^{\infty}\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)$.

Integration gives $\dfrac{\sin z}{z} = C \displaystyle\prod_{-\infty}^{\infty}\left(1 - \frac{z}{n\pi}\right)e^{\frac{z}{n\pi}}$ where $C = \lim\limits_{z \to 0} \dfrac{\sin z}{z} = 1$. Thus by pairing the terms, we obtain $\sin z = z\displaystyle\prod_1^\infty \left(1 - \frac{z^2}{n^2\pi^2}\right)$. By writing $z\pi$ for z we find also that $\displaystyle\prod_1^\infty\left(1 - \frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi z}$. In particular $\dfrac{2}{\pi} = \dfrac{1.3.3.5.5.7\ldots}{2.2.4.4.6.6\ldots}$ (*Wallis's Formula.*)

11.4. The Convergence of Infinite Integrals (*Real*). An integral is called *infinite* if the integrand $f(x)$ becomes infinite within the range or if the range is infinite. It has been shown that these two cases are not theoretically distinct (*Chap. V*). To determine whether a given integral is convergent or divergent, we should first consider a suitable approximation to $f(x)$ in the neighbourhoods of its infinities or (when the range is infinite) when x is large. The conditions for convergence are simplified when the integrand is of constant sign in a critical part of the range.

11.41. Infinite Integrals with Positive Integrands. The convergence of many integrals occurring in applications may be determined by using the following results obtained in *Chapter V*.

(i) $\displaystyle\int^\infty x^\beta e^{-\alpha x}\,dx$ converges for $\alpha > 0$, all β; $\alpha = 0$, $\beta < -1$; and otherwise diverges.

(ii) $\displaystyle\int^\infty x^\beta (\log x)^\alpha\,dx$ converges for $\beta < -1$, all α; $\beta = -1, \alpha < -1$; and otherwise diverges.

(iii) $\int_0^c x^\beta \left\{\log\left(\frac{1}{x}\right)\right\}^\alpha dx$ converges for $\beta > -1$, all α; $\beta = -1$, $\alpha < -1$; and otherwise diverges

(where to avoid a possible infinity $0 < c < 1$ in the last).

Example. $\int_0^\infty \frac{x^m(\log x)^3\,dx}{x^{2n}+1}$.

At the upper limit, there is convergence or divergence with $\int^\infty x^{m-2n}(\log x)^3\,dx$ which converges only when $2n > m + 1$.

At $x = 0$, the integral converges or diverges with $\int_0 x^m(\log x)^3\,dx$, which converges only if $m > -1$.

Thus the given integral converges only when $2n > m + 1 > 0$.

Notes. (i) The three integrals given above are changed into one another by obvious changes of variable, and the results given are not theoretically distinct.

(ii) If $f(x)$ tends to a *non-zero* limit k when $x \to \infty$, the integral $\int^\infty f(x)dx$ cannot be convergent; for (taking $k > 0$), when x is large $f(x) > k_1$ where k_1 is any fixed number > 0 and $< k$. The integral diverges since $\int^\infty k_1\,dx$ diverges. It is not necessary, however, for convergence that $\lim f(x)$ should exist (even when $f(x)$ is of constant sign); $f(x)$ may oscillate and whilst $f(x)$ (if > 0) must have zero for its lower limit, its upper limit may be infinite.

Example. $\int^\infty \frac{e^x\,dx}{e^{4x}\sin^2 x + \cos^2 x}$.

When x is a multiple of π, the integrand $f(x)$ is e^x and therefore its upper limit is ∞.

However if $n\pi < x < (n+1)\pi$ (n being a positive integer),

$$\frac{e^{n\pi}}{e^{4(n+1)\pi}\sin^2 x + \cos^2 x} < f(x) < \frac{e^{(n+1)\pi}}{e^{4n\pi}\sin^2 x + \cos^2 x}.$$

Also
$$\int_{n\pi}^{(n+1)\pi} \frac{dx}{A\cos^2 x + B\sin^2 x} = \frac{\pi}{\sqrt{(AB)}}\ (A,\ B > 0).$$

Therefore
$$\frac{\pi e^{n\pi}}{e^{2(n+1)\pi}} < \int_{n\pi}^{(n+1)\pi} f(x)dx < \frac{\pi e^{(n+1)\pi}}{e^{2n\pi}}.$$

Thus the integral converges or diverges with the series whose general term is $e^{-n\pi}$ (a geometric series of ratio < 1).

The integral converges. (*See Bromwich, Infinite Series, App 433*.)

11.42. The Absolute Convergence of Infinite Integrals. When $\int_a^\infty |f(x)|dx$ converges, the integral $\int_a^\infty f(x)dx$ must also converge, and the latter integral is then said to be *absolutely* convergent.

Example. $\int^\infty \frac{\sin ax}{x^a}$ is absolutely convergent if $a > 1$, since $\int^\infty \frac{dx}{x^a}$ is convergent and $|\sin ax| < 1$.

11.43. Convergence of Infinite Integrals in General. The best-known tests for integrals when the integrand is not of constant sign near a critical value and when the convergence is not absolute are analogous to

the Abel and Dirichlet Tests for series. To establish them, we use what is appropriately called the *Abel Lemma for integrals.*

11.44. The Abel Lemma for Integrals. If (i) $f(x) \geq 0$ and $f(x)$ steadily decreases as x increases in $a < x < b$, (ii) $f(x)$ is continuous and possesses a derivative in $a < x < b$, (iii) $\phi(x)$ is continuous in $a \leq x \leq b$, (iv) H, h are the upper and lower limits of the integral $\int_a^\xi \phi(x)dx$ as ξ varies from a to b,

then
$$hf(a) < \int_a^b f(x)\phi(x)dx < Hf(a).$$

Let $\psi(x) = \int_a^x \phi(x)dx$; then h, H are finite (since $\phi(x)$ is continuous).

Now $\int_a^b f(x)\phi(x)dx = f(b)\psi(b) - \int_a^b f'(x)\psi(x)dx$ since $f'(x)$ exists.

But $f'(x) < 0$ and $f(b) > 0$; therefore
$$hf(b) - h\int_a^b f'(x)dx < \int_a^b f(x)\phi(x)dx < Hf(b) - H\int_a^b f'(x)dx$$
i.e.
$$hf(a) < \int_a^b f(x)\phi(x) < Hf(a).$$

Notes. (i) The lemma may be established without assuming the existence of $f'(x)$ (*Ref. Hardy, Messenger of Mathematics*, 36 (1906).)

(ii) Since $\psi(x)$ is continuous, there is at least one value c in the interval for which $\psi(x)$ is equal to any given number between h, H inclusive. Therefore
$$\int_a^b f(x)\phi(x)dx = f(a)\int_a^c \phi(x)dx$$
for some c in $a < c < b$. This is known as *Bonnet's Theorem.*

Putting $f(x) - f(b)$ for $f(x)$ in this result, we find
$$\int_a^b f(x)\phi(x)dx = \int_a^b f(b)\phi(x)dx + (f(a) - f(b))\int_a^c \phi(x)dx.$$
$$= f(a)\int_a^c \phi(x)dx + f(b)\int_c^b \phi(x)dx.$$

If $f(x)$ is an increasing monotone, the *same* result is obtained by writing $-f(x)$ for $f(x)$. This relation is known as *Du Bois-Reymond's Theorem* or *The Second Mean Value Theorem* (for Integrals).

(iii) The *First Mean Value Theorem* (*Weierstrass*) for integrals is simply that $\int_a^b f(x)\phi(x)dx = \phi(c)\int_a^b f(x)dx$ when $f(x) > 0$ and follows immediately from the definition of the integral, if $\phi(x)$ is continuous. For the corresponding Weierstrassian Mean Value Theorem in complex integration, reference may be made to *Bieberbach, Funktionentheorie*, V, § 5.

11.45. The Dirichlet Test for Convergence. (*Hardy.*) If (i) $\int_a^\infty \phi(x)dx$ oscillates finitely (or converges).

(ii) $f(x)$ decreases steadily to zero as x increases to infinity, then $\int_a^\infty f(x)\phi(x)dx$ converges.

By the Abel Lemma $\int_{X_1}^{X_2} f(x)\phi(x)dx = Hf(X_1)$ where H is the upper limit of $\left|\int_{X_1}^{x} \phi(x)dx\right|$ as x varies from X_1 to X_2. But $\int_{X_1}^{X_2}\phi(x)dx$ has an upper bound K for all X_1, X_2 such that $a \leq X_1 < X_2$. Also given $\varepsilon\ (>0)$, we can find X_0 such that $|f(x)| < \varepsilon$ for all $x > X_0$. Thus an X_0 exists for which $\left|\int_{X_1}^{X_2} f(x)\phi(x)dx\right| \leq \varepsilon K\ (X_1, X_2 > X_0)$.

$$\text{i.e. } \int_a^\infty f(x)\phi(x)dx \text{ converges.}$$

11.46. The Abel Test for Convergence. (*Hardy.*) If (i) $\int_a^\infty \phi(x)dx$ is convergent,

(ii) $f(x)$ is monotonic and bounded as $x \to \infty$, then $\int_a^\infty f(x)\phi(x)dx$ is convergent.

For $f(x)$, being bounded and monotonic, must tend to a limit l as x tends to infinity. Put $f(x) - l$ for $f(x)$ in the Dirichlet test if $f(x)$ decreases and put $l - f(x)$ for $f(x)$ if $f(x)$ increases. Then

$$\int_a^\infty (f(x) - l)\phi(x)dx$$

must converge and therefore also $\int_a^\infty f(x)\phi(x)dx$ since $l\int_a^\infty \phi(x)dx$ is given to be convergent.

Examples. (i) $\int_1^\infty \frac{\sin x}{x^p}dx$ converges if $0 < p$ by the Dirichlet Test. It converges *absolutely* when $p > 1$.

(ii) $\int_0^c \frac{\sin x}{x^p}dx$ converges absolutely if $p < 2$. $\int_0^c \frac{\cos x}{x^p}dx$ converges absolutely when $p < 1$.

(iii) $\int_c^\infty \frac{dx}{(\sin x)^{1/3}(\log x)^p}\ (c > 1)$.

$$\int_0^\pi \frac{dx}{(\sin x)^{1/3}} = -\int_\pi^{2\pi} \frac{dx}{(\sin x)^{1/3}} = \int_{2\pi}^{3\pi} \frac{dx}{(\sin x)^{1/3}} = \dots$$

Also $\sin^{-1/3} x = x^{-1/3}(1 + O(x^2))$ near $x = 0$ and is of a similar form near $x = \pi, 2\pi, \dots$; the integral $\int_0^\pi \frac{dx}{(\sin x)^{1/3}}$ converges, and the integral $\int_c^\infty \frac{dx}{(\sin x)^{1/3}}$ oscillates finitely.

Also $(\log x)^{-p} \to 0$ monotonically if $p > 0$.

Therefore by the Dirichlet Test, the given integral converges if $p > 0$.

11.5. Uniform Convergence of Infinite Integrals. Consider the infinite integral $\int_a^\infty f(x, \alpha)dx$ which involves a parameter α. If, given $\varepsilon\ (>0)$, we can find X_0 such that $\left|\int_{X_1}^{X_2} f(x, \alpha)dx\right| < \varepsilon$ for all $X_1, X_2 > X_0$,

the integral converges. But if X_0 can be found *independent of* α in the range $\alpha_1 \leq \alpha \leq \alpha_2$, the convergence is said to be *uniform* in $\alpha_1 \leq \alpha \leq \alpha_2$.

Example. Let $I = \int_0^\infty \frac{\alpha\, dx}{\alpha^2 + x^2}$.

If $\alpha > 0$, $I = \lim_{X \to \infty} \left(\arc\tan \frac{X}{\alpha}\right) = \frac{1}{2}\pi$; if $\alpha = 0$, $I = 0$. This is sufficient to show that the convergence is not uniform in $0 \leq \alpha < \alpha_1$. It is, however, uniformly convergent in $0 < \alpha_1 \leq \alpha \leq \alpha_2$, for we can choose X_0 to satisfy the inequality $X_0 > \alpha_2/(\tan \varepsilon)$ $(\varepsilon > 0)$; then

$$|\arc\tan(X_2/\alpha) - \arc\tan(X_1/\alpha)| < \pi/2 - \arc\tan(X_0/\alpha) < \varepsilon$$

for all α in this interval.

Similarly $\int_0^\infty \frac{\beta\, dx}{1 + \beta^2 x^2}$ which is equal to $\frac{1}{2}\pi$, 0 or $-\frac{1}{2}\pi$ according as $\beta >$, $=$ or < 0, is not uniformly convergent in an interval that includes $\beta = 0$.

11.51. The M-Test for the Uniform Convergence of Integrals.

If (i) $\int^\infty M(x)dx$ is convergent, where $M(x) > 0$,

(ii) $|f(x, \alpha)| \leq M(x)$ throughout the interval $\alpha_1 \leq \alpha \leq \alpha_2$, then $\int^\infty f(x, \alpha)dx$ is uniformly (and absolutely) convergent in $\alpha_1 < \alpha < \alpha_2$.

For, given ε, we can find X_0 such that $\int_{X_1}^{X_2} M(x)dx < \varepsilon$ for all $X_1, X_2 > X_0$, and X_0 is obviously independent of α.

Thus, in this interval of α, $\left|\int_{X_1}^{X_2} f(x, \alpha)dx\right| < \int_{X_1}^{X_2} M(x)dx < \varepsilon$. The convergence is therefore uniform (and absolute).

11.52. The Dirichlet Test for Uniform Convergence of Integrals.

If (i) $\int^\infty \phi(x, \alpha)dx$ oscillates between finite limits U, L throughout the interval $\alpha_1 < \alpha < \alpha_2$ (U, L independent of α),

(ii) $f(x, \alpha) > 0$ and tends steadily and *uniformly* to zero in the interval, then $\int^\infty f(x, \alpha)\phi(x, \alpha)dx$ converges uniformly in the interval. For, by the Abel Lemma, $\left|\int_{X_1}^{X_2} f(x, \alpha)\phi(x, \alpha)dx\right| < Hf(X_1, \alpha)$, where H is the greater of $|U|$, $|L|$, and is independent of α. Also, given ε, we can find X_0 (independent of α) such that $f(X_1, \alpha) < \varepsilon$ for $X_1 > X_0$.

Thus $\left|\int_{X_1}^{X_2} f(x, \alpha)\phi(x, \alpha)dx\right| < \varepsilon H$ and the convergence is uniform.

Notes. (i) If $\phi(x, \alpha)$ does not involve α, it is sufficient to state that $\int^\infty \phi(x)dx$ oscillates finitely (or is convergent).

(ii) The theorem remains true (obviously) if $\int^\infty \phi(x, \alpha)dx$ is uniformly convergent in the interval.

11.53. The Abel Test for Uniform Convergence of Integrals.

If (i) $\int^{\infty} \phi(x, \alpha)dx$ is uniformly convergent in $\alpha_1 < \alpha < \alpha_2$,

(ii) $f(x, \alpha) > 0$ and steadily decreases as x tends to infinity for every α in the interval and has an upper bound K for all α in the interval. Then $\int^{\infty} f(x, \alpha)\phi(x, \alpha)dx$ converges uniformly in the interval.

For $\left|\int_{X}^{X_2} f(x, \alpha)\phi(x, \alpha)dx\right| < Hf(X_1, \alpha) \le HK$ where H is the upper limit of $\left|\int_{X_1}^{x} \phi(x, \alpha)dx\right|$ as x varies from X_1 to X_2. But since $\int^{\infty} \phi(x, \alpha)dx$ is uniformly convergent, we can find X_0, independent of α, such that $H < \epsilon$. The number K is also independent of α. The convergence is therefore uniform.

Notes. (i) It is not necessary that, in the Abel Test, the function $f(x, \alpha)$ should tend uniformly to a limit.

(ii) The formula for change of variable in an infinite integral may easily be shown to be unaltered, if the change of variable is appropriate (see *Hardy, Pure Mathematics, VIII*); and so the tests we have given for the convergence (ordinary or uniform) of $\int^{\infty} f(x)dx$ are applicable, with suitable modifications, to integrals in which the integrand becomes infinite at a point of the range: if, for example, $x = a$ were such a point, the transformation $t(x - a) = 1$ transforms the integral $\int_a f(x)dx$ into an integral of the type $\int^{\infty} \phi(t)dt$.

Examples. (i) $\int_0^{\infty} e^{-t}t^{\alpha-1}\,dt$.

$\int_1^{\infty} e^{-t}t^{\alpha-1}\,dt$ converges for all finite α; also since $t^{\alpha-1} \le t^{k-1}$ $(t \ge 1)$, for all $\alpha \le k$ the convergence is uniform for $\alpha \le k$ (any finite number).

Again $\int_0^1 e^{-t}t^{\alpha-1}\,dt$ is uniformly convergent in $0 < \alpha_1 \le \alpha \le k$, since $e^{-t}t^{\alpha-1} < t^{\alpha_1-1}$ when t is small and $\int_0^1 t^{\alpha_1-1}\,dt$ converges.

If α is complex and equal to $u + iv$, the convergence is still uniform if $0 < a \le R(\alpha)$, since $|t^{u+iv-1}| = |t|^{u-1}$.

(ii) $\int_1^{\infty} \frac{\cos \alpha x}{x^p}\,dx$.

By the M-Test, the integral is uniformly convergent for all finite α, whenever $\int_1^{\infty} \frac{dx}{x^p}$ converges, i.e. when $p > 1$.

(iii) $\int_1^{\infty} e^{-\alpha x}\frac{\sin x}{\sqrt{x}}\,dx$.

$\int_1^{\infty} \frac{\sin x}{\sqrt{x}}\,dx$ is convergent by the Dirichlet Test of ordinary convergence. The function $e^{-\alpha x}$ $(\alpha > 0)$ is non-increasing and bounded (< 1) as x tends to infinity. By the Abel Test, the integral converges uniformly in $0 < \alpha < \alpha_1$.

More generally, $\int_a^\infty e^{-\alpha x}\phi(x)dx$ converges uniformly in $0 \leq \alpha \leq \alpha_1$, when $\int_a^\infty \phi(x)dx$ converges.

(iv) $\int_1^\infty \frac{x \sin \alpha x\, dx}{x^2 + a^2}$ (α real).

$\int_1^x \sin \alpha x\, dx = \frac{1}{\alpha}(\cos \alpha - \cos \alpha x)$, which oscillates between $\frac{1}{\alpha}(\cos \alpha - 1)$ and $\frac{1}{\alpha}(\cos \alpha + 1)$ and these are bounded if $\alpha = 0$ is excluded from the interval. Also $x/(x^2 + a^2)$ tends steadily to zero when $x > a$ as $x \to \infty$, and therefore the given integral is uniformly convergent in each of the intervals $\alpha_1 \leq \alpha \leq \alpha_2 < 0$; $0 < \alpha_1 \leq \alpha \leq \alpha_2$.

11.54. Continuity of a Uniformly Convergent Integral.
If (i) $f(x, \alpha)$ is a continuous function of both variables x, α in $\alpha_1 \leq \alpha \leq \alpha_2$ and for $a \leq x$, and (ii) $\int_a^\infty f(x, \alpha)dx$ is uniformly convergent in $\alpha_1 \leq \alpha \leq \alpha_2$, then $\int_a^\infty f(x, \alpha)dx$ is a continuous function of α in $\alpha_1 \leq \alpha \leq \alpha_2$.

Let α, α_0 belong to the interval; then

$$\left|\int_a^\infty f(x, \alpha)dx - \int_a^\infty f(x, \alpha_0)dx\right| \leq \left|\int_a^{x_1}(f(x, \alpha) - f(x, \alpha_0))dx\right| + \left|\int_{x_1}^\infty f(x, \alpha)dx\right| + \left|\int_{x_1}^\infty f(x, \alpha_0)dx\right|$$

Given ε, we can find x_0 such that $\left|\int_{x_1}^\infty f(x, \alpha)dx\right| < \varepsilon$ for all $x_1 > x_0$ and for all α in $\alpha_1 \leq \alpha \leq \alpha_2$ (because the integral is uniformly convergent). Now a continuous function of two variables is *uniformly* continuous and therefore we can find δ (> 0) such that $|f(x, \alpha) - f(x, \alpha_0)| < \varepsilon$ for all α that satisfy $|\alpha - \alpha_0| < \delta$, where x is any number in $a \leq x \leq x_1$ (x_1 having being fixed and is independent of α).

Thus $\left|\int_a^\infty f(x, \alpha)dx - \int_a^\infty f(x, \alpha_0)dx\right| < \varepsilon(x_1 - a) + 2\varepsilon$ for all α in $|\alpha - \alpha_0| < \delta$, i.e. the integral is a continuous function of α.

Example. $\int_0^\infty e^{-\alpha x}\frac{\sin x}{x}dx$.

Since $\int_0^\infty \frac{\sin x}{x}dx$ converges, the given integral converges uniformly, by Abel's Test, in the interval $0 \leq \alpha \leq \alpha_1$; for $e^{-\alpha x}$ is a non-increasing monotone for every x in $0 \leq \alpha \leq \alpha_1$ and is bounded for all x.

Therefore $\lim_{\alpha \to 0}\int_0^\infty e^{-\alpha x}\frac{\sin x}{x}dx = \int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$ (§ *10·832*).

More generally, $\lim_{\alpha \to 0}\int_0^\infty e^{-\alpha x}\phi(x)dx = \int_0^\infty \phi(x)dx$ when the integral on the right

converges. This example of the use of the continuity theorem resembles Abel's Theorem on the continuity of power series.

Note. If $\beta = 1/a$ and $\beta = 0$ belongs to the interval of uniform convergence of the transformed integral we may say that $\alpha = \infty$ belongs to the interval of uniform convergence of the original integral.

11.55. Repeated Integrals. I. *Finite Intervals.* If $f(x, \alpha)$ is a continuous function of both variables in $\alpha_1 < \alpha < \alpha_2$, $a < x < b$, then

$$\int_a^b\left\{\int_{\alpha_1}^{\alpha_2} f(x, \alpha)d\alpha\right\}dx = \int_{\alpha_1}^{\alpha_2}\left\{\int_a^b f(x, \alpha)dx\right\}d\alpha$$

since each repeated integral is equal to the double integral

$$\iint f(x, \alpha)dx\, d\alpha$$

over the rectangle determined by the intervals.

The repeated integrals still exist and are equal when $f(x, \alpha)$ is continuous in the rectangle except at a finite number of points or along a finite number of simple curves, provided $f(x, \alpha)$ is *bounded*.

II. *One Interval Infinite.* If (i) $f(x, \alpha)$ is a continuous function of both variables x, α in $\alpha_1 \le \alpha \le \alpha_2$ and for $a \le x$,

(ii) $\int_a^\infty f(x, \alpha)dx$ is uniformly convergent in $\alpha_1 \le \alpha \le \alpha_2$,

then $\int_{c_1}^{c_2}\left\{\int_a^\infty f(x, \alpha)dx\right\}d\alpha = \int_a^\infty\left\{\int_{c_1}^{c_2} f(x, \alpha)d\alpha\right\}dx$ $(\alpha_1 < c_1 < c_2 < \alpha_2)$.

Given ε, we can find x_0 independent of α such that $\left|\int_{x_1}^\infty f(x, \alpha)dx\right| < \varepsilon$ for all $x_1 > x_0$; and therefore also $\left|\int_{x_1}^{x_2} f(x, \alpha)dx\right| < \varepsilon$ $(x_1, x_2 > x_0)$.

Now $\int_{x_1}^{x_2}\left\{\int_{c_1}^{c_2} f(x, \alpha)d\alpha\right\}dx = \int_{c_1}^{c_2}\left\{\int_{x_1}^{x_2} f(x, \alpha)dx\right\}d\alpha$ (by I above) and therefore $\left|\int_{x_1}^{x_2}\left\{\int_{c_1}^{c_2} f(x, \alpha)d\alpha\right\}dx\right|$ is less than $\varepsilon(c_2 - c_1)$ however large x_2 may be, i.e. $\int_{x_1}^\infty\left\{\int_{c_1}^{c_2} f(x, \alpha)d\alpha\right\}dx$ exists and its modulus is less than $\varepsilon(c_2 - c_1)$.

Also
$$\left|\int_{c_1}^{c_2}\left\{\int_{x_1}^\infty f(x, \alpha)dx\right\}d\alpha\right| < \varepsilon(c_2 - c_1).$$

Thus
$$\left|\int_{c_1}^{c_2}\left\{\int_a^\infty f(x, \alpha)dx\right\}d\alpha - \int_a^\infty\left\{\int_{c_1}^{c_2} f(x, \alpha)d\alpha\right\}dx\right.$$
$$= \left|\int_{c_1}^{c_2}\left\{\int_{x_1}^\infty f(x, \alpha)dx\right\}d\alpha - \int_{x_1}^\infty\left\{\int_{c_1}^{c_2} f(x, \alpha)d\alpha\right\}dx\right| < 2\varepsilon(c_2 - c_1).$$

The required result follows.

Example $\int_0^\infty \frac{\cos \alpha x}{x^2 + 1}dx$ is uniformly convergent in any interval of α by the M Test.

By contour integration its value is $\frac{1}{2}\pi e^{-|\alpha|}$.

Integration from 0 to α (> 0) gives $\int_0^\infty \frac{\sin \alpha x}{x(1 + x^2)}dx = \frac{1}{2}\pi(1 - e^{-\alpha})$ ($\alpha > 0$) and similarly $\int_0^\infty \frac{\sin \alpha x\, dx}{x(1 + x^2)}dx = \frac{\pi}{2}(e^{\alpha} - 1)$ ($\alpha < 0$). The case $\alpha < 0$ of course follows obviously from the case $\alpha > 0$.

Note. The result still holds when $\int_a^\infty f(x, \alpha)dx$ ceases to be uniformly convergent in the neighbourhood of a finite number of points, provided the integral is bounded (i.e. is boundedly convergent).

III. *Both Intervals Infinite.* We have already determined conditions under which

$$\int_{\alpha_1}^{\alpha_2}\left\{\int_a^\infty f(x, \alpha)dx\right\}d\alpha = \int_a^\infty\left\{\int_{\alpha_1}^{\alpha_2} f(x, \alpha)d\alpha\right\}dx.$$

This result may sometimes be extended to the case when $\alpha_2 \to \infty$. Thus

1. If (i) $f(x, \alpha)$ is a continuous function of both variables in $\alpha_1 \leq \alpha \leq \alpha_2$, $a \leq x \leq b$, where α_2, b may be as large as we please ;

(ii) $\int_a^\infty f(x, \alpha)dx$ is uniformly convergent in $\alpha_1 \leq \alpha \leq \alpha_2$;

(iii) $\int_{\alpha_1}^\infty f(x, \alpha)d\alpha$ is uniformly convergent in $a \leq x \leq b$;

(iv) $\int_a^\infty\left\{\int_{\alpha_1}^{\alpha} f(x, \alpha)d\alpha\right\}dx$ converges uniformly for all $\alpha > \alpha_1$ including $\alpha = \infty$, then $\int_{\alpha_1}^\infty\left\{\int_a^\infty f(x, \alpha)dx\right\}d\alpha = \int_a^\infty\left\{\int_{\alpha_1}^\infty f(x, \alpha)dx\right\}dx$. (*Ref. Gibson, Calculus, 182*).

For $\int_{\alpha_1}^\infty\left\{\int_a^\infty f(x, \alpha)dx\right\}d\alpha = \lim_{\alpha_2\to\infty}\int_{\alpha_1}^{\alpha_2}\left\{\int_a^\infty f(x, \alpha)dx\right\}d\alpha$

$$= \lim_{\alpha_2\to\infty}\int_a^\infty\left\{\int_{\alpha_1}^{\alpha_2} f(x, \alpha)d\alpha\right\}dx \text{ (using (i), ii))}$$

$$= \int_a^\infty \lim_{\alpha_2\to\infty}\left\{\int_{\alpha_1}^{\alpha_2} f(x, \alpha)d\alpha\right\}dx \text{ (using (iii), (iv))} = \int_a^\infty\left\{\int_{\alpha_1}^\infty f(x, \alpha)d\alpha\right\}dx.$$

2. Let (i) $f(x, \alpha) > 0$.

(ii) $\int_a^b\left\{\int_{\alpha_1}^\infty f(x, \alpha)d\alpha\right\}dx = \int_{\alpha_1}^\infty\left\{\int_a^b f(x, \alpha)dx\right\}d\alpha$ for all b, however large.

(iii) $\int_a^\infty\left\{\int_{\alpha_1}^{\alpha_2} f(x, \alpha)d\alpha\right\}dx = \int_{\alpha_1}^{\alpha_2}\left\{\int_a^\infty f(x, \alpha)dx\right\}d\alpha$ for all α_2, however large.

Then $(I_1 =)\int_a^\infty\left\{\int_{\alpha_1}^\infty f(x, \alpha)d\alpha\right\}dx = \int_{\alpha_1}^\infty\left\{\int_a^\infty f(x, \alpha)dx\right\}d\alpha (= I_2)$ if either I_1 or I_2 converges. (*Ref. Titchmarsh, Theory of Functions. 1.85.*)

Suppose that I_1 exists.

Since $f(x, \alpha) \geq 0$, then $\int_{\alpha_1}^{\alpha_2} f(x, \alpha)d\alpha < \int_{\alpha_1}^{\infty} f(x, \alpha)d\alpha$ and therefore $\int_{\alpha_1}^{\alpha_2}\left\{\int_a^\infty f(x, \alpha)dx\right\}d\alpha$ which equals $\int_a^\infty\left\{\int_{\alpha_1}^{\alpha_2} f(x, \alpha)d\alpha\right\}dx$ (by (iii)) is

$$< \int_a^\infty\left\{\int_{\alpha_1}^{\infty} f(x, \alpha)d\alpha\right\}dx, \text{ i.e. } \leq I_1.$$

Thus I_2 exists and is $\leq I_1$. Similarly $I_1 \leq I_2$, i.e. $I_1 = I_2$.

Examples. (i) Let $f(x, \alpha) = x^{p-1}\alpha^{p+q-1}e^{-(1+x)\alpha}$.
Here $f(x, \alpha) > 0$ for $x > 0$, $\alpha > 0$.

Denote $\int_0^\infty e^{-t}t^{p-1}dt$ by $\Gamma(p)$; then the integral for $\Gamma(p)$ is uniformly convergent in $0 < p_1 \leq p \leq p_2$ where p_1 may be as small as we please and p_2 as large as we please.

$$\int_0^\infty f(x, \alpha)dx = \alpha^{p+q-1}e^{-\alpha}\int_0^\infty x^{p-1}e^{-\alpha x}dx = \Gamma(p)\alpha^{q-1}e^{-\alpha} \text{ if } \alpha > 0 \text{ and } p > 0,$$

and integration with respect to α of this integral is legitimate (by uniform convergence) if $q > 0$ and the interval of α is $0 < \alpha_1 < \alpha < \alpha_2$.

Similarly $\int_0^\infty f(x, \alpha)d\alpha = \frac{x^{p-1}}{(1+x)^{p+q}}\Gamma(p+q)$ when $0 < a \leq x \leq b$. Thus the conditions (ii), (iii) of Theorem 2, above, are satisfied by uniform convergence for the intervals $0 < a < x \leq b$ and $0 < \alpha_1 < \alpha < \alpha_2$ respectively.

But $\int_0^\infty\left\{\int_0^\infty f(x, \alpha)dx\right\}d\alpha$ is equal to $\Gamma(p)\Gamma(q)$ $(p, q > 0)$ and therefore

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)\int_0^\infty \frac{x^{p-1}dx}{(1+x)^{p+q}} \quad (p, q > 0).$$ → Say $x<s$. what then?

By writing $\frac{x}{1-x}$ for x we find $\int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ $(p, q > 0)$.

(ii) Let $f(x, \alpha) = \frac{1}{x}e^{-b\alpha}\sin ax\cos\alpha x$ $(a > 0,\ b > 0)$.

$$F(\alpha) = \int_0^\infty \frac{e^{-b\alpha}\sin ax\cos\alpha x}{x}dx = \frac{e^{-b\alpha}}{2}\int_0^\infty \frac{\sin(a+\alpha)x + \sin(a-\alpha)x}{x}dx$$

$$= \frac{\pi}{2}e^{-b\alpha}\ (\alpha < a);\ \frac{\pi}{4}e^{-b\alpha}\ (\alpha = a),\ 0\ (\alpha > a)$$

the integral being uniformly convergent except near $\alpha = a$, where it is boundedly convergent.

Thus $\int_0^\infty F(\alpha)d\alpha$ exists and is equal to $\int_0^a \frac{\pi}{2}e^{-b\alpha}d\alpha = \frac{\pi}{2b}(1 - e^{-ab})$.

Again $\int_0^\infty\left\{\int_0^\alpha f(x, \alpha)d\alpha\right\}dx = \int_0^\infty\left\{\frac{e^{-b\alpha}(x\sin\alpha x - b\cos\alpha x) + b}{x^2 + b^2}\right\}\frac{\sin ax}{x}dx$

But $\frac{e^{-b\alpha}}{(x^2+b^2)}(x\sin\alpha x - b\cos\alpha x) \to 0$ uniformly when $\alpha \to \infty$, all x, and

therefore $\int_0^\infty \left\{ \int_a^\alpha f(x, \alpha) d\alpha \right\} dx$ is uniformly convergent since $\int_0^\infty \frac{\sin ax}{x(x^2 + b^2)} dx$ converges,

i.e.
$$\int_0^\infty \frac{\sin ax \, dx}{x(x^2 + b^2)} = \frac{\pi}{2b^2} (1 - e^{-ab}).$$

Also $\int_0^\infty \frac{x \sin ax}{x^2 + b^2} dx = \int_0^\infty \left\{ \frac{\sin ax}{x} - \frac{b^2 \sin ax}{x(x^2 + b^2)} \right\} dx = \frac{\pi}{2} e^{-ab}$ where b may now be zero.

Notes. (i) Much of the difficulty of expressing in simple form the conditions under which the inversion of a double limit (involving integration) is removed when Lebesgue Integrals are used. It may be shown, for example, that

(i) if $|S(x, \alpha)| < K$ (constant) in $a < x < b$ (all α), then
$$\lim_{\alpha \to \infty} \int_a^b S(x, \alpha) dx = \int_a^b \lim_{\alpha \to \infty} S(x, \alpha) dx.$$

(ii) if $|S(x, \alpha)| < \phi(x)$ (all α), then
$$\lim_{\alpha \to \infty} \int^\infty S(x, \alpha) dx = \int^\infty \lim_{\alpha \to \infty} S(x, \alpha) dx \text{ if } \int^\infty \phi(x) dx \text{ converges.}$$

In particular $\int^\infty \{ \int^\infty S(x, \alpha) dx \} d\alpha = \int^\infty \{ \int^\infty S(x, \alpha) d\alpha \} dx$

(ii) In Theorem 2, the result will hold for any function $f(x, \alpha)$ if one of the repeated (doubly infinite) integrals is absolutely convergent. For the convergence, say, of $\int_a^\infty \left\{ \int_a^\infty |f(x, \alpha)| d\alpha \right\} dx$ implies the convergence of $\int_a^\infty \left\{ \int_a^\infty F(x, \alpha) d\alpha \right\} dx$ and $\int_a^\infty \left\{ \int_a^\infty G(x, \alpha) d\alpha \right\} dx$ where F, G are the *positive* functions determined by $F + G = |f|$, $F - G = f$. The theorem is then applicable to F, G and so the result is true for $F - G$ $(= f)$. If $f(x, \alpha)$ is complex and equal to $u + iv$, then the absolute convergence implies absolute convergence for the functions u, v. The theorem is true then for u, v and therefore for $u + iv$.

11.56. Differentiation of Infinite Integrals. If (i) $f_\alpha(x, \alpha)$ is a continuous function of x, α in $\alpha_1 < \alpha < \alpha_2$ and for all $x > a$,

(ii) $\int_a^\infty f_\alpha(x, \alpha) dx$ is uniformly convergent in $\alpha_1 < \alpha < \alpha_2$,

(iii) $\int_a^\infty f(x, \alpha) dx$ converges, then
$$\frac{d}{d\alpha} \left\{ \int_a^\infty f(x, \alpha) dx \right\} = \int_a^\infty f_\alpha(x, \alpha) dx$$
in this interval.

Denote $\int_a^\infty f_\alpha(x, \alpha) dx$ by $F(\alpha)$ and $\int_a^\infty f(x, \alpha) dx$ by $G(\alpha)$. Then
$$\int_{c_1}^{c_2} F(\alpha) d\alpha = \int_a^\infty \left\{ f(x, c_2) - f(x, c_1) \right\} dx \quad (\alpha_1 < c_1 < c_2 < \alpha_2)$$
because of the uniform convergence in the interval
$$= G(c_2) - G(c_1) \text{ since } G(\alpha) \text{ converges.}$$

Thus
$$\frac{dG}{d\alpha} = F(\alpha) \text{ or } \frac{d}{d\alpha} \int_a^\infty f(x, \alpha) dx = \int_a^\infty f_\alpha(x, \alpha) dx.$$

Examples. (i) $\displaystyle\int_0^\infty e^{-\alpha x}\sin x\,dx = -\left\{\frac{e^{-\alpha x}(\cos x + \alpha\sin x)}{1+\alpha^2}\right\}_0^\infty = \frac{1}{1+\alpha^2}$

$(\alpha \geq \alpha_1 > 0).$

The convergence is uniform in $0 < \alpha_1 \leq \alpha < \alpha_2$ since $\dfrac{\cos x + \alpha\sin x}{1+\alpha^2}$ is bounded and $e^{-\alpha x}$ decreases steadily to zero. Also $\alpha = \infty$ belongs to the interval of uniform convergence.

Integration gives $\displaystyle\int_0^\infty \frac{e^{-\alpha_1 x} - e^{-\alpha_2 x}}{x}\sin x\,dx = \text{arc tan } \alpha_2 - \text{arc tan } \alpha_1\ (\alpha_1, \alpha_2 > 0)$

Let $\alpha_2 \to \infty$, then $\displaystyle\int_0^\infty e^{-\alpha_1 x}\frac{\sin x}{x}\,dx = \pi/2 - \text{arc tan } \alpha_1\ (\alpha_1 > 0).$

But the integral on the left is uniformly convergent for $\alpha_1 \geq 0$ by Abel's Test. Therefore (putting $\alpha_1 = 0$)

$$\int_0^\infty \frac{\sin x}{x}\,dx = \tfrac{1}{2}\pi.$$

Ex. (ii) Let $I_p = \displaystyle\int_0^\infty e^{-x^2-\lambda/x^2}x^p\,dx.$

The integral cannot converge (at the lower limit) if $\lambda < 0$. Let

$$J = \int_0^1 e^{-x^2-\lambda/x^2}x^p\,dx \text{ and } K = \int_1^\infty e^{-x^2-\lambda/x^2}x^p\,dx.$$

$J < \displaystyle\int_0^1 e^{-x^2}x^p\,dx$ which converges if $p > -1$, and the convergence is uniform for $0 \leq \lambda$ if $p > -1$, since $e^{-\lambda/x^2} \leq 1$.

Also $e^{-x^2-\lambda/x^2} < e^{-\lambda_0/x^2}$ when $\lambda > \lambda_0 > 0$ and therefore J is uniformly convergent in $\lambda > \lambda_0 > 0$ for all p, since $\displaystyle\int_0^1 e^{-\lambda_0/x^2}x^p\,dx$ converges. Also since $e^{-x^2-\lambda/x^2}x^p < e^{-x^2}x^p$, K is uniformly convergent for $\lambda \geq 0$, all p.

$I_0 = \displaystyle\int_0^\infty e^{-x^2-\lambda/x^2}\,dx$ converges uniformly for $\lambda \geq 0$ and the integral obtained by differentiating with respect to λ, viz. $-I_{-2}$ is uniformly convergent for $\lambda \geq \lambda_0 > 0$.

Therefore
$$\begin{aligned}\frac{dI_0}{d\lambda} &= -\int_0^\infty e^{-x^2-\lambda/x^2}x^{-2}\,dx\ (\lambda \geq \lambda_0 > 0)\\ &= -\int_0^\infty e^{-1/y^2-\lambda y^2}\,dy\ (y = 1/x)\\ &= -\int_0^\infty e^{-u^2-\lambda/u^2}\frac{du}{\sqrt{\lambda}}\ (u = y\sqrt{\lambda}) = -\frac{1}{\sqrt{\lambda}}I_0\end{aligned}$$

i.e. $I_0 = Ce^{-2\sqrt{\lambda}}$; but I_0 converges uniformly for $\lambda \geq 0$, and therefore $C = \displaystyle\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}$ (*Chapter XII, § 12.24*).

Thus $I_0 = \dfrac{\sqrt{\pi}}{2}e^{-2\sqrt{\lambda}}$ and $I_{-2} = \displaystyle\int_0^\infty e^{-x^2-\lambda/x^2}\frac{dx}{x^2} = \frac{1}{\sqrt{\lambda}}I_0 = \frac{\sqrt{\pi}}{2\sqrt{\lambda}}e^{-2\sqrt{\lambda}} (\lambda > 0).$

Also $I_{-2p} = \lambda^{-p-\frac{1}{2}} I_{2p} = (-1)^p \dfrac{d^p I_0}{d\lambda^p}$; and in particular

$$I_2 = \lambda^{3/2} \frac{\sqrt{\pi}}{2} \frac{d^2}{d\lambda^2}\left(e^{-2\sqrt{\lambda}}\right) = \frac{\sqrt{\pi}}{4} e^{-2\sqrt{\lambda}} (1 + 2\sqrt{\lambda}) \quad (\lambda > 0)$$

Putting $\lambda = a^2 b^2$ and writing ax for x we find

$$\int_0^\infty e^{-a^2x^2 - b^2/x^2} dx = \frac{\sqrt{\pi}}{2a} e^{-2ab} \quad (a > 0,\ b > 0).$$

Ex. (iii) *Frullani's Integrals.* Let (i) $\phi(u) \to A$ when $u \to \infty$ and $\phi(u) \to B$ when $u \to 0$,

(ii) $\phi(u)$ be continuous (all $u > 0$), and possess a derivative;

then
$$\int_0^\infty \frac{\phi(bx) - \phi(ax)}{x} dx = (A - B) \log \frac{b}{a} \quad (b > a > 0).$$

$\displaystyle\int_{x_1}^{x_2} \phi'(\lambda x) dx = \frac{1}{\lambda} \{\phi(\lambda x_2) - \phi(\lambda x_1)\}$ and therefore the integral $\int^\infty \phi'(\lambda x)dx$ is uniformly convergent for $\lambda \geq a$, and $\int_0 \phi'(\lambda x) dx$ is uniformly convergent for $b \geq \lambda \geq a$.

Therefore
$$\int_0^\infty \left\{\int_a^b \phi'(\lambda x) d\lambda\right\} dx = \int_a^b \left\{\int_0^\infty \phi'(\lambda x) dx\right\} d\lambda.$$

But $\phi'(\lambda x) = \dfrac{1}{x}\dfrac{\partial \phi(u)}{\partial \lambda} = \dfrac{1}{\lambda}\dfrac{\partial \phi(u)}{\partial x}$ where $u = \lambda x$.

Thus
$$\int_0^\infty \frac{\phi(bx) - \phi(ax)}{x} dx = \int_a^b \frac{A - B}{\lambda} d\lambda = (A - B) \log \frac{b}{a}.$$

For example,
$$\int_0^\infty \frac{e^{-bx} - e^{-ax}}{x} dx = \log \frac{a}{b}.$$

(iv) Evaluate $\displaystyle\int_0^\infty \frac{1}{x^2}\{e^{-ax}(1 + (a + c)x) - e^{-bx}(1 + (b + c)x)\} dx$.

The integral converges at $x = \infty$, if $a, b > 0$. Near $x = 0$, the integrand is $O(1)$ and the integral converges.

Integration by parts gives $J + K$ where

$$J = \left\{-\frac{e^{-ax}(1 + (a + c)x) - e^{-bx}(1 + (b + c)x)}{x}\right\}_0^\infty = 0,$$

$$K = \int_0^\infty \left\{\frac{c}{x}(e^{-ax} - e^{-bx}) - a(a + c)e^{-ax} + b(b + c)e^{-bx}\right\} dx = c \log \frac{b}{a} + b - a.$$

For example,
$$\int_0^\infty \left\{\frac{e^{-x} - e^{-2x}}{x^2} - \frac{e^{-x}}{x}\right\} dx = 1 - 2 \log 2.$$

11.57. Tannery's Theorem for Integrals. If (i) $f(x, n) \to g(x)$ uniformly in any fixed interval of x when $n \to \infty$,

(ii) $|f(x, n)| < M(x)$, all n (so that also $|g(x)| < M(x)$),

(iii) $\displaystyle\int_a^\infty M(x) dx$ converges,

(iv) $\displaystyle\lim_{n \to \infty} p(n) = \infty$,

then $\displaystyle\lim_{n \to \infty} \int_a^{p(n)} f(x, n) dx = \int_a^\infty g(x) dx$. (*Bromwich, Infinite Series*, § *174*.)

Given ε, we can choose x_0 so that $\int_{x_1}^{\infty} M(x)dx < \varepsilon$ for all $x_1 > x_0$, and n can be chosen sufficiently large to ensure that $p(n) > x_0$. In the interval $a < x < x_0$, n_0 can be chosen sufficiently large to ensure that $|f(x, n) - g(x)| < \varepsilon/(x_0 - a)$ for all $n > n_0$ (and n_0 is independent of x in the interval).

Therefore $\int_a^{x_0} |f - g|dx < \varepsilon$ for all $n > n_0$.

Also $\left|\int_{x_0}^{\infty} f(x, n)dx\right| < \int_{x_0}^{\infty} M(x)dx < \varepsilon$ and $\left|\int_{x_0}^{\infty} g(x)dx\right| < \varepsilon$, similarly.

Thus $$\left|\int_a^{p(n)} f(x, n)dx - \int_a^{\infty} g(x)dx\right| < \int_a^{x_0} |f - g|dx + \int_{x_0}^{\infty} |f|dx + \int_{x_0}^{\infty} |g|dx$$
$$< 3\varepsilon$$

i.e. $$\lim_{n\to\infty} \int_a^{p(n)} f(x, n)dx = \int_a^{\infty} g(x)dx.$$

Example. Prove $\lim\limits_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{\alpha}\, dx = \int_0^{\infty} e^{-x} x^{\alpha}\, dx$ ($\mathbf{R}(\alpha) > \rho_0 > -1$).

$$\frac{d}{dt}\left\{e^t\left(1 - \frac{t}{n}\right)^n\right\} = -\frac{te^t}{n}\left(1 - \frac{t}{n}\right)^{n-1}$$

and therefore as t increases from 0 to x ($< n$), we have

$$0 < -\frac{d}{dt}\left\{e^t\left(1 - \frac{t}{n}\right)^n\right\} < \frac{t}{n}e^x, \text{ i.e. } 0 < 1 - e^x\left(1 - \frac{x}{n}\right)^n < \frac{x^2}{2n}e^x.$$

This gives $0 < e^{-x} - \left(1 - \frac{x}{n}\right)^n < \frac{x^2}{2n}$; but $\frac{x^2}{2n}$ tends uniformly to zero as $n \to \infty$ in any fixed interval of x, and therefore $\left(1 - \frac{x}{n}\right)^n$ tends uniformly to e^{-x} in such an interval.

Again $0 < \left|\left(1 - \frac{x}{n}\right)^n x^{\alpha}\right| < e^{-x}x^{\rho}$ where $\rho = \mathbf{R}(\alpha)$ $(0 < x < n)$, and $\int_0^{\infty} e^{-x}x^{\rho}\, dx$ converges if $\rho > -1$.

The conditions of Tannery's Theorem are satisfied and the result follows.

Note. Tannery's Theorem, of which the above is an analogue, refers to series (and products). (*Ref. Bromwich, Infinite Series*, § 49.)

11.58. Integration of Series when Infinite Integrals are involved. The result

$$\int_a^b \left\{\sum_1^{\infty} u_n(x)\right\}dx = \sum_1^{\infty} \int_a^b u_n(x)dx$$

which has been proved under certain conditions of uniform convergence and continuity may no longer be true when

(i) there is an infinity of $u_n(x)$ or $\Sigma u_n(x)$ within the interval $a < x < b$,
(ii) the interval is infinite,
(iii) the series ceases to be uniformly convergent at one or more points.

If, however, (i) $\int_a^c \{\Sigma u_n(x)\}dx = \Sigma \int_a^c u_n(x)dx$ for all $c < b$,

(ii) $u_n(x) > 0$ (all x, n), then

$$\int_a^b \{\Sigma u_n(x)\}dx (= I_1) = \Sigma \int_a^b u_n(x)dx (= I_2)$$

if either I_1 or I_2 converges.

Suppose, for example, that I_1 converges to the value S_1; then $\Sigma \int_a^c u_n(x)dx = \int_a^c \{\Sigma u_n(x)\}dx < S_1$ for all c.

Therefore since $u_n > 0$, I_2 exists and has a value $S_2 < S_1$. By similar reasoning $S_1 < S_2$. Therefore $S_1 = S_2$.

The case of the infinite interval is obtained by putting ∞ for b (or $-\infty$ for a).

When there is more than one point of discontinuity within an interval, the interval may be subdivided so as to bring the point of discontinuity to the end point of a sub-interval.

Examples. (i) $\displaystyle\int_0^1 \frac{\log\left(\frac{1}{x}\right)}{2 - x}dx.$

$\dfrac{\log\left(\frac{1}{x}\right)}{2 - x} = \log\left(\dfrac{1}{x}\right)\left\{\dfrac{1}{2} + \dfrac{x}{2^2} + \dfrac{x^2}{2^3} + \dots\right\}$, the series within the bracket being uniformly convergent for $|x| < 2$ and so for the interval $(0, 1)$. Since $x = 0$ is the only discontinuity within the interval, we have

$$\int_\epsilon^1 \log\left(\frac{1}{x}\right)\left\{\frac{1}{2} + \frac{x}{2^2} + \dots\right\}dx = \int_\epsilon^1 \frac{\log\left(\frac{1}{x}\right)dx}{2 - x}$$

and therefore
$$\int_\epsilon^1 \frac{\log\left(\frac{1}{x}\right)}{2 - x}dx = \sum_1^\infty \int_\epsilon^1 \frac{x^{n-1}}{2^n}\log\left(\frac{1}{x}\right)dx.$$

But every term of the integrand is > 0 and $\displaystyle\int_0^1 \frac{\log\left(\frac{1}{x}\right)dx}{2 - x}$ converges; therefore,

applying the theorem, we find $\displaystyle\int_0^1 \frac{\log\left(\frac{1}{x}\right)}{2 - x}dx = \sum_1^\infty \frac{1}{n^2 . 2^n}$ $(= S$, say).

Putting $2x$ for x we obtain

$$S = \int_0^{\frac{1}{2}} \frac{\log\left(\frac{1}{x}\right)}{1 - x}dx - (\log 2)^2.$$

But $\displaystyle\int_0^{\frac{1}{2}} \frac{\log\left(\frac{1}{x}\right)}{1 - x}dx = \int_{\frac{1}{2}}^1 \frac{\log\left(\frac{1}{1 - x}\right)}{x}dx = \lim_{\epsilon \to 0}\int_{\frac{1}{2}}^{1-\epsilon} (1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \dots)\,dx.$

The series for $\log\left(\dfrac{1}{1 - x}\right)$ is uniformly convergent for the interval of integration and 1 belongs to the interval of the integrated series.

Thus $\int_0^1 \frac{\log\left(\frac{1}{x}\right) dx}{1-x} = \lim_{\epsilon \to 0}\left(x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \dots\right)_\epsilon^1 = \sum_1^\infty \frac{1}{n^2} = S = \frac{\pi^2}{6}$.

i.e. $$\int_0^1 \frac{\log\left(\frac{1}{x}\right) dx}{2-x} = \sum_1^\infty \frac{1}{n^2}\cdot\frac{1}{2^n} = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2.$$

(ii) $\int_0^1 \frac{\log\left(\frac{1}{x}\right)}{1-x} dx$ similarly is equal to $-\sum_1^\infty \int_0^1 x^{n-1} \log x \, dx = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.

Notes. (i) The theorem is applicable to the series $\Sigma u_n(x)$ when $u_n(x)$ is not necessarily > 0, provided one of the integrals I_1, I_2 is absolutely convergent. For let $w_n(x) = u_n(x)$ when $u_n(x) > 0$ and $w_n(x) = 0$ when $u_n(x) < 0$, and let $v_n(x) = -u_n(x)$ when $u_n(x) < 0$ and $v_n(x) = 0$ when $u_n(x) > 0$. Then the conditions of the theorem are satisfied by the series $\Sigma w_n(x)$ and $\Sigma v_n(x)$ and therefore the process is valid for $\Sigma w_n(x) - \Sigma v_n(x)$, i.e. for $\Sigma u_n(x)$.

Example. $\int_0^\infty \frac{\sin ax}{e^x - 1} dx$ (a real).

$\frac{1}{e^x - 1} = e^{-x} + e^{-2x} + \dots$ ($x > \delta > 0$), the series being uniformly convergent in the interval except at $x = 0$.

But at $x = 0$, the given integrand is $O(1)$ and it therefore converges there.

Also $\int_\delta^\infty \frac{|\sin ax|}{e^x - 1} dx < \int_\delta^\infty \frac{dx}{e^x - 1}$ so that the given integral is absolutely convergent. The integration term-by-term is legitimate for 0 to ∞.

Thus $\int_0^\infty \frac{\sin ax}{e^x - 1} dx = \sum_1^\infty \int_0^\infty e^{-nx} \sin ax \, dx = \frac{a}{1+a^2} + \frac{a}{4+a^2} + \frac{a}{9+a^2} + \dots$

It may be proved by contour integration that

$$\int_0^\infty \frac{\sin ax}{e^x - 1} dx = \frac{\pi}{2}\,\frac{e^{2\pi a} + 1}{e^{2\pi a} - 1} - \frac{1}{2a} \quad (\S\ 10 \cdot 86,\ Ex.\ \text{viii})$$

and we thus obtain the verification

$$\frac{\pi}{e^{2\pi a} - 1} = \frac{1}{2a} - \frac{\pi}{2} + \frac{a}{1+a^2} + \frac{a}{4+a^2} + \dots$$

or $$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{2t}{t^2 + 4\pi^2} + \frac{2t}{t^2 + 16\pi^2} + \dots \quad (\S\ 11 \cdot 71,\ Ex.\ \text{iv}).$$

(ii) The theorem may also be extended to complex functions if one of the integrals I_1, I_2 is absolutely convergent.

(iii) If $\sum_1^\infty v_n(x)$ converges uniformly in $a \leq x < b$ and $v_n(x)$ is continuous in this interval, then term-by-term integration is legitimate for the series $\sum_1^\infty \phi(x) v_n(x)$, where $\phi(x)$ is continuous except at one point (say $x = b$) provided $\int_a^b \phi(x) dx$ is absolutely convergent. (*Ref. Bromwich, Infinite Series*, § 175.) For, given ϵ, we can find m, independent of x such that $\left|\sum_m^{m+p} v_n(x)\right| < \epsilon$ for all $p > 0$, and therefore $\left|\sum_m^{m+p} \int_a^b \phi(x) v_n(x) dx\right| < \epsilon K$ where $K = \int_a^b |\phi(x)| dx$.

Thus $\sum_1^\infty \int_a^b \phi(x) v_n(x) dx$ converges to a value S.

Also $|S - \int_a^b \phi(x) (\sum_1^{m-1} v_n(x)) dx| = |S - \sum_1^{m-1} \int_a^b \phi(x) v_n(x)\, dx| < \epsilon K$

Therefore $\int_a^b (\phi(x) \Sigma v_n(x)) dx$ also converges to S.

11.6. Asymptotic Expansions. Consider the convergent integral $I = \int_0^\infty \frac{e^{-xt}\, dt}{1+t^2}$ $(x > 0)$. The integrand is

$$e^{-xt}\left\{1 - t^2 + t^4 \ldots (-1)^n t^{2n} + (-1)^{n+1} \frac{t^{2n+2}}{(1+t^2)}\right\}$$

Also
$$\int_0^\infty e^{-xt} t^m\, dt = (m!)/x^{m+1}$$

i.e.
$$I = \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} \ldots (-1)^n \frac{(2n)!}{x^{2n+1}} + R_n, \text{ where}$$

$$R_n = \int_0^\infty \frac{e^{-xt}(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt$$

so that if x is fixed, $|R_n| \to \infty$ when $n \to \infty$; for the series obviously does not converge.

However, if n is fixed, $|R_n| < \int_0^\infty t^{2n+2} e^{-xt}\, dt$, i.e. $< \frac{(2n+2)!}{x^{2n+3}}$ which tends to zero as $x \to \infty$.

The error in taking the sum of the first $(n+1)$ terms of the series $\sum_0^n (-1)^r \frac{(2r)!}{x^{2r+1}}$ (non-convergent) as the value of the integral is less than the next term (which tends to zero as $x \to \infty$). A close approximation to the value of the integral is therefore obtained when x is large.

For example $\int_0^\infty \frac{e^{-10t}\, dt}{1+t^2} = \frac{1}{10} - \frac{2}{10^3} + \frac{24}{10^5} = 0{\cdot}09824$ approximately, the error being less than 0·000072.

Such an expansion is called *Asymptotic.*

11.61. Definition of Asymptotic Expansion. A series $\sum_0^\infty \frac{a_n}{x^n}$ is said to be an asymptotic expansion (whether convergent or not) of a function $F(x)$ if $F(x) - \sum_0^n \frac{a_n}{x^n} = O\left(\frac{1}{x^{n+1}}\right)$ when n is fixed and x is large; and we write

$$F(x) \sim \sum_0^\infty \frac{a_n}{x^n}$$

11.62. Addition of Asymptotic Expansions. If $F(x) \sim \sum_0^\infty \frac{a_n}{x^n}$ and $G(x) \sim \sum_0^\infty \frac{b_n}{x^n}$, then $F(x) + G(x) \sim \sum_0^\infty \frac{a_n + b_n}{x^n}$ for

$$F(x) + G(x) - \sum_0^n \frac{a_n + b_n}{x^n} = \left(F(x) - \sum_0^n \frac{a_n}{x^n}\right) + \left(G(x) - \sum_0^n \frac{b_n}{x^n}\right) = O\left(\frac{1}{x^{n+1}}\right).$$

11.63. Multiplication of Asymptotic Expansions. If $F(x) \sim \sum_0^\infty \frac{a_n}{x^n}$, $G(x) \sim \sum_0^\infty \frac{b_n}{x^n}$ then $F(x)G(x) \sim \sum_0^\infty \frac{(a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0)}{x^n}$, for

$$\left\{\sum_0^n \frac{a_n}{x^n} + E_n\right\}\left\{\sum_0^n \frac{b_n}{x^n} + L_n\right\} = \sum_0^n \frac{(a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0)}{x^n} + O\left(\frac{1}{x^{n+1}}\right)$$

when
$$E_n,\ L_n = O\left(\frac{1}{x^{n+1}}\right).$$

11.64. Substitution of one Asymptotic Expansion in another. If $y = F(x)$, then $\phi(y) \equiv \alpha_0 + \alpha_1 y + \alpha_2 y^2 + \ldots$ is an asymptotic expansion if $\{\phi(y) - \sum_0^n \alpha_r y^r\} = O(y^{n+1})$. If it is required to find $\phi(F(x))$ as an asymptotic expansion in negative powers of x, it is therefore necessary that $F(x)$ should be of the form $\frac{a_1}{x} + \frac{a_2}{x^2} + \ldots$ Substitute for y in $\phi(y)$ and let the series obtained by rearranging in powers of $\frac{1}{x}$ be $\sum_0^\infty \frac{\rho_n}{x^n}$ where $\rho_0 = \alpha_0$, $\rho_1 = \alpha_1 a_1, \ldots$; then

$$\left\{\sum_0^n \alpha_r \left(\frac{a_1}{x} + \ldots + \frac{a_n}{x^n}\right)^r - \sum_0^n \frac{\rho_r}{x^r}\right\} = O\left(\frac{1}{x^{n+1}}\right),$$

$$\left\{\sum_0^n \alpha_r (F(x))^r - \sum_0^n \alpha_r \left(\frac{a_1}{x} + \ldots + \frac{a_n}{x^n}\right)^r\right\} = O\left(\frac{1}{x^{n+1}}\right)$$

since
$$F(x) = \left(\sum_1^n \frac{a_n}{x^n}\right) + O\left(\frac{1}{x^{n+1}}\right).$$

Also
$$\{\phi(F(x)) - \sum_0^n \alpha_r (F(x))^r\} = O(F^{n+1}) = O\left(\frac{1}{x^{n+1}}\right)$$

since
$$\phi(F) \sim \sum_0^\infty \alpha_r F^r$$

i.e.
$$\left\{\phi(F(x)) - \sum_0^n \frac{\rho_r}{x^r}\right\} = O\left(\frac{1}{x^{n+1}}\right).$$

The rearranged series is therefore the asymptotic expansion of $\phi(F(x))$.

If, however, $F(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots$ $(a_0 \neq 0)$, $F(x) \to a_0$ as

$x \to \infty$ and the series for $\phi(F(x))$ in powers of $F(x)$, if merely asymptotic (i.e. not convergent), will not give a correct result.

Suppose then that $\sum_0^\infty \alpha_r y^r$ has a finite radius of convergence R; then $F(x) = a_0 + F_1(x)$ may be substituted for y in $\sum_0^\infty \alpha_r y^r$ and rearranged in powers of F_1, provided $|a_0| < R$ $\quad x < R$ (since $F_1(x) \to 0$ as $x \to \infty$).

We thus obtain $\phi(F(x)) = \beta_0 + \beta_1 F_1 + \beta_2 F_1^2 + \dots$, where the series is now convergent and x large. $F_1(x)$ is expressible as an asymptotic series and its first term is a_1/x. Therefore in this case, $\phi(F(x))$ is also obtained by rearranging in powers of $1/x$.

11.65. Division by an Asymptotic Expansion. Suppose that $F(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots$, where $a_0 \neq 0$. Then

$$1/F(x) = \frac{1}{a_0}(1 - G(x) + G^2(x) - \dots)$$

where $G(x) \sim \sum_1^\infty \frac{a_n}{x^n}$, the series being convergent since $G(x) \to 0$ as $x \to \infty$. The constant term in $G(x)$ is zero and therefore the above series can be rearranged in powers of $1/x$ to give the asymptotic expansion of $1/F(x)$.

By applying the rule for multiplication, we deduce that $H(x)/F(x)$ can be similarly expanded, when $H(x)$, $F(x)$ are expanded in asymptotic series and the constant term of $F(x)$ is not zero.

11.66. Integration of Asymptotic Expansions. Let $F(x) \sim \frac{a_2}{x^2} + \frac{a_3}{x^3} + \dots$ (the terms a_0, a_1 being absent). Then $F(x) = \sum_2^n \frac{a_r}{x^r} + \frac{\lambda_n}{x^{n+1}}$, where $|\lambda_n| < \epsilon$ for all large x,

i.e. $$\int_x^\infty F(x)dx = \frac{a_2}{x} + \frac{a_3}{2x^2} + \dots + \frac{a_n}{(n-1)x^{n-1}} + R_n$$

where $|R_n| < \frac{\epsilon}{nx^n}$, i.e. the asymptotic expansion of $\int_x^\infty F(x)dx$ is obtained by term-by-term integration.

An asymptotic series cannot however be *differentiated* term-by-term to give a correct result without further investigation.

Examples. (i) $\int_x^\infty e^{x^2-t^2}\,dt$ (x real and positive).

If $I_m = \int_x^\infty e^{x^2-t^2} t^{-m}\,dt$, then $I_m = \frac{1}{2x^{m+1}} - \frac{m+1}{2} I_{m+2}$ and the integrals are convergent.

Therefore $\int_x^\infty e^{x^2-t^2}\,dt = \frac{1}{2x} - \frac{1}{2^2x^3} + \frac{1.3}{2^3x^5} - \dots + (-1)^n \frac{1.3.5\dots(2n-1)}{2^{n+1}} \cdot \frac{1}{x^{2n+1}} - K$

where $|K| = \frac{1.3.5\dots(2n+1)}{2^{n+1}}\left\{\int_x^\infty e^{x^2-t^2}t^{-2n-2}\,dt\right\}$

$$< \frac{1.3.5\dots(2n+1)}{2^{n+2}x^{2n+3}} \text{ since } \frac{1}{t^{2n+2}} < \frac{t}{x^{2n+3}} \text{ and } \int_x^\infty e^{x^2-t^2}2t\,dt = 1.$$

Thus $$\int_x^\infty e^{x^2-t^2}\,dt \sim \frac{1}{2x} - \frac{1}{4x^3} + \dots + (-1)^n \frac{(2n)!}{n!\,2^{2n+1}x^{2n+1}} + \dots$$

Also using the result $\int_0^\infty e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}$ (§ *12.24*), we may write

$$\int_0^x e^{-t^2}\,dt \sim \frac{1}{2}\sqrt{\pi} - e^{-x^2}\left(\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} - \dots\right)$$

(ii) $\int_{n\pi}^\infty \frac{\sin x}{x}\,dx$ (n a positive integer).

If $I_m = \int_{n\pi}^\infty \frac{\sin x}{x^m}\,dx$ $(m > 0)$, we obtain by integrating twice that

$$I_m = \frac{(-1)^n}{(n\pi)^m} - m(m+1)I_{m+2}.$$

Also $|I_m| < \frac{1}{(n\pi)^m} + m\left|\int_{n\pi}^\infty \frac{\cos x\,dx}{x^{m+1}}\right| < \frac{2}{(n\pi)^m}$ (using $|\cos x| < 1$)

Thus $\int_{n\pi}^\infty \frac{\sin x}{x}\,dx = u_0 + u_1 + u_2 + \dots + u_m + R$ where

$$u_m = (-1)^{m+n}\frac{(2m)!}{(n\pi)^{2m+1}} \text{ and } R = (-1)^{m+1}(2m+2)!\,I_{2m+3}.$$

Also $$|R| < \frac{2\{(2m+2)!\}}{(n\pi)^{2m+3}} = 2|u_{m+1}|.$$

Thus $\int_{n\pi}^\infty \frac{\sin x}{x}\,dx \sim \frac{(-1)^n}{n\pi}\left(1 - \frac{2!}{(n\pi)^2} + \frac{4!}{(n\pi)^4} - \dots\right)$, the error in stopping at any particular term being less than twice the succeeding term.

For example, $\int_{6\pi}^\infty \frac{\sin x}{x}\,dx = 0{\cdot}052762$ (correct to 6 decimals) so that since $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\pi$, we find $\int_0^{6\pi} \frac{\sin x}{x}\,dx = 1{\cdot}518034$.

Notes. (i) If a function $G(x)$ is expressible in the form $\phi(x)F(x)$ where

$$F(x) \sim a_0 + a_1/x + a_2/x^2 + \dots$$

then $\phi(x)(a_0 + a_1/x + a_2/x^2 \dots)$ may be called an asymptotic expansion of $G(x)$.

(ii) The theory remains the same when the variable is complex and tends to infinity in a given direction. It should be noted, however, that a non-convergent series $\Sigma a_n/z^n$ cannot represent the same analytic function of z asymptotically for *all* directions.

(iii) If $f(x)$ possesses derivatives of all orders near $x = 0$, then Maclaurin's Theorem (with a remainder) shows that

$$f(x) = f(0) + xf'(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}\left\{f^n(0) + \frac{x}{n+1}f^{(n+1)}(\theta x)\right\}$$

where $0 < \theta < 1$, and therefore for a fixed n

$$f(x) = f(0) + xf'(0) + \ldots + \frac{x^n}{n!}f^n(0) + O(x^{n+1})$$

and it follows that an asymptotic expansion of $f\left(\frac{1}{x}\right)$ is $f(0) + \frac{f'(0)}{x} + \frac{f''(0)}{2!x^2} + \ldots$ for x large. (*Bowman, Bessel Functions*, § 77.)

Example. Let $f(x) = \int_0^\infty e^{-t}\{t(1+xt)\}^{-\frac{1}{2}}\,dt$ $(x > 0)$. The integral for $f(x)$ and also all those obtained by differentiation with respect to x are uniformly convergent for all $x \geqslant 0$. We therefore obtain $f(0) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt = \Gamma(\frac{1}{2}) = \sqrt{\pi}$.

and $$f^{(n)}(0) = \frac{1.3\ldots(2n-1)}{2^n}(-1)^n\Gamma(n+\tfrac{1}{2}) \quad \text{(Ch. XII).}$$

Therefore $$f\left(\frac{1}{x}\right) = \int_0^\infty e^{-t}\left\{\frac{x}{t(x+t)}\right\}^{-\frac{1}{2}}dt$$

$$\sim \sqrt{\pi}\left(1 - \frac{1^2}{4x} + \frac{1^2.3^2}{4.8}\cdot\frac{1}{x^2} - \frac{1^2.3^2.5^2}{4.8.12}\cdot\frac{1}{x^3} + \ldots\right).$$

11.67. Non-Convergent Series. Definitions by various writers (such as *Euler, Borel, Césaro, Riesz*) have been given of the 'sum' of a non-convergent series. To be useful such definitions must be consistent, i.e. must be applicable to convergent series and give the actual sum. As an illustration we shall consider briefly the Césaro method.

11.68. Series Summable (C 1). If $S_n = a_1 + a_2 + \ldots + a_n$ does not converge but the sequence $\frac{1}{n}(S_1 + S_2 + \ldots + S_n)$ converges to a limit S, then the infinite series $\sum_1^\infty a_n$ is said to be summable (C 1) to the value S.

For consistency it must be shown that if $S_n \to S$, then

$$\frac{1}{n}(S_1 + S_2 + \ldots S_n)$$

must also tend to S.

Denote $S_n - S$ by c_n, then $c_n \to 0$.

Consider $C_n = \dfrac{c_1 + c_2 + \ldots + c_n}{n}$.

Given ε, we can find m such that $|c_r| < \varepsilon$ for all $r > m$.

Taking $n > m$, we have

$$C_n = \frac{c_1 + c_2 + \ldots + c_{m-1}}{n} + \frac{c_m + c_{m+1} + \ldots + c_n}{n}.$$

Keeping m fixed (for the moment), the term

$$\left|\frac{c_m + \ldots + c_n}{n}\right| < \frac{n - m + 1}{n}\varepsilon < \varepsilon$$

since $n > m - 1$. Also $\left|\frac{c_1 + c_2 + \ldots c_{m-1}}{n}\right| < \frac{(m-1)}{n}k$ where k is the upper bound of c_r (all r), and n can be chosen sufficiently large to ensure that $\frac{(m-1)}{n}k < \varepsilon$. Thus m can be chosen and then n_0 so that $|C_n| < 2\varepsilon$ for all $n > n_0$. Therefore $C_n \to 0$

i.e.
$$\frac{1}{n}(S_1 + S_2 + \ldots + S_n) \to S$$

Notes (i) *Hardy* has shown that if $\sum_1^\infty a_n$ is summable $(C\,1)$, then $\sum_1^\infty a_n$ is convergent if $a_n = O(1/n)$. (*Proc. L.M.S. 2, VIII, 302–4*.)

(ii) A series $\sum_1^\infty a_n$ is said to be summable (C_r) to the value S, if

$$\lim S(n, r)/A(n, r) \to S$$

where
$$S(n, r) = a_n + ra_{n-1} + {}^{r+1}C_2 a_{n-2} + {}^{r+2}C_3 a_{n-3} + \ldots,$$
$$A(n, r) = {}^{n+r-1}C_r.$$

Examples. (i) If $s_n = 1 - 1 + 1 - 1 \ldots + (-1)^{n-1}$, $s_n = \frac{1}{2}(1 - (-1)^n)$, $s_1 + s_2 + \ldots + s_n = \frac{1}{2}n$ (n even), $\frac{1}{2}(n+1)$ (n odd), and therefore the series $1 - 1 + 1 - 1 + 1 \ldots$ is summable $(C\,1)$ to $\frac{1}{2}$.

(ii) $\sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\theta = \frac{1 - \cos 2n\theta}{2\sin\theta}$ $(\theta \neq 2m\pi)$ and therefore the series is oscillatory.

$\sum_1^n s_n = \frac{n}{2\sin\theta} - \frac{\cos(n+1)\theta\sin n\theta}{2\sin^2\theta}$ and therefore the series is summable $(C\,1)$ to $\frac{1}{2}\operatorname{cosec}\theta$

Examples XI

? 1. If $\sum_1^\infty a_n = s$, show that $\sum_1^\infty b_n = s$, where $b_n = \frac{a_1 + 2a_2 + \ldots + na_n}{n(n+1)}$.

? 2. If $a_n > 0$, all n, show that $\sum_1^\infty (a_1 a_2 \ldots a_n)^{\frac{1}{n}} < e\sum_1^\infty a_n$, when $\sum_1^\infty a_n$ is convergent (and a_n are not all zero).

Determine whether the infinite series whose general terms are given in *Examples 3–15* are convergent or not.

3. $\dfrac{1}{\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\log n}$ 4. $\dfrac{(\log n)^n}{n^{\log n}}$ 5. $n^{-(\log\log n)^{-1}}$

6. $n^{\frac{1}{n}} - 1$ 7. $\dfrac{1}{n^n - 1}$ 8. $n^{\frac{1}{n}} - 1 - \dfrac{\log n}{n}$

9. $(\log n)^{-\log n}$ 10. $(\log n)^{-\log\log n}$ 11. $n^{-1-1/n}$

12. $a^{\frac{1}{n}} - 1 - \dfrac{1}{n}$ $(a > 0)$ 13. $\dfrac{(n!)^2}{(2n+2)!}$

14. $\dfrac{3.12.21\ldots(9n+3)}{1.4.7\ldots(3n+4)}\,\dfrac{(2n+1)!}{n!}$

15. $\dfrac{\alpha(\alpha+1)\ldots(\alpha+n)\beta(\beta+1)\ldots(\beta+n)}{\gamma(\gamma+1)\ldots(\gamma+n)\delta(\delta+1)\ldots(\delta+n)}$

16. Show that if $\sum_1^\infty a_n$ is convergent ($a_n > 0$), so also is $\sum_1^\infty (a_n a_{n+1})^{\frac{1}{2}}$, and that the converse is true if a_n is monotonic.

Discuss the convergence of the series whose general terms are given in *Examples 17–20*.

17. $\dfrac{1.4.7\ldots(3n+1)}{1.7.12\ldots(6n-1)}\,\dfrac{(2n+1)!x^n}{2.5\ldots(3n+2)n!}$

18. $\dfrac{\alpha(\alpha+1)\ldots(\alpha+n)\beta(\beta+1)\ldots(\beta+n)x^{n-1}}{p(p-1)\ldots(p+n)q(q+1)\ldots(q+n)}$

19. $\dfrac{(n!)^3\{(n+2)!\}^3\{(n+4)!\}^3.2^{6n}x^n}{(2n+1)!\{(2n+5)!\}^3}$

20. $\dfrac{(2n)!\{(3n)!\}^2(4n)!x^n}{(6n+1)!\{(2n+1)!\}^3}$

Prove the results given in *Examples 21–7*.

21. $\lim\limits_{n\to\infty}\left(\dfrac{n^3}{1+n^4}+\dfrac{2n^3}{2^4+n^4}+\ldots+\dfrac{n^3}{2n^4}\right)=\dfrac{\pi}{8}$

22. $\lim\limits_{n\to\infty}\dfrac{1}{n}\left\{\log\left(1+\dfrac{1}{n}\right)+\log\left(1+\dfrac{2}{n}\right)+\ldots+\log 2\right\}=2\log 2-1$

23. $\lim\limits_{n\to\infty}\dfrac{1}{n}\left\{\dfrac{(2n)!}{n!}\right\}^{\frac{1}{n}}=\dfrac{4}{e}$

24. $\lim\limits_{n\to\infty}\left\{\left(\dfrac{n}{n}\right)^n+\left(\dfrac{n-1}{n}\right)^n+\ldots+\left(\dfrac{1}{n}\right)^n\right\}=\dfrac{e}{e-1}$

25. $\lim\limits_{n\to\infty}\left\{\dfrac{1}{\sqrt{(n^2+n)}}+\dfrac{1}{\sqrt{(n^2+2n)}}+\ldots+\dfrac{1}{\sqrt{(2n^2)}}\right\}=2(\sqrt{2}-1)$

26. $\sum\limits_1^\infty\dfrac{n}{(4n^2-1)(16n^2-1)}=\dfrac{1}{12}(1-\log 2)$

27. $\sum\limits_1^\infty\dfrac{1}{n(4n^2-1)(16n^2-1)}=\dfrac{7}{2}-\dfrac{10}{3}\log 2$

28. If the positive function $f(x, y)$ has a lower limit $g(\xi)$ and an upper limit $G(\xi)$ when $y = \xi - x$ and x varies from 0 to ξ, and if $\xi G(\xi)$, $\xi g(\xi)$ tend steadily to zero as $\xi \to \infty$, show that the double series $\Sigma\Sigma f(m, n)$ converges if the integral $\int^\infty G(\xi)\xi\,d\xi$ converges; but the series diverges if the integral $\int^\infty g(\xi)\xi\,d\xi$ diverges. (*Bromwich.*)

Establish the convergence or divergence of the series given in *Examples 29–31*.

29. $\Sigma\Sigma e^{-m^2-n^2}$ 30. $\Sigma\Sigma(m^2+n^2)^{-1}\{\log(m^2+n^2)\}^{-p}$

31. $\Sigma\Sigma\dfrac{1}{(m^2+mn+n^2)^\mu}$

32. If $S(x, y) = \sum\limits_0^\infty\sum\limits_0^\infty a_{pq}x^py^q$ and $\lambda(k) = \varlimsup\limits_{n\to\infty}\{\max\limits_{p+q=n} a_{pq}k^q\}^{\frac{1}{n}}$, show that the double power series is absolutely convergent within the region bounded by

$$|x|\lambda(|y|/|x|) = 1. \quad (\textit{Lemaire.})$$ §4·61 n.k

33. Prove that the double series $\sum\limits_0^\infty\sum\limits_0^\infty\dfrac{(m+n)!}{m!n!}x^my^n$ is absolutely convergent for $|x|+|y|<1$ to the sum $(1-x-y)^{-1}$.

34. Show that the double series $\sum_0^\infty\sum_0^\infty \left\{\frac{(m+n)!}{m!n!}\right\}^2 x^m y^n$ is absolutely convergent for $|x|^{\frac12}+|y|^{\frac12}<1$ and that its sum is $(1-2x-2y+x^2-2xy+y^2)^{-\frac12}$. (*Daniell.*)

35. Prove that the series $\sum_1^\infty(-1)^{n-1}\frac{n^{r-1}+p_1n^{r-2}+\dots+p_{r-1}}{n^r+q_1n^{r-1}+\dots+q_r}$ is convergent (not absolutely) and that $\sum_1^\infty(-1)^{n-1}\frac{n^{r-2}+p_1n^{r-3}+\dots+p_{r-2}}{n^r+q_1n^{r-1}+\dots+q_r}$ is absolutely convergent.

Discuss the convergence of the series whose general terms are given in *Examples 36–43.*

36. $\frac{(-1)^n\left(1+\frac1n\right)}{\log n}$ 37. $\frac{1}{n(1+n^2c^2)}$ 38. $\frac1n\sin\frac1n$

39. $n^{\frac1n}2^{-n}$ 40. $n^{\left(2-\frac{3n}{2n}\right)}$ 41. $\frac1n\sin^2(n\theta)$

42. $(-1)^n\frac{\alpha(\alpha+1)\dots(\alpha+n)}{\gamma(\gamma+1)\dots(\gamma+n)}$

43. $(-1)^{n+1}\frac{1.3.5\dots(2n-3).2.5.8\dots(3n-4)}{(n-1)!1.7.13\dots(6n-11)}$

44. The series $1-\frac12+\frac13-\frac14+\dots(=\log 2)$ is deranged into the series $1-\frac12-\frac14+\frac13+\frac15+\frac17+\frac19-\frac16-\frac18\dots-\frac1{20}+\frac1{11}+\dots+\frac1{41}-\frac1{22}\dots$ where a group of terms of one sign contains twice as many members as the previous group of the opposite sign. Show that the sum oscillates between $\frac12\log 2$ and $\frac32\log 2$.

Determine the limits of the functions given in *Examples 45–50* when n tends to ∞ and $x>0$, and obtain the intervals of uniform convergence.

45. $\frac{nx}{1+nx}$ 46. $\frac{nx}{1+nx+n^2x^2}$ 47. $\frac{nx^n}{1+x^{2n}}$

48. $\frac{x^n}{n(1+x^{2n})}$ 49. $\frac{\sin\frac1n\sin\frac xn}{\sin^2\left(\frac1n\right)+x\cos^2\left(\frac1n\right)}$ 50. $\tan^{-1}(nx)$

For the functions given in *Examples 51–4*, find the values of

$$\lim_{n\to\infty}\int_0^c f(x,n)dx \text{ and } \int_0^c\lim_{n\to\infty}f(x,n)dx\ (c>0).$$

51. $\frac{n^2x}{1+n^4x^2}$ 52. $\frac{x}{1+n^4x^2}$ 53. $ne^{-nx}\sin nx$ 54. nxe^{-nx^2}

Discuss the uniform convergence with respect to x of the series given in *Examples 55–62.*

55. $\sum_1^\infty\frac1{n^x}$ 56. $\sum_1^\infty(-1)^{n-1}\frac{x^n}{n(1+x^n)}$ 57. $\sum_1^\infty\frac{\cos^n x\sin nx}{n}$

58. $\sum_1^\infty\frac{1}{n^2+n^4x^2}$ 59. $\sum_0^\infty e^{nx}\cos nx$ 60. $\sum_0^\infty x^n\cos n\theta$

61. $\sum_1^\infty\frac{e^{-nx}}{n^p}$ 62. $\sum_1^\infty(-1)^{n-1}\frac{e^{-nx}}{n^p}$

63. Show that $\lim_{x\to1-0}\sum_1^\infty\frac{(-1)^{n-1}x^n}{n(1+x^n)}=\frac12\log 2.$

64. Prove that $\dfrac{\arcsin x}{\sqrt{(1-x^2)}}=x+\dfrac{2}{3}x^3+\dfrac{2.4}{3.5}x^5+\dots$ $(|x|<1)$ and deduce that (i) $(\arcsin x)^2=\frac{1}{2}\sum_1^\infty\dfrac{\{(n-1)!\}^2}{(2n)!}(2x)^{2n}$ $(|x|<1)$.

(ii) $\dfrac{\pi^2}{9}=1+\dfrac{1}{6}\cdot\dfrac{1}{2}+\dfrac{1.2}{6.10}\cdot\dfrac{1}{2}+\dots$

(iii) $\dfrac{2\pi\sqrt{3}}{9}=1+\dfrac{1}{6}+\dfrac{1.2}{6.10}+\dfrac{1.2.3}{6.10.14}+\dots$

Prove the results given in *Examples 65–72*.

65. $\displaystyle\int_0^\pi\cos(x\cos\theta)d\theta=\pi\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2 4^2}-\dots\right)$.

66. $\displaystyle\int_0^1\frac{1}{x}\log(1+x)dx=\frac{\pi^2}{12}$.

67. $\displaystyle\int_0^\pi\log(1-2a\cos x+a^2)dx=0$ $(|a|\le 1)$, $2\pi\log|a|$ $(|a|>1)$.

68. $\displaystyle\sum_0^\infty\frac{x^n}{(\alpha n+1)(\alpha n+2)\dots(\alpha n+m+1)}=\frac{1}{m!}\int_0^1\frac{(1-t)^m\,dt}{1-xt^\alpha}$ $(\alpha>0,\ |x|<1)$.

69. $1-({}^4C_2)^{-1}+({}^6C_2)^{-1}-({}^{10}C_2)^{-1}+\dots=10-2\pi-4\log 2$.

70. $\dfrac{2x\arctan x+\log(1+x^2)}{1+x^2}=2(S_2x^2-S_4x^4+\dots)$ $(|x|<1)$, where $S_r=\sum_1^r(1/n)$.

71. $\frac{1}{2}(\arctan x)\log(1+x^2)=\frac{1}{2}(1+\frac{1}{3})x^3-\frac{1}{3}(1+\frac{1}{3}+\frac{1}{5})x^5+\dots$ $(|x|<1)$

72. $\frac{1}{4}\pi\log 2=\frac{1}{2}(1+\frac{1}{3})-\frac{1}{3}(1+\frac{1}{3}+\frac{1}{5})+\dots$

Discuss the convergence of the infinite products given in *Examples 73–80*.

73. $\displaystyle\prod_1^\infty\left(1+\frac{(-1)^{n-1}}{nx-1}\right)$ **74.** $\displaystyle\prod_1^\infty\left(\frac{\alpha+n}{\beta+n}\right)$ **75.** $\displaystyle\prod_2^\infty\left(1+\frac{(-1)^n x}{n\log n}\right)$

76. $\displaystyle\prod_1^\infty\sec\frac{x}{n}$ **77.** $\displaystyle\prod_1^\infty\cosh\frac{x}{n}$ **78.** $\displaystyle\prod_1^\infty\left(1+(-1)^n\sinh\frac{x}{n}\right)$

79. $\displaystyle\prod_1^\infty\left(\frac{x+x^{2n}}{x^{2n}+1}\right)$ **80.** $\displaystyle\prod_1^\infty\left(\frac{1+x^n+x^{2n}}{1+x^{2n}}\right)$

81. Show that if $u_{2r-1}=\dfrac{1}{r^{1/3}}+\dfrac{1}{r^{2/3}}+\dfrac{2}{r}$, $u_{2r}=-\dfrac{1}{r^{1/3}}-\dfrac{1}{r}$, the series $\sum_1^\infty u_n$, $\sum_1^\infty u_n^2$ are divergent but the product $\prod_1^\infty(1+u_n)$ is convergent.

82. If $u_{2r-1}=\dfrac{a}{r^\alpha}+\dfrac{b}{r^{2\alpha}}+\dfrac{c}{r^{3\alpha}}$, $u_{2r}=-\dfrac{a}{r^\alpha}+\dfrac{a^2-b}{r^{2\alpha}}-\dfrac{a^3-2ab+c}{r^{3\alpha}}$, show that the series $\sum_1^\infty u_n$, $\sum_1^\infty u_n^2$ diverge if $\alpha\le\frac{1}{2}$, but the product $\prod_1^\infty(1+u_n)$ converges if $\alpha>1/3$.

83. Prove that $\displaystyle\prod_0^\infty(1+x^{2^n})=\frac{1}{1-x}$ if $|x|<1$.

Prove the results given in *Examples 84–7*.

84. $\displaystyle\sum_1^\infty\frac{\sin n\theta\cos n\phi}{n}=\tfrac{1}{2}(\pi-\theta)$, $(\theta>\phi)$; $-\tfrac{1}{2}\theta\ (\theta<\phi)$; $\tfrac{1}{4}(\pi-2\theta)\ (\theta=\phi)$. $(0<\theta,\ \phi<\pi)$

85. $\sum_1^\infty (-1)^{n-1}\frac{\sin^3 n\theta}{n} = 0,\ (0 \le \theta \le \frac13\pi),\ (\theta = \pi),\ (\frac53\pi < \theta < 2\pi)$: $\frac14\pi$, $(\theta = \frac13\pi)$: $-\frac18\pi$, $(\theta = \frac53\pi)$: $\frac18\pi$, $(\frac13\pi < \theta < \pi)$: $-\frac14\pi$, $(\pi < \theta < \frac53\pi)$.

86. $\sum_1^\infty \frac{\cos n\theta}{(2\cos\theta)^n} = \cos 2\theta$, $\sum_1^\infty \frac{\sin n\theta}{(2\cos\theta)^n} = \sin 2\theta$ $(|\cos\theta| > \frac12)$

87 $\sum_1^\infty \frac{\cos n\theta \cos n\phi}{n^2} = \frac14(\theta^2 + \phi^2) - \frac12\pi\theta + \frac16\pi^2$ or $\frac14(\theta^2 + \phi^2) - \frac12\pi\phi + \frac16\pi^2$, according as $\theta > \phi$ or $\theta < \phi$ where $0 \le \theta, \phi \le \pi$.

88. If $f(z) = \sum_0^\infty a_n z^n$ and $z\ (= re^{i\theta})$ is a point on the circle $|z| = r < R$ where R is the radius of convergence of the power series

$$\int_0^{2\pi} \mathbf{R}(f(z))\cos n\theta\, d\theta = \pi r^n \mathbf{R}(a_n),\ \int_0^{2\pi} \mathbf{R}(f(z))\sin n\theta\, d\theta = -\pi r^n \mathbf{I}(a_n) \text{ and}$$

$$\int_0^{2\pi} \mathbf{R}(f(z))d\theta = 2\pi\mathbf{R}(a_0), \text{ and deduce that}$$

$$\text{(i) } |a_n| r^n \le \frac{1}{\pi}\int_0^{2\pi} |\mathbf{R}(f(z))| d\theta \quad (n > 0)$$

$$\text{(ii) } |a_n| r^n \le \text{Max}\,\{4\,\text{Max}\,\mathbf{R}(f(z)),\ 0\} - 2\mathbf{R}(f(0))$$

89. If $f(z)$ is analytic on and within the circle $|z| < R$, show that for a point $z_0\ (= re^{i\theta})$ within the circle

$$\mathbf{R}(f(z_0)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\mathbf{R}(f(z))d\phi$$

where $z\ (= Re^{i\phi})$ is a point on the circle $|z| = R$ (*Poisson.*)

Obtain the Lagrangian Expansions given in *Examples 90–5*

90. If $z = 1 + tz^a$, $\log z = t + \frac{(2a-1)}{2!}t^2 + \frac{(3a-1)(3a-2)}{3!}t^3 + \dots$

$$\left(a \ne 1,\ |t| < \frac{(a-1)^{a-1}}{a^a}\right)$$

91. If $\log z = tz$, $z = 1 + t + \frac{3t^2}{2!} + \frac{4^2t^3}{3!} + \dots\ (|t| < 1/e)$

92. If $z(1-z)^3 = t$, $z = t + \frac{6}{2!}t^2 + \frac{9.10}{3!}t^3 + \frac{12.13.14}{4!}t^4 + \dots\ (|t| < 3^3/4^4)$

93. If $ze^{2z} = t$, $e^z = 1 + t - \frac{3}{2!}t^2 + \frac{5^2}{3!}t^3 - \dots\ (|t| < 1/2e)$

94. If $z(1+z) = t$, $(1+z)^n = 1 + nt + \frac{n(n-3)}{2!}t^2 + \frac{n(n-4)(n-5)}{3!}t^3 + \dots$ $(|t| < \frac14)$

95. If $z = t\sqrt{(1+z)}$; $z = t + \frac12 t^2 + \frac18 t^3 + \sum_2^\infty (-1)^{n+1}\frac{1.3\ \dots\ (2n-3)}{2.4\ \dots\ 2n}\frac{t^{2n+1}}{2^{2n}}$ $(|t| < 2)$

Obtain the expansions given in *Examples 96–101.*

96. $\sec z = 4\pi\left(\frac{1}{\pi^2 - 4z^2} - \frac{3}{9\pi^2 - 4z^2} + \frac{5}{25\pi^2 - 4z^2} - \dots\right)$

97. $\operatorname{cosech} z = \frac{1}{z} - 2z\left(\frac{1}{z^2 + \pi^2} - \frac{1}{z^2 + 4\pi^2} + \frac{1}{z^2 + 9\pi^2} - \dots\right)$

98. $\operatorname{sech} x = 4\pi\left(\frac{1}{\pi^2+4x^2}-\frac{3}{9\pi^2+4x^2}+\frac{5}{25\pi^2+4x^2}-\dots\right)$

99. $\coth x = \frac{1}{x}+2x\left(\frac{1}{x^2+\pi^2}+\frac{1}{x^2+4\pi^2}+\frac{1}{x^2+9\pi^2}+\dots\right)$

100. $\pi z\cot\pi z = 1+2z^2\sum_{1}^{\infty}\frac{1}{z^2-n^2}$

101. $\pi\tan\frac{1}{2}\pi z = \sum_{0}^{\infty}\frac{4z}{(2n+1)^2-z^2}$

102. Show that $\cos\pi z = \prod_{1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2}\right)$

Discuss the convergence of the integrals given in *Examples 103–28.*

103. $\int_0^\infty \frac{e^{-x}(\log x)^{\frac{1}{2}}\,dx}{\sqrt{x}(1+x^2)}$ 104. $\int_0^\infty \frac{(\log x)^{1/3}}{x^{\frac{1}{2}}(1+x^a)}\,dx$

105. $\int_0^\infty \frac{x^\alpha e^{-\beta x}(\log x)^\gamma}{1+x^2}\,dx$ 106. $\int_0^\infty \frac{x\,dx}{(\sin x)^{1/3}(x^2+a^2)}$

107. $\int_0^\infty x^2 e^{-x^6\sin^2 x}\,dx$ 108. $\int_0^\infty x^3 e^{-x^6\sin^2 x}\,dx$

109. $\int_0^\infty \frac{e^{-px}\,dx}{(\sin x)^{1/5}}$ 110. $\int_0^\infty e^{-px}\log(\cos^2 x)\,dx$

111. $\int_\pi^\infty \frac{dx}{(\log x)^2(\sin x)^{1/3}}$ 112. $\int_\pi^\infty \frac{dx}{x^2(\log x)^4(\sin x)^{1/5}}$

113. $\int_0^\infty \frac{x^\alpha\cos\beta x\,dx}{1+x^\gamma}$ 114. $\int_0^1 \frac{\log(1-x)\,dx}{\sqrt{(1-x)}}$ 115. $\int_0^1 \frac{x\log x\,dx}{(1+x)^3}$

116. $\int_0^\infty \frac{\sin\alpha x\sin\beta x\sin\gamma x}{x^2}\,dx$ 117. $\int_0^\infty \frac{\sin(ax^2+bx)\,dx}{x}$

118. $\int_0^\infty \frac{\sin(x^m)\,dx}{x^p}$ 119. $\int_0^\infty \frac{e^{\cos x}\sin 2x}{x^\alpha}\,dx$

120. $\int_0^1 \frac{x^a-1}{\log x}\,dx$ 121. $\int_0^\infty \frac{x\cosh ax}{\sinh x}\,dx$

122. $\int_0^\infty \left\{5e^{-x}-\frac{x+2}{x}(e^{-2x}-e^{-3x})\right\}\frac{dx}{x}$

123. $\int_0^\infty \frac{x^a+x^{-a}}{1+x^2}\,dx$ 124. $\int_0^\infty \frac{\cosh\alpha x\cosh\beta x}{\cosh x}\,dx$

125. $\int_0^{\frac{1}{2}\pi} \frac{(\tan x)^a\,dx}{a\cos^2 x+b\sin^2 x}\ (ab>0)$ 126. $\int^\infty \frac{dx}{1+x^2\sin^2 x}$

127. $\int^\infty \frac{dx}{1+x^3\sin^2 x}$ 128. $\int_0^1 (\log x)\log(1+x)\,dx$

129. If $f(x)$ is monotonic decreasing and tends to zero when x tends to infinity and if $\int^\infty f(x)dx$ converges, prove that $\lim_{x\to\infty} xf(x)=0$. Deduce that if $\int^\infty f(x)dx$ converges so also does $\int^\infty xf'(x)dx$.

130. If $f(x)$ is an odd function of x show that

$$\int_0^\infty f(\sin x)\,\frac{dx}{x} = \int_0^{\frac12\pi} f(\sin x)\,\frac{dx}{\sin x}$$

if both integrals converge.

Prove the results given in *Examples 131–2.*

131. $\int_0^\infty \frac{\sin^{2n+1} x}{x}\,dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$ **132.** $\int_0^\infty \log(\cos^2 x)\frac{\sin x}{x}\,dx = -\pi\log 2$

133. If $f(x)$ is an even function of x, show that

$$\int_0^\infty f(\sin x)\,\frac{dx}{x^2} = \int_0^{\frac12\pi} f(\sin x)\,\frac{dx}{\sin^2 x}$$

if the integrals converge.

Prove the results given in *Examples 134–47.*

134. $\int_0^\infty \log(\cos^2 x)\,\frac{dx}{x^2} = -\pi$

135. $\int_0^\infty \{\log(\cos^2 x)\}\{\log(\sin^2 x)\}\frac{dx}{x^2} = 2\pi(2\log 2 - 1)$

136. $\int_0^\infty \log\left(1 + \frac{a^2}{x^2}\right)dx = \pi a$ **137.** $\int_0^\infty \frac{(\tan^{-1} x)^2}{x^2} = \pi\log 2$

138. $\int_0^\infty \frac{\{\log(1+x^2)\}^2}{x^2}\,dx = 4\pi\log 2$

139. $\int_0^\infty \frac{\sin ax \sin bx \sin cx}{x^3}\,dx = \frac12\pi bc \ (a > b + c)$;

$\frac18\pi(2bc + 2ca + 2ab - a^2 - b^2 - c^2) \ (a < b + c), \ (a > b > c > 0)$

140. $\int_0^\infty \frac{\sin ax \sin x}{x^2}\,dx = \frac12\pi a \ (0 < a < 1); \ \frac12\pi \ (a > 1)$

141. $\int_0^\infty \frac{\sin ax \sin^2 x}{x^3}\,dx = \frac18\pi a(4 - a) \ (0 < a < 2); \ \frac12\pi \ (a > 2)$

142. $\int_0^\infty \frac{\sin ax \sin^3 x}{x^3}\,dx = \frac{\pi a}{24}(9 - a^2) \ (0 < a < 1); \ \frac{\pi}{48}(a^3 - 9a^2 + 27a - 3)$
$(1 < a < 3); \ \frac12\pi \ (a > 3)$

143. $\int_0^{\frac12\pi} \log(\cot\tfrac12 x)dx = \int_0^{\frac12\pi}\frac{x\,dx}{\sin x} = 2\int_0^1 \frac{(\tan^{-1} x)}{x}\,dx$

$$= 2\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots\right)$$

144. $\int_0^1 \frac{\log(1-x)dx}{\sqrt{(1-x)}} = -4$ **145.** $\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{1}{12}\pi^2$

146. $\int_0^1 x^x\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \dots$

147. $\int_0^1 \frac{x^a - 1}{\log x}\,dx = \log(a + 1) \ (a > -1)$

148. By means of the change of variable $x = \sqrt{\lambda}e^u$, $t = 2\sqrt{\lambda}\sinh u$ $(\lambda > 0)$, prove that

$$\text{(i)} \int_0^\infty f(x^2 + \lambda^2/x^2)dx = \int_0^\infty f(t^2 + 2\lambda)dt$$

$$\text{(ii)} \int_0^\infty \left(x^2 - \frac{\lambda^2}{x^2}\right) f\left(x^2 + \frac{\lambda^2}{x^2}\right)dx = \int_0^\infty t^2 f(t^2 + 2\lambda)dt$$

if the integrals are convergent.

149. Deduce from *Example 148* that $(\lambda > 0)$

$$\text{(i)} \int_0^\infty e^{-x^2-\lambda^2/x^2}dx = \frac{\sqrt{\pi}}{2}e^{-2\lambda}; \quad \text{(ii)} \int_0^\infty x^2 e^{-x^2-\lambda^2/x^2}dx = \frac{\sqrt{\pi}}{4}e^{-2\lambda}(1 + 2\lambda);$$

$$\text{(iii)} \int_0^\infty \frac{1}{x^2}e^{-x^2-\lambda^2/x^2}dx = \frac{\sqrt{\pi}}{\lambda}e^{-2\lambda}$$

150. Show that $\int_0^\infty x\sin(x^3 - ax)dx$ is uniformly convergent for any finite interval of a.

Prove the results given in *Examples 151–66.*

151. $\int_0^\infty \frac{1}{x^2}[e^{-ax}\{1 + x(a + b)\} - e^{-bx}\{1 + x(b + b)\}]dx = (b - a) + b\log\frac{b}{a}$ $(a, b > 0)$

152. $\int_0^\infty \frac{1}{x^2}[e^{-ax}\{1 + (a - b)x\} - e^{-bx}]dx = b - a - b\log\frac{b}{a}$ $(a, b > 0)$

153. $\int_0^\infty \frac{1}{x^2}[e^{-ax}(1 - \frac{1}{2}x) - e^{-x}\{1 + (\frac{1}{2} - a)x\}]dx = (a + \frac{1}{2})\log a + 1 - a$

$(a > 0)$

154. $\int_0^\infty \frac{1}{x}\{(k_2 - k_3)e^{-ax} + (k_3 - k_1)e^{-bx} + (k_1 - k_2)e^{-cx}\}dx$

$= -(k_2 - k_3)\log a + (k_1 - k_3)\log b + (k_2 - k_1)\log c$ $(a, b, c > 0)$

155. $\int_0^\infty \frac{1}{x^2}\{e^{-ax}(A(1 + ax) + A_1x) + e^{-bx}(B(1 + bx) + B_1x)$

$+ e^{-cx}(C(1 + cx) + C_1x)\}dx = -Aa - Bb - Cc - \log(a^{A_1}b^{B_1}c^{C_1})$

where $A + B + C = A_1 + B_1 + C_1 = 0$ $(a, b, c > 0)$

156 $\int_0^\infty \frac{1}{x}\{(b - c)e^{-ax} + (c - a)e^{-bx} + (a - b)e^{-cx}\}dx = \log\left(\frac{a^cb^ac^b}{c^aa^bb^c}\right)$

$(a, b, c > 0)$

157. $\int_0^\infty \frac{1}{x^2}[\Sigma e^{-ax}\{b - c + x\{a(b - c) + \log b/c\}\}]dx = 0$ $(a, b, c > 0)$

158. $\int_0^\infty \frac{1}{x}(e^{-\frac{1}{2}ax} - e^{-\frac{1}{2}bx})^2dx = 2\log\left(\frac{a + b}{2\sqrt{ab}}\right)$ $(a, b > 0)$

159. $\int_0^\infty \frac{1}{x^2}\{(b - c)e^{-ax} + (c - a)e^{-bx} + (a - b)e^{-cx}\}dx$

$= a(b - c)\log a + b(c - a)\log b + c(a - b)\log c$ $(a, b, c > 0)$

160. $\int_0^\infty \frac{1}{x^2}(e^{-\frac{1}{2}ax} - e^{-\frac{1}{2}bx})^2dx = a\log a + b\log b - (a + b)\log\left(\frac{a + b}{2}\right)$

$(a, b > 0)$

161. $\int_0^\infty \frac{e^{-x}\sinh^2 \frac12 ax}{x^2}\,dx = \frac14 \log\{(1-a)^{1-a}(1+a)^{1+a}\}$ $(|a| < 1)$

162. $\int_0^\infty \frac{1}{x^3}\left[e^{-ax}\{1+x(a+k_1)+x^2(\frac12 a^2+ak_1+k_2)\}\right.$

$$\left. - e^{-bx}\{1+x(b+k_1)+x^2(\tfrac12 b^2+bk_1+k_2)\}\right]dx$$

$$= \tfrac14(b^2-a^2)+k_1(b-a)+k_2\log\frac{b}{a}\quad (a,\ b > 0)$$

163. $\int_0^\infty \frac{1}{x^3}\{e^{-ax}(1+(a-b)x+\frac12(a-b)^2x^2) - e^{-bx}\}dx$

$$= \tfrac14(b-a)(a-3b)+\tfrac12 b^2\log\frac{b}{a}\quad (a, b > 0)$$

164. $\int_0^\infty \frac{1}{x^3}\{e^{-x}(2x^2+6x+9) - 9e^{-x/3}\}dx = -\frac12\log 3$

165. $\int_0^\infty \frac{1}{x^3}\{e^{-x}(2-x^2) - e^{-2x}(2+4x+3x^2)\}dx = 0$

166. $\int_0^\infty \left\{\frac{e^{-ax}-e^{-bx}}{x^2}+\frac{a-b}{2}\,\frac{e^{-ax}+e^{-bx}}{x}\right\}dx = \frac12 ab\log\frac{b}{a} - \frac14(b^2-a^2)$

$$(a, b > 0)$$

167. If $J_0(\xi)=\frac1\pi\int_0^\pi\cos(\xi\sin\theta)d\theta$, prove that $\int_0^\infty \frac1x J_0(xt)\sin x\,dx$ is equal to $\frac12\pi$ if $0 < t < 1$ and equal to $\arcsin(1/t)$ if $t > 1$.

168. If $J_1(\xi)=\frac1\pi\int_0^\pi\cos(\theta-\xi\sin\theta)d\theta$, prove that $\int_0^\infty\frac1x J_1(xt)\sin x\,dx$ is equal to $\frac1t\{1-\sqrt{(1-t^2)}\}$ if $0 < t < 1$ and equal to $1/t$ if $t > 1$.

Establish the asymptotic expansions in *Examples 169–73.*

169. $\int_x^\infty \frac{e^{-t}\,dt}{t}\sim e^{-x}\left(\frac1x-\frac1{x^2}+\frac{2!}{x^3}-\frac{3!}{x^4}+\dots\right)$

170. $\int_x^\infty\frac{\cos t\,dt}{\sqrt t}\sim\frac{\cos x}{\sqrt x}\left\{\frac1{2x}-\frac{1.3.5}{(2x)^3}+\frac{1.3.5.7.9}{(2x)^5}-\dots\right\}$

$$-\frac{\sin x}{\sqrt x}\left\{1-\frac{1.3}{(2x)^2}+\frac{1.3.5.7}{(2x)^4}-\dots\right\}$$

171. $\int_x^\infty\frac{\sin t\,dt}{\sqrt t}\sim\frac{\cos x}{\sqrt x}\left\{1-\frac{1.3}{(2x)^2}+\frac{1.3.5.7}{(2x)^4}-\dots\right\}$

$$+\frac{\sin x}{\sqrt x}\left\{\frac1{2x}-\frac{1.3.5}{(2x)^3}+\frac{1.3.5.7.9}{(2x)^5}-\dots\right\}$$

172. $\int_0^x\frac{e^t\,dt}{\sqrt t}\sim\frac{e^x}{\sqrt x}\left(1+\frac1{2x}+\frac{1.3}{4x^2}+\frac{1.3.5}{8x^3}+\dots\right)$

173. $\int_0^\infty\frac{e^{-t}\,dt}{\sqrt{t}(x^2+t^2)}\sim\sqrt\pi\left(\frac1{x^2}-\frac{1.3}{2^2x^4}+\frac{1.3.5.7}{2^4x^6}-\dots\right)$

Find the sum of the series given in *Examples 174–8*, showing that they are summable $(C\,1)$.

174. $1+0+0+\dots+0-1+0+0+\dots+0+1+0+0+\dots+0-1+0+0\dots$, where $+1$ is followed by p zeros and -1 by q zeros.

175. $1 + \cos\theta + \cos 2\theta + \ldots$ 176. $\cos\theta + \cos 3\theta + \cos 5\theta + \ldots$
177. $1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 + 1 \ldots$
178. $\frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - 1 + \frac{1}{5} + \frac{1}{6} - 1 + \frac{1}{7} \ldots$

Solutions

1. $\sum_1^n b_n = \dfrac{s_1 + s_2 + \ldots + s_n}{n+1} \to s$ where $s_n = \sum_1^n a_n$.

2. $(a_1 a_2 \ldots a_n)^{1/n} = \dfrac{1}{(n!)^{1/n}}(a_1\, 2a_2\, 3a_3 \ldots na_n)^{1/n} < \dfrac{n+1}{(n!)^{1/n}} b_n$ (*Example 1*) $< eb_n$ since $\left(1 + \dfrac{1}{r}\right)^r < e$ ($r = 1$ to n).

3. D 4. D 5. $D\left(u_n > \dfrac{1}{n}\right)$ 6. $D\left(u_n > \dfrac{1}{n}\right)$

7. $D\ (u_n \to \infty)$ 8. $C\left(u_n = O\left\{\left(\dfrac{\log n}{n}\right)^2\right\}\right)$ 9. $C\left(u_n < \dfrac{1}{n^2}\right)$

10. $D\left(u_n > \dfrac{1}{n}\right)$ 11. D 12. D except when $a = e$.

13. $C\ (u_n/u_{n+1} \to 4)$ 14. D 15. C when $\gamma + \delta - \alpha - \beta > 1$.
16. $2(a_n a_{n+1})^{\frac{1}{2}} < a_n + a_{n+1}$, $a_{n+1} < (a_n a_{n+1})^{\frac{1}{2}}$ if $a_{n+1} < a_n$
17. C when $|x| < 3/2$.
18. C when $|x| < 1$, $x = 1$, $p + q - \alpha - \beta > 1$;
$x = -1$, $p - q - \alpha - \beta > 0$
19. C when $|x| < 8$. 20. C when $|x| < 4$.

21. $f(n) = n^2\displaystyle\int_1^n \frac{x\,dx}{x^4 + n^4} + O\left(\frac{n^2}{1 + n^4}\right)$ 22. Consider $\displaystyle\int_0^n \log\left(2 - \frac{x}{n}\right)dx$
23. Use *Example 22.*
24. Use *Tannery's Theorem for Series.* (*Bromwich, Infinite Series*, § 49)

25. Take $\displaystyle\int_1^n \frac{dx}{\sqrt{(n^2 + nx)}}$.

26. $\sum_1^n = \frac{3}{2}S_{4n} + \frac{1}{2}S_{2n} - \frac{1}{2}S_n + \frac{1}{2} + \dfrac{1}{12(2n+1)} - \dfrac{1}{6(4n+1)}$ where $S_r = \sum_1^r 1/r$
27. $\sum_1^n = \frac{3}{2}S_{4n} + 2S_{2n} + \frac{3}{2}S_n + \frac{7}{6} + \dfrac{1}{3(2n+1)} - \dfrac{8}{3(4n+1)}$ where $S_r = \sum_1^r 1/r$
28. *Bromwich, Infinite Series*, § 32. 29. C 30. $C\ (p > 1)$ $D\ (p < 1)$
31. $C\ (\lambda > 1)$, $D\ (\lambda < 1)$
32. *Lemaire, Bull. des Sci. Math.*, 20, 1896, 286.
33. Use test of *Example 32.*
34. Use *Example 32*, for the sum, expand $(1 - 2hz + h^2)^{-\frac{1}{2}}$ in powers of h, where $h = x - y$, $z = (x - y)(x - y)$, and use Leibniz's Theorem for the nth derivative of $(z - 1)^n(z + 1)^n$.
36. C
37. Abs. and unif. C in the intervals $k_1 < c < s_1 < 0 < s_2 < c < k_2$
38. C 39. C 40. C
41. D except when $\theta = m\pi$. 42. C when $\alpha < \gamma$. 43. D
44. If T_r is the sum of r terms, $T_m = S_{2m_1} - \frac{1}{2}S_{m_1} - \frac{1}{2}S_{m_2}$, where $m = 2^{2n+1} - 1$, $m_1 = \frac{1}{2}(2^{2n+2} - 1)$, $m_2 = \frac{1}{2}(2^{2n} - 1)$; $T_{m'} = \frac{1}{2}S_{\frac{1}{2}m'} - \frac{1}{2}S_{\frac{1}{4}m'}$ where $m' = 2^{2n} - 1$ and $S_r = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}$; also $T_m > T_r > T_{m'}$ when $m > r > m'$. Thus $T_m \to \frac{3}{2}\log 2$, $T_{m'} \to \frac{1}{2}\log 2$.
45. $0\ (x = 0)$; $1\ (x > 0)$; unif. C in $0 < \varepsilon < x$.
46. $0\ (x > 0)$; unif. C for $0 < \varepsilon < x$.
47. $0\ (x < 1)$; $\infty\ (x > 1)$. unif. C in an interval not containing $x = 1$.

48. 0, unif. C in $0 < x$.
49. a $(x = 0)$, 0 $(x \neq 0)$; unif. C for $0 < s < x$.
50. 0 $(x = 0)$, $\frac{1}{2}\pi$ $(x \neq 0)$; unif. C for $0 < s < x$.
51. $0, 0$ 52. $0, 0$ 53. $\frac{1}{2}, 0$ 54. $\frac{1}{2}, 0$
55. $x > 1 + s > 1$ 56. $-1 + s < x$
57. Any interval that excludes $m\pi$ 58. Any finite interval of x.
59. When $|c| < 1$, all x. 60. $|x| < 1 - s < 1$
61. $0 < s < x$, all p; $0 < x$, $p > 1$ 62. $0 < s < x$, all p; $0 < x$, $p > 0$
64. If y is the series, then $(1 - x^2)y' - xy = 1$, i.e. $\dfrac{d}{dx}\{y\sqrt{(1 - x^2)}\} = (1 - x^2)^{-\frac{1}{2}}$.
65. Integrate $1 - \dfrac{x^2 \cos^2 \theta}{2!} + \dfrac{x^4 \cos^4 \theta}{4!} - \dots$

66. Integrate the power series for $\dfrac{1}{x}\log(1 + x)$.

67. Integrate the series for $\log(1 - 2a \cos x + a^2)$ in powers of a, when $|a| < 1$; when $|a| > 1$, take $a' = 1/a$.

68. Take the power series for $(1 - x^m)^{-1}$, when $|x| < 1$, $|t| < 1$; integrate and use Abel's Theorem for $x = 1$ or -1.

69. Use previous example with $m = 3$, $\alpha = 2$.
73. C when $nx - 1$ is never zero $(\alpha \neq 0)$. 74. D except when $\alpha = \beta$.
75. C all finite x. 76. C all x. 77. C all x. 78. C all x
79. C $(|x| > 1)$; $D(|x| < 1)$ 80. C except when $x = \pm 1$.
85. Take $\sin^5 n\theta = \frac{5}{8} \sin n\theta - \frac{1}{16} \sin 5n\theta$.
86. *See Titchmarsh, Theory of Functions*, 2.53.
89. *Goursat, Cours d'Analyse, III*, 508. 103. C all α. 104. C, $a > \frac{1}{2}$
105. C, $\alpha > -1$, $\gamma > -1$ and (i) $\beta > 0$ or (ii) $\beta = 0$, $\delta > \alpha + 1$
106. C (Dirichlet's Test) 107. C 108. D 109. C, $p > 0$
110. C, $p > 0$ 111. C, $p > 0$ 112. C, $p > 0$
113. C, $\alpha > -1$, $\gamma > \alpha$ 114-17. C 118. C, $1 - m < p < 1 + m$
119. C, $a > 0$ 120. C, $a > -1$ 121. C, $|a| < 1$ 122. C
123. C, $|a| < 1$ 124. C, $|\alpha| + |\beta| < 1$ 125. C, $|a| < 1$
126. C 127. D 128. C

129. $\displaystyle\int_{x_1}^{x_2} f(x)dx > (x_2 - x_1)f(x_2)$ and therefore if $\lim xf(x) \neq 0$, the integral is not convergent. $\int xf'(x)dx = xf(x) - \int f(x)dx$

130. Divide the interval at $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, . . . and change the variable in each interval to obtain 0, $\frac{1}{2}\pi$ as the limits of integration in each. Use the relation

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_1^\infty (-1)^n\left(\frac{1}{x - n\pi} + \frac{1}{x + n\pi}\right)$$

133. *See Example 130* and use the relation

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_1^\infty \left(\frac{1}{(x - n\pi)^2} + \frac{1}{(x + n\pi)^2}\right)$$

136. Integrate $\displaystyle\int_0^\infty \frac{a\,dx}{x^2 + a^2} = \frac{\pi}{2}$ $(a > 0)$.

137. Integrate $\displaystyle\int_0^\infty \frac{dx}{(1 + a^2x^2)(1 + b^2x^2)} = \frac{\pi}{2(a + b)}$ first with regard to a, then with regard to b and put $a = b = 1$.

138. Integrate $\displaystyle\int^\infty \frac{ab\,dx}{(x^2 + a^2)(x^2 + b^2)}$ as in *Example 137*.

139. $\int_0^\infty \frac{1}{x}\sin ax \cos bx \cos cx \, dx = \frac{1}{2}\pi \ (a > b + c)$ and $\frac{1}{4}\pi \ (a = b + c)$. Integrate first with regard to b and then with regard to c.

140. $\int_0^\infty \frac{\cos ax \sin x \, dx}{x} = \frac{1}{2}\pi \ (0 < a < 1)$ and $0 \ (a > 1)$. Integrate with respect to a.

141. $\int_0^\infty \frac{\cos ax \sin^2 x}{x^2} dx = \frac{\pi}{4}(2 - a) \ (0 < a < 2)$ and $0 \ (a > 2)$ (see *Example 140*). Integrate with respect to a.

142. Use $\int_0^\infty \frac{\cos ax \sin^3 x}{x^3} dx = \frac{\pi}{8}(3 - a^2)(0 < a < 1)$; $\frac{\pi}{16}(3 - a)^2 (1 < a < 3)$; $0 \ (a > 3)$ from previous example.

143. Integrate the series for $\frac{1}{x}\tan^{-1} x$. Substitution of $x = \tan \frac{1}{2}\theta$ gives $\int \frac{\theta}{\sin \theta} d\theta$ and integration by parts of the latter gives $\int \log \cot \frac{1}{2}\theta \, d\theta$.

144. Use the result $\frac{d}{dx}[2\sqrt{(1 - x)}\{2 - \log(1 - x)\}] = \{\log(1 - x)\}/(1 - x)^{\frac{1}{2}}$.

145. Integration of $\sum_0^\infty (-1)^n x^n \log x$ term by term is valid since $\int_0^1 |\log x| \left(\frac{1}{1 - x}\right) dx$ converges.

146. $x^x = \sum_0^\infty \frac{x^n}{n!}(\log x)^n$ when $x > 0$ and $x^x \to 1$ when $x \to 0+$. Also $|x \log x| < e^{-1}$ for all x in $0 < x \leq 1$.

147. $\int_0^1 x^a \, dx = (a + 1)^{-1}$ for $a > -1 + \delta > -1$. Integrate from 0 to a.

160. Use the result $\frac{d}{dx}\left\{\left(\frac{a}{3x^3} + \frac{1}{x}\right)\cos(x^3 - ax)\right\} - \frac{a^2}{3x^3}\sin(x^3 - ax)$

$$= 3x \sin(x^3 - ax) - \left(\frac{1}{x^2} + \frac{a}{x^3}\right)\cos(x^3 - ax).$$

161. The integrand is $\frac{k}{x}(e^{-ax} - e^{-bx}) = \frac{d}{dx}\left(-\frac{e^{-ax} - e^{-bx}}{x}\right)$.

169. Take $t = x + n$. **174.** $(p + 1)/(p + q + 2)$ **175.** $\frac{1}{2}$

176. 0 **177.** 0 **178.** 0

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CHAPTER XII

BERNOULLIAN POLYNOMIALS. GAMMA AND BETA FUNCTIONS.

12. The Bernoullian Numbers and Polynomials. The function $\dfrac{1}{e^t - 1}$ may be expanded in a series of rational functions as follows:

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \sum_1^\infty \frac{2t}{t^2 + 4n^2\pi^2}, \quad (\textit{See § 11.33.})$$

the only infinities being poles at $t = \pm 2n\pi i$, $t = 0$.

If $|t| < 2\pi$, each of the terms in the series on the right can be expanded in powers of t and the double series is absolutely convergent since $\sum_1^\infty \dfrac{1}{(4n^2\pi^2 - |t|^2)}$ is a convergent series (of positive terms). Thus we may write

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + B_1\frac{t}{2!} - B_2\frac{t^3}{4!} + B_3\frac{t^5}{6!} - \ldots \quad (|t| < 2\pi)$$

where B_1, B_2, B_3, . . . are called the *Bernoullian Numbers*.

Since
$$\frac{1}{t^2 + 4n^2\pi^2} = \frac{1}{(2n\pi)^2} - \frac{t^2}{(2n\pi)^4} + \frac{t^4}{(2n\pi)^6} - \ldots$$

then
$$B_1 = \frac{1}{\pi^2}\sum_1^\infty \frac{1}{n^2}; \; B_2 = \frac{3}{\pi^4}\sum_1^\infty \frac{1}{n^4}, \ldots, \; B_r = \frac{2(2r)!}{(2\pi)^{2r}}\sum_1^\infty \frac{1}{n^{2r}}.$$

Notes. (i) We may verify that $\dfrac{t}{e^t - 1} + \frac{1}{2}t$ is an *even* function of t by noting that $\dfrac{-t}{e^{-t} - 1} - \frac{1}{2}t = \dfrac{t}{e^t - 1} + \frac{1}{2}t$.

(ii) More than one notation has been used for the Bernoullian Numbers. The Bernoullian Number B_n is sometimes taken to be the coefficient of $t^n/n!$ in the expansion of $t/(e^t - 1)$. If this be denoted by B_n', we have $B_0' = 1$, $B_1' = -\frac{1}{2}$, $B_2' = B_1$, $B_3' = 0$, . . .

$$B_{2m}' = (-1)^{m-1}B_m, \; B_{2m+1}' = 0 \; (m > 0).$$

It has also been taken as the numerical coefficient of $t^n/(n + 1)!$ in the expansion of $(e^t - 1)^{-1} - t^{-1} + \frac{1}{2}$, and if this be denoted by B_n'', we have $B''_1 = B_1$, $B''_2 = 0$, $B_3'' = B_2$, $B_4'' = 0$, . . ., $B_{2m-1}'' = B_m$, $B_{2m}'' = 0$.

(iii) The values of the lower Bernoullian Numbers may be found directly from the definition, i.e. from the identity

$$\left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots\right)\left(1 - \frac{t}{2} + B_1\frac{t^2}{2!} - B_2\frac{t^4}{4!} + B_3\frac{t^6}{6!} - \ldots\right) = 1$$

Thus it will be found that

$$B_1 = \frac{1}{6},\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ B_4 = \frac{1}{30},\ B_5 = \frac{5}{66},\ B_6 = \frac{691}{2730},\ B_7 = \frac{7}{6},$$

$$B_8 = \frac{3617}{510},\ B_9 = \frac{43867}{798},\ B_{10} = \frac{174611}{330}.$$

Examples. (i) Expand $\frac{1}{2}z\cot\frac{1}{2}z$ in powers of z.

$$\tfrac{1}{2}z\cot\tfrac{1}{2}z = \tfrac{1}{2}iz\frac{e^{iz}+1}{e^{iz}-1} = \frac{iz}{2}\left(1+\frac{2}{e^{iz}-1}\right) = \frac{iz}{2}\left(\frac{2}{iz}+\sum_1^\infty(-1)^{r-1}2B_r\frac{(iz)^{2r-1}}{(2r)!}\right)$$

$$= 1 - B_1\frac{z^2}{2!} - B_2\frac{z^4}{4!} - \dots \quad (|z| < 2\pi).$$

Other expansions of a similar type will be found in *Examples XII, 1–6.*

(ii) Show that $B_r = \dfrac{4r}{(2\pi)^{2r}}\displaystyle\int_0^\infty \frac{t^{2r-1}\,dt}{e^t-1}$.

By contour integration $\displaystyle\int_0^\infty \frac{\sin at}{e^t-1}\,dt = \frac{\pi}{2}\frac{e^{2\pi a}+1}{e^{2\pi a}-1} - \frac{1}{2a}$ *(see § 10.86).*

The integral obtained by differentiating the integrand on the left a finite number of times with regard to a is uniformly convergent for all a since the integral $\displaystyle\int_0^\infty \frac{t^r\,dt}{e^t-1}$ is convergent ($r > 0$). Equating coefficients of a^{2r-1} on both sides, we find that

$$\frac{(-1)^{r-1}}{(2r-1)!}\int_0^\infty \frac{t^{2r-1}\,dt}{e^t-1}$$

is the coefficient of a^{2r-1} in $\dfrac{\pi}{2}\left(1+\dfrac{2}{e^{2\pi a}-1}-\dfrac{1}{a\pi}\right)$

i.e. $$B_r = \frac{4r}{(2\pi)^{2r}}\int_0^\infty \frac{t^{2r-1}\,dt}{e^t-1}.$$

B_r is also equal to $\dfrac{4r}{(2\pi)^{2r}}\displaystyle\int_0^1 \frac{(\log 1/t)^{2r-1}\,dt}{1-t}$ (writing t for e^{-t}).

These formulae may also be obtained by expanding $(e^t-1)^{-1}$ in powers of e^{-t} and using the theorems of *§ 11.58.*

It follows from the above that $\displaystyle\sum_1^\infty \frac{1}{n^{2r}} = \frac{1}{(2r-1)!}\int_0^\infty \frac{t^{2r-1}\,dt}{e^t-1}$.

12.01. Bernoullian Polynomials. Definition. The function $t\dfrac{e^{zt}-1}{e^t-1}$ may also be expanded in powers of t in the interval $|t| < 2\pi$. The coefficient of t^n is obviously a *polynomial* in z of degree n, and we may write

$$t\frac{e^{zt}-1}{e^t-1} = \sum_1^\infty \phi_n(z)\frac{t^n}{n!}$$

where $\phi_n(z)$ is called the *Bernoullian Polynomial of degree n.*

12·02 $\phi_n(z) = z^n - \frac{1}{2}nz^{n-1} + {}^nC_2B_1z^{n-2} - {}^nC_4B_2z^{n-4} + \dots$ The last term is $(-1)^{\frac{1}{2}n}\frac{1}{2}n(n-1)B_{\frac{1}{2}n-1}z^2$ if n is even (>2) and is $(-1)^{\frac{1}{2}(n+1)}nB_{\frac{1}{2}(n-1)}z$ if n is odd (>1).

For $\sum_1^\infty \phi_n(z)\frac{t^n}{n!} = \left(zt + \frac{z^2t^2}{2!} + \frac{z^3t^3}{3!} + \dots\right)\left(1 - \frac{1}{2}t + B_1\frac{t^2}{2!} - B_2\frac{t^4}{4!} + \dots\right)$

and the result follows by equating the coefficients of t^n.

We find

$\phi_1 = z$; $\phi_2 = z(z-1)$; $\phi_3 = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z = z(z-\frac{1}{2})(z-1) = \frac{1}{2}\phi_2\phi_2'$;
$\phi_4 = z^4 - 2z^3 + z^2 = z^2(z-1)^2 = \phi_2^2$.
$\phi_5 = z^5 - \frac{5}{2}z^4 + \frac{5}{3}z^3 - \frac{1}{6}z = z(z-\frac{1}{2})(z-1)\{z(z-1) - \frac{1}{3}\} = \frac{1}{2}\phi_2\phi_2'(\phi_2 - \frac{1}{3})$;
$\phi_6 = z^6 - 3z^5 + \frac{5}{2}z^4 - \frac{1}{2}z^2 = z^2(z-1)^2\{z(z-1) - \frac{1}{2}\} = \phi_2^2(\phi_2 - \frac{1}{2})$.

Corollary. $\phi_n(0) = 0$ for all values of n.

Note. The general Bernoullian Polynomial $B_n^{m}(z)$ of order m and degree n is defined to be the coefficient of $\frac{t^n}{n!}$ in the expansion of $\frac{t^me^{zt}}{(e^t-1)^m}$. Thus if we denote the polynomial of the first order by $B_n(z)$, we have

$$\sum_1^\infty \phi_n(z)\frac{t^n}{n!} = \sum_0^\infty B_n(z)\frac{t^n}{n!} - 1 + \frac{1}{2}t + \sum_1^\infty (-1)^nB_n\frac{t^{2n}}{(2n)!}$$

i.e. $B_0(z) = 1$; $B_1(z) = \phi_1(z) - \frac{1}{2}$; $B_2(z) = \phi_2(z) + B_1$; . . . ;
$B_{2m+1}(z) = \phi_{2m+1}(z)$, $B_{2m}(z) = \phi_{2m}(z) + (-1)^{m-1}B_m$.

12·03 $\phi_n(z+1) - \phi_n(z) = nz^{n-1}$. This *difference* relation is obtained by equating the coefficients of t^n in the identity

$$t\frac{e^{(z+1)t} - 1}{e^t - 1} - t\frac{e^{zt} - 1}{e^t - 1} = te^{zt}$$

12·04 $\phi_n(z) = (-1)^n\phi_n(1-z)$ $(n > 1)$. For

$$\sum_1^\infty \phi_n(1-z)\frac{(-t)^n}{n!} = -t\frac{e^{-t(1-z)} - 1}{e^{-t} - 1} = t\frac{e^{zt} - 1}{e^t - 1} - t = \sum_1^\infty \phi_n(z)\frac{t^n}{n!} - t.$$

Corollary: $\phi_n(1) = \phi_n(0) = 0$, all $n > 1$; $\phi_n(\frac{1}{2}) = 0$, n *odd* (>1). Thus $\phi_n(z)$ contains the factor $z(z-1)$ $(n > 1)$ and the factor $z(z-1)(z-\frac{1}{2})$, n *odd* (>1).

12·05 $\phi_{2m}' = 2m\phi_{2m-1}$ $(m \geq 1)$;
$$\phi_{2m+1}' = (2m+1)\{\phi_{2m} + (-1)^{m-1}B_m\} \ (m \geq 1)$$

For $\sum_1^\infty \phi_n'(z)\frac{t^n}{n!} = \frac{t^2e^{zt}}{e^t - 1} = t^2\frac{e^{zt} - 1}{e^t - 1} + \frac{t^2}{e^t - 1}$

$$= \left\{\sum_2^\infty \phi_n(z)\frac{t^{n+1}}{n!}\right\} + \left\{\sum_1^\infty (-1)^{m-1}B_m\frac{t^{2m+1}}{(2m)!}\right\} + (z - \tfrac{1}{2})t^2 + t$$

i.e. $\phi_1'(z) = 1$; $\phi_2' = 2z - 1$; $\phi_3' = 3(\phi_2 + B_1)$; $\phi_4' = 4\phi_3$; . . .

Corollary 1. ϕ_n contains the factor $z^2(z-1)^2$ when n is even (>2).
For $\phi_{2m}'(0) = 2m\phi_{2m-1}(0) = 0$; $\phi_{2m}'(1) = 2m\phi_{2m-1}(1) = 0$.

Corollary 2. ϕ_{2m} is of the form $\theta^2P_{m-2}(\theta)$ and ϕ_{2m+1} is of the form $\frac{1}{2}\theta\theta'Q_{m-1}(\theta)$ where θ is $z(z-1)$ and P_r, Q_r are polynomials of degree r in θ.

We have already seen that it is true for ϕ_3, ϕ_4, ϕ_5, ϕ_6.

Assume that it is true for ϕ_{2m-1}, ϕ_{2m}.

Then $\phi'_{2m+1} = (2m+1)(\theta^2 P_{m-2} + (-1)^{m-1}B_m)$

$= \text{Even polynomial in } \theta' \text{ since } \theta'^2 = 4\theta + 1$

i.e. $\phi_{2m+1} = \text{Odd polynomial in } \theta' \text{ since } \theta'' = 2, \text{ and } \theta' \text{ is a factor,}$

$= \frac{1}{2}\theta\theta' Q_{m-1}(\theta) \text{ since } \theta\theta' \text{ must be a factor.}$

Also $\phi'_{2m+2} = (2m+2)\frac{1}{2}\theta\theta' Q_{m-1}(\theta)$

i.e. $\phi_{2m+2} = \theta^2 P_{m-2}(\theta) \text{ since } \theta^2 \text{ must be a factor.}$

Thus $\phi_7 = \frac{1}{2}\theta\theta'(\theta^2 - \theta + \frac{1}{3})\,;\ \phi_8 = \theta^2(\theta^2 - \frac{4}{3}\theta + \frac{2}{3}).$

Corollary 3. If we consider real values of z, ϕ_{2m} does not vanish between 0, 1, and ϕ_{2m+1} does not vanish between 0, $\frac{1}{2}$ and between $\frac{1}{2}$, 1. The result is true for ϕ_3, ϕ_4. Assume that it is true up to ϕ_{2m-1}.

Now $\phi'_{2m} = 2m\phi_{2m-1}$ which vanishes *only* at 0, $\frac{1}{2}$, 1; thus ϕ_{2m} has *one* maximum (or one minimum) between 0, 1 and we know this to be at $z = \frac{1}{2}$. But $\phi_{2m}(0) = \phi_{2m}(1) = 0$ and therefore ϕ_{2m} cannot vanish between 0, 1.

It follows that the equation $\phi_{2m} - (-1)^{m-1}B_m = 0$ has one root at most between 0, $\frac{1}{2}$, and one at most between $\frac{1}{2}$ and 1; i.e. ϕ_{2m+1} has one maximum (or minimum) at most between 0, $\frac{1}{2}$ and one at most between $\frac{1}{2}$ and 1. But $\phi_{2m+1}(0) = \phi_{2m+1}(\frac{1}{2}) = \phi_{2m+1}(1) = 0$; and therefore

FIG. 1

ϕ_{2m+1} has one maximum (or minimum) at least within these intervals. Thus ϕ_{2m+1} does not vanish within $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$.

The graphs of ϕ_n for $n = 2, 3, 4, 5, 6$ are shown in *Fig.* 1.

Note. $|\phi_{2m}(\frac{1}{2})| > B_m$: $\phi_{2m}(z)$ has the same sign as $(-1)^m$ in (0, 1).

12.06. $1^n + 2^n + 3^n + \ldots + r^n = \dfrac{1}{n+1}\phi_{n+1}(r+1)\ (n > 0).$

This follows from the equations:

$$\sum_1^r \{\phi_{n+1}(r+1) - \phi_{n+1}(r)\} = (n+1)\sum_1^r r^n \text{ and } \phi_{n+1}(1) = 0.$$

Thus, for example,

$$1^4 + 2^4 + 3^4 + \ldots + r^4 = \tfrac{1}{30}r(r+1)(2r+1)(3r^2+3r-1)\,;$$

$$1^5 + 2^5 + 3^5 + \ldots + r^5 = \tfrac{1}{12}r^2(r+1)^2(2r^2+2r-1).$$

12.07. *The Euler-Maclaurin Summation Formula* (*for a Polynomial*). We have proved in the last paragraph that

$1^n + 2^n + \ldots + (r-1)^n$

$$= \frac{r^{n+1}}{n+1} - \frac{1}{2}r^n + \frac{nB_1}{2!}r^{n-1} - \frac{n(n-1)(n-2)}{4!}B_2 r^{n-3} + \ldots$$

where the last term involves r^2 (when n is odd) and r (when n is even).

If $f(x)$ is any polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_n$, then

$$\sum_{x=1}^{x=r-1} f(x) = a_0\sum_1^{r-1} x^n + a_1\sum_1^{r-1} x^{n-1} + \ldots + a_{n-1}\sum_1^{r-1} x + a_n(r-1)$$

$$= a_0\left(\frac{r^{n+1}}{n+1} - \frac{1}{2}r^n + \frac{nB_1}{2!}r^{n-1} - \frac{n(n-1)(n-2)}{4!}B_2r^{n-3} + \ldots\right)$$

$$+ a_1\left(\frac{r^n}{n} - \frac{1}{2}r^{n-1} + \frac{(n-1)B_1}{2!}r^{n-2} - \frac{(n-1)(n-2)(n-3)}{4!}B_2r^{n-4} + \ldots\right)$$

$$+ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$+ a_{n-1}\left(\frac{r^2}{2} - \frac{1}{2}r\right) + a_n(r-1)$$

$$= \int_0^r f(x)dx - \frac{1}{2}\{f(r) + f(0)\} + \frac{B_1}{2!}\{f'(r) - f'(0)\} - \frac{B_2}{4!}\{f'''(r) - f'''(0)\} + \ldots$$

i.e. $$\sum_{x=1}^{x=r-1} f(x) = \int^r f(x)dx - \frac{1}{2}f(r) + \frac{B_1}{2!}f'(r) - \frac{B_2}{4!}f'''(r) + \ldots + C$$

the constant being adjusted so that $(r-1)$ is a factor.

Example. Find $\sum_{x=1}^{r-1}(2x^3 - 3x^2 + 1)$.

The sum is $(\frac{1}{2}r^4 - r^3 + r) - \frac{1}{2}(2r^3 - 3r^2) + \frac{1}{6}\cdot\frac{1}{2}(6r^2 - 6r) + C$
$= \frac{1}{2}r^4 - 2r^3 + 2r^2 + \frac{1}{2}r - 1$, (making $(r-1)$ a factor)

12.1. The General Euler-Maclaurin Summation Formula. In § *11.31*, we have shown that

$$\phi^n(0)\{f(a+h) - f(a)\} = \sum_{m=1}^{m=n}(-1)^{m-1}h^m\{\phi^{(n-m)}(1)f^{(m)}(a+h) - \phi^{(n-m)}(0)f^{(m)}(a)\} + (-1)^nh^{n+1}\int_0^1 \phi(t)f^{(n+1)}(a+h-th)dt$$

where $\phi(t)$ is a polynomial of degree n and $f(t)$ is analytic on the line joining $t=0$ to $t=1$.

If $\phi(t)$ is taken to be $\phi_{2n}(t)$ (the Bernoullian Polynomial of degree $2n$), we have

$$\phi_{2n}(t) = t^{2n} - nt^{2n-1} + \frac{(2n)!}{2!(2n-2)!}B_1t^{2n-2} - \frac{(2n)!}{4!(2n-4)!}B_2t^{2n-4} + \ldots + (-1)^n\frac{(2n)!}{(2n-2)!2!}B_{n-1}t^2.$$

Therefore

$$\phi_{2n}^{(2n)}(0) = (2n)!\,;\ \phi_{2n}^{(2n-1)}(0) = -\frac{(2n)!}{2}\,;\ \phi_{2n}^{(2n-2)}(0) = \frac{(2n)!}{2!}B_1\,;$$

$$\ldots\ \phi_{2n}^{(2n-2r-1)}(0) = 0\,;\ \phi_{2n}^{(2n-2r)}(0) = (-1)^{r-1}\frac{(2n)!}{(2r)!}B_r\,;\ \ldots$$

$$\phi_{2n}''(0) = (-1)^{n-2}\frac{(2n)!}{(2n-2)!}B_{n-1}\,,\ \phi_{2n}'(0) = 0\,;\ \phi_{2n}(0) = 0$$

Also, since $\phi_{2n}(t) = \phi_{2n}(1-t)$, we have

$$\phi_{2n}(1) = 0\,;\ \phi_{2n}'(1) = 0\,;\ \phi_{2n}'''(1) = 0\,;\ \ldots\,;\ \phi_{2n}^{(2n-3)}(1) = 0\,;$$

$$\phi_{2n}^{(2n-1)}(1) = \frac{(2n)!}{2}\,;\ \phi_{2n}^{(2r)}(1) = \phi_{2n}^{(2r)}(0),\ (r = 1 \text{ to } n).$$

Thus (writing $2n$ for n in Darboux's formula),

$$f(a+h) - f(a) = \sum_{m=1}^{2n}(-1)^{m-1}\frac{h^m}{(2n)!}\{\phi_{2n}^{(2n-m)}(1)f^{(m)}(a+h) - \phi_{2n}^{(2n-m)}(0)f^{(m)}(a)\} + \frac{h^{2n+1}}{(2n)!}\int_0^1 \phi_{2n}(t)f^{(2n+1)}(a+th)dt.$$

The series of $2n$ terms on the right is

$$\frac{h}{2}\{f'(a+h) + f'(a)\} - \frac{h^2B_1}{2!}\{f''(a+h) - f''(a)\} + \frac{h^4B_2}{4!}\{f^{(iv)}(a+h) - f^{(iv)}(a)\}$$

$$\ldots (-1)^{m-1}\frac{h^{2m}B_m}{(2m)!}\{f^{(2m)}(a+h) - f^{(2m)}(a)\}$$

$$+ \ldots (-1)^{n-2}\frac{h^{2n-2}}{(2n-2)!}B_{n-1}\{f^{(2n-2)}(a+h) - f^{(2n-2)}(a)\}$$

i.e.
$$f(a+h) - f(a) = \tfrac{1}{2}h\{f'(a+h) + f'(a)\} - \sum_1^{n-1}(-1)^{m-1}\frac{h^{2m}B_m}{(2m)!}\{f^{(2m)}(a+h) - f^{(2m)}(a)\} + \frac{h^{2n+1}}{(2n)!}\int_0^1 \phi_{2n}(t)f^{(2n+1)}(a+th)dt.$$

Denote $f'(x)$ by $F(x)$ and take $h = 1$.

Then
$$\int_a^{a+1} F(x)dx = \tfrac{1}{2}\{F(a+1) + F(a)\} - \sum_1^{n-1}(-1)^{m-1}\frac{B_m}{(2m)!}\{F^{(2m-1)}(a+1) - F^{(2m-1)}(a)\} + \frac{1}{(2n)!}\int_0^1 \phi_{2n}(t)F^{(2n)}(a+t)dt.$$

Similarly we may obtain formulae by putting $a + r$ for a; and if we add the above equation to those for $r = 1, 2, \ldots, s - 1$ we find

$$\int_a^{a+s} F(x)dx = \tfrac{1}{2}\{F(a+s) + F(a)\} + F(a+1) + F(a+2) + \ldots$$

$$+ F(a+s-1) - \sum_{m=1}^{n-1}\frac{(-1)^{m-1}}{(2m)!}B_m\{F^{(2m-1)}(a+s) - F^{(2m-1)}(a)\} + R_n$$

where R_n

$$= \frac{1}{(2n!)}\int_0^1 \phi_{2n}(t)\{F^{(2n)}(a+t) + F^{(2n)}(a+t+1) + \ldots + F^{(2n)}(a+t+s-1)\}\,dt$$

or

$$F(a) + F(a+1) + \ldots + F(a+s)$$

$$= \int_a^{a+s} F(x)dx + \tfrac{1}{2}\{F(a+s) + F(a)\}$$

$$+ \sum_1^{n-1}\frac{(-1)^{m-1}}{(2m)!}B_m\{F^{(2m-1)}(a+s) - F^{(2m-1)}(a)\} - R_n$$

The above formula may be used either to determine an approximate value of $\int_a^b F(x)dx$ or to determine an approximate value of the series $\overset{s}{\underset{0}{\Sigma}}F(a+r)$.

Suppose that $F^{(2n)}(x)$, $F^{(2n+2)}(x)$ have a constant sign in $(a, a+s)$ and that these signs are the same.

$$R_{n+1} = \frac{1}{(2n+2)!}\int_0^1 \phi_{2n+2}(t)\left\{\sum_0^{s-1}F^{(2n+2)}(a+t+r)\right\}dt$$

and

$$R_n = \frac{1}{(2n)!}\int_0^1 \phi_{2n}(t)\left\{\sum_0^{s-1}F^{(2n)}(a+t+r)\right\}dt.$$

But ϕ_{2n}, ϕ_{2n+2} have constant signs in $(0, 1)$ and these signs are opposite; so that R_n, R_{n+1} have opposite signs.

But $\qquad R_n - R_{n+1} = \dfrac{(-1)^n}{(2n)!}B_n\{F^{(2n-1)}(a+s) - F^{(2n-1)}(a)\}$

i.e. $\qquad |R_n| < \dfrac{B_n}{(2n)!}|F^{(2n-1)}(a+s) - F^{(2n-1)}(a)|$

The series giving $\overset{s}{\underset{0}{\Sigma}}F(a+r)$ is therefore asymptotic.

Writing x for $a + s$, we may take the summation formula as

$$F(a) + F(a+1) + \ldots + F(x) \sim \int^x F(x)dx + \tfrac{1}{2}F(x)$$

$$+ \sum_1^{\infty}\frac{(-1)^{m-1}}{(2m)!}B_m F^{(2m-1)}(x) + C$$

where C is independent of x.

Examples. (i) Let $F(x) = \frac{1}{x}$ and $n = 1$. Take $x = n$; then

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \sim \log n + \frac{1}{2n} - \frac{B_1}{2n^2} + \frac{B_2}{4n^4} - \frac{B_3}{6n^6} + \ldots$$

where $\gamma = \lim\limits_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right)$, i.e. is *Euler's Constant*.

Let $n = 10$; $\log 10 = 2{\cdot}30258509$.

$\sum\limits_1^{10}\frac{1}{r} = 2{\cdot}92896825\ldots$; $\frac{1}{2n} = 0{\cdot}05$;

$$\frac{B_1}{2n^2} = 0{\cdot}00083333\ldots, \quad \frac{B_2}{4n^4} = 0{\cdot}00000083\ldots$$

$\frac{B_3}{6n^6} < 10^{-8}$, i.e. $\gamma = 0{\cdot}5772157$ (correct to 7 places).

(ii) Let $F(x) = \frac{1}{x^2}$ and we find similarly that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \sim C - \frac{1}{n} + \frac{1}{2n^2} - \frac{B_1}{n^3} + \frac{B_2}{n^5} - \frac{B_3}{n^7} + \ldots$$

where $C = \sum\limits_1^{\infty}\frac{1}{r^2}\left(= \frac{\pi^2}{6}\right)$.

Taking $n = 10$, we find that $\frac{\pi^2}{6} = 1{\cdot}644934\ldots$

(iii) Find $\sum\limits_1^{\infty}\frac{1}{n^3}$.

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3} = C - \frac{1}{2n^2} + \frac{1}{2n^3} + \sum_1 \frac{(-1)^m(2m+1)B_m}{2n^{2m+2}}$$

$\sum\limits_1^{10}\frac{1}{n^3}$	$= 1{\cdot}19753198567$	$\frac{1}{2n^3}$	$= 0{\cdot}00050000000$
$\frac{1}{2n^2}$	$= \quad 50000000$	$\frac{5}{2}B_2 \times 10^{-6} =$	8333
$\frac{3}{2}B_1 \times 10^{-4}$	$= \quad 2500000$		$0{\cdot}00050008333$
$\frac{7}{2}B_3 \times 10^{-8}$	$= \quad 83$		
	$1{\cdot}20255608050$		$C = 1{\cdot}2020569032\ldots$

(iv) Find $\frac{1}{5^3} + \frac{1}{9^3} + \frac{1}{13^3} + \ldots$

Taking $n = 6$,

$$\frac{1}{5^3} + \frac{1}{9^3} + \frac{1}{13^3} + \frac{1}{17^3} + \frac{1}{21^3} + \frac{1}{25^3} = C - \frac{1}{4 \times 25} + \frac{1}{2 \times 25^3} - \frac{4B_1}{25^3}$$

the error being less than $\frac{64B_2}{25^4}$, i.e. $< 10^{-6}$.

$\sum\limits_1^{6}\frac{1}{(4n+1)^3} = 0{\cdot}0655906$, and we find $C = 0{\cdot}074833\ldots$

(v) Find an approximation to $n!$ when n is large.

$$\log(n!) \sim C + \int^n \log x\,dx + \tfrac{1}{2}\log n + \sum_1 \frac{(-1)^{m-1}B_m}{2m(2m-1)}\cdot\frac{1}{n^{2m-1}}$$

$$\sim C + (n + \tfrac{1}{2})\log n - n + \frac{B_1}{1.2\,n} - \frac{B_2}{3.4\,n^3} + \ldots$$

where $\qquad C = \lim\limits_{n\to\infty}\{\log(n!) + n - (n + \tfrac{1}{2})\log n\}.$

The value of this limit may be deduced from the sine product

$$\frac{\sin \pi x}{\pi x} = \lim_{n\to\infty} (1 - x^2)(1 - x^2/2^2) \ldots (1 - x^2/n^2) \quad (\S\ \mathit{11.45}).$$

If $x = \frac{1}{2}$, we have $\dfrac{\pi}{2} = \lim \dfrac{2.2.4.4.6.6 \ldots 2n.2n}{1.3.3.5.5.7 \ldots (2n-1)(2n+1)}$ (*Wallis's Formula*)

$$= \lim \frac{2^{4n}(n!)^4}{(2n+1)\{(2n)!\}^2}$$

Now $e^C = \lim \left\{\dfrac{(n!)e^n}{n^{n+\frac{1}{2}}}\right\} = \lim \left\{\dfrac{(2n)!e^{2n}}{(2n)^{2n+\frac{1}{2}}}\right\}$ (putting $2n$ for n).

Squaring the first form of the limit for e^C and dividing by the second we find

$$e^C = \lim \left\{\frac{(n!)^2 2^{2n+\frac{1}{2}}}{(2n)!n^{\frac{1}{2}}}\right\}.$$

Thus $\dfrac{2e^{2C}}{\pi} = \lim \dfrac{2}{n}(2n+1) = 4$. Thus $e^{2C} = 2\pi$ and $C = \frac{1}{2}\log(2\pi)$, i.e.

$$\log(n!) \sim (n+\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log(2\pi) + \frac{B_1}{1.2.n} - \frac{B_2}{3.4\,n^3} + \ldots \text{ (\textit{Stirling's Series})}$$

Since the exponential series is convergent, we obtain an asymptotic series for $n!$ by expanding e^S in powers of $\dfrac{1}{n}$ where $S = \sum_1 (-1)^{m-1}\dfrac{B_m}{2m(2m-1)n^{2m-1}}$. It will be found that

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{(2\pi n)}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \ldots\right).$$

Note. Since $\phi_n(1) = 0$ $(n > 1)$, the Bernoullian numbers are obtained from the equations

$^nC_2B_1 - {}^nC_4B_2 + \ldots = \frac{1}{2}n - 1$ (last term on the left involves $^nC_{n-2}$ or $^nC_{n-1}$).

Thus $3B_1 = \frac{1}{2}$, $10B_1 - 5B_2 = \frac{3}{2}$, $21B_1 - 35B_2 + 7B_3 = \frac{5}{2}$, ...

$$^{2n+1}C_2B_1 - {}^{2n+1}C_4B_2 + \ldots + (-1)^{n-1}(2n+1)B_n = \tfrac{1}{2}(2n-1).$$

It follows that B_n when expressed in its lowest terms cannot have a denominator greater than $2.3.5.7 \ldots (2n+1)$. A more exact result has, however, been given by Staudt (and Clausen) by which it is shown that

$$B_n = \text{Integer} + (-1)^n\left(\frac{1}{2} + \Sigma \frac{1}{2m+1}\right)$$

where the summation extends to every value of m which is a factor of n (including 1 and n) and is such that $(2m + 1)$ is prime. (*Rado, Journ. L.M.S. 9, 2, p. 85.*)

Thus $B_4 = -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$; $B_8 = 1 - (\frac{1}{2} + \frac{1}{3} + \frac{1}{17})$;

$B_6 = -1 + (\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13})$.

The theorem may be utilised to determine higher values of B_n.

Thus
$$B_{10} = \frac{2(20!)}{(2\pi)^{20}}\left(1 + \frac{1}{2^{20}} + \frac{1}{3^{20}} + \ldots\right).$$

Using Stirling's Series we find that $\dfrac{2(20!)}{(2\pi)^{20}} = 529{\cdot}11$ (error $< 0{\cdot}01$).

Also $1 + \dfrac{1}{2^{20}} + \dfrac{1}{3^{20}} + \ldots < 1 + \dfrac{1}{2^{19}}$ since the sum is $< 1 + \dfrac{1}{2^{19} - 1}$, i.e. the integral part of B_{10} is 529

But $B_{10} = \text{Integer} + \left(\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{11}\right) = \text{Integer} + \dfrac{41}{330} = 529\dfrac{41}{330}$.

12.2. Gamma Functions. Euler's Definition. The integral $\int_0^\infty e^{-t}t^{x-1}\,dt$ is convergent when x is real and positive and therefore defines a function of x for all x in the interval $0 < \varepsilon < x < G$. It is called the *Gamma Function* and is written $\Gamma(x)$. This is known as Euler's definition and it is in this form that the function frequently occurs in applications.

The above integral is also convergent when x is complex, provided $\mathbf{R}(x) > 0$, so that the function is also defined by the integral for the region $\varepsilon < \mathbf{R}(x) < G$. In a subsequent paragraph we shall give another definition (Weierstrass's) of the Gamma Function which is consistent with the above but which exists over a larger domain.

12.21. The Relation $\Gamma(1 + x) = x\Gamma(x)$ $(x > 0)$. Integration by parts gives

$$\Gamma(1 + x) = -\left(e^{-t}t^x\right)_0^\infty + x\Gamma(x)$$
$$= x\Gamma(x) \quad (x > 0).$$

Thus if n is a positive integer

$$\Gamma(n + 1) = n!\int_0^\infty e^{-t}\,dt = n!$$

so that $\Gamma(1 + x)$ is an extension of the meaning of $n!$ to all values of $x > -1$. In particular $\Gamma(1) = 1$, so that $0!$ may be interpreted as 1. This formula enables us to express $\Gamma(\lambda)$ when $\lambda > 1$ as a multiple of $\Gamma(f)$ where f is the fractional part of λ (assumed not integral).

Example. $\Gamma(5\frac{1}{2}) = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})$.

12.22. The Beta Function. The integral $B(p, q) = \int_0^1 x^{p-1}(1 - x)^{q-1}\,dx$ is convergent if $p > 0$, $q > 0$. It therefore defines a function for the region $p > \varepsilon > 0$, $q > \varepsilon' > 0$, the function being called a *Beta Function*.

For complex p, q the integral is defined for

$$\mathbf{R}(p) > \varepsilon > 0, \quad \mathbf{R}(q) > \varepsilon' > 0.$$

By writing $1 - x$ for x we see that $B(p, q) = B(q, p)$.

Note. The *indefinite* integral of $x^{p-1}(1 - x)^{q-1}$ can be determined theoretically in terms of elementary functions when

(i) p (or q) is an integer, (ii) $p + q$ is an integer and p, q rational.

Case (i) is obvious. For case (ii) take $x = \xi/(1 + \xi)$, $\xi = u^s$, $p = r/s$. The *definite* integral from 0 to 1 should of course be always evaluated in terms of Gamma Functions. (*See* § *12.24 below.*)

12.23. Relations connecting Beta Functions of Different Arguments. By integration by parts or otherwise we may find various relations connecting Beta Functions of different arguments.

Examples. (i) Integration by parts gives

$$B(p, q) = \left\{\frac{x^p}{p}(1 - x)^{q-1}\right\}_0^1 + \frac{q-1}{p}B(p + 1, q - 1) \quad (p > 0, q > 1)$$

i.e.

$$pB(p, q) = (q - 1)B(p + 1, q - 1).$$

Thus when n is an integer (> 0)

$$B(p, n) = \frac{n-1}{p}B(p+1, n-1) = \frac{(n-1)!}{p(p+1)\ldots(p+n-1)}$$

(See also § 12.22, Note.)

(ii) $B(p+1, q) = \int_0^1 x^{p-1}(1-1+x)(1-x)^{q-1}\,dx = B(p, q) - B(p, q+1).$

(iii) Using (i) and (ii) we find

$$pB(p, q+1) = qB(p+1, q) = q\{B(p, q) - B(p, q+1)\}$$

i.e.
$$B(p, q+1) = \frac{q}{p+q}B(p, q).$$

(iv) Prove $B(p, 1-p) = \dfrac{\pi}{\sin p\pi}$ $(0 < p < 1)$.

Put $x = u/(1+u)$ in the integral for $B(p, q)$ and we find

$$B(p, 1-p) = \int_0^\infty \frac{u^{p-1}\,du}{1+u} = \frac{\pi}{\sin p\pi} \text{ (by contour integration, } 0 < p < 1).$$

* **12.24.** $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. $\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

$$\Gamma(p) = \int_0^\infty e^{-t}t^{p-1}\,dt = 2\int_0^\infty e^{-x^2}x^{2p-1}\,dx \text{ (writing } x^2 = t).$$

Therefore $\Gamma(p)\Gamma(q) = 4 \lim\limits_{R\to\infty} \iint_D e^{-x^2-y^2}x^{2p-1}y^{2q-1}\,dx\,dy$ where D is the square determined by $0 < x < R$, $0 < y < R$. The first quadrant of the circle $x^2 + y^2 = R^2$ lies between the square of side R and the square of side $\frac{1}{2}R$. The integrand is positive and the double integral exists when R (and therefore $\frac{1}{2}R$) tends to infinity. Therefore the limit of the double integral over the first quadrant of the circle also exists when $R \to \infty$ and its value is the same as that for the square when $R \to \infty$,

i.e. $\Gamma(p)\Gamma(q) = 4 \lim\limits_{R\to\infty} \iint_{D'} e^{-r^2}r^{2p+2q-1}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\,r\,dr\,d\theta$

where D' is the quadrant and $x = r\cos\theta$, $y = r\sin\theta$,

i.e. $\Gamma(p)\Gamma(q) = \Gamma(p+q)\int_0^1 u^{q-1}(1-u)^{p-1}\,du$ (taking $u = \sin^2\theta$)

$$= \Gamma(p+q)B(p, q).$$

Examples. (i) $\Gamma(p)\Gamma(1-p) = B(p, 1-p) = \pi/(\sin p\pi)$. (§ 12.23 (iv).)

This has been proved only for $0 < p < 1$. It will be shown below that the formula is true for the general Gamma Function (except when p is an integer positive or negative).

(ii) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, since $\{\Gamma(\frac{1}{2})\}^2 = \pi$ and $\Gamma(\frac{1}{2}) > 0$. It may be observed that

$$B(\tfrac{1}{2}, \tfrac{1}{2}) = \int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = 2\int_0^{\pi/2} d\theta = \pi.$$

Thus
$$\int_0^\infty e^{-t}t^{\frac{1}{2}-1}\,dt = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}.$$

12.25. $2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2x)$. (*Reduplication Formula*).

$$\frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})} = B(x, \tfrac{1}{2}) = \int_0^1 \frac{t^{x-1}\,dt}{\sqrt{(1-t)}};$$

$$\frac{\{\Gamma(x)\}^2}{\Gamma(2x)} = B(x, x) = \int_0^1 t^{x-1}(1-t)^{x-1}\,dt.$$

In the latter integral put $v = 2t - 1$; then $t(1-t) = \frac{1}{2}\sqrt{(1-v^2)}$. Thus

$$B(x, x) = \frac{1}{2^{2x-1}}\int_{-1}^{1}(1-v^2)^{x-1}\,dv = \frac{1}{2^{2x-1}}\int_0^1(1-w)^{x-1}w^{-\frac{1}{2}}\,dw \quad (w = v^2)$$

$$= \frac{1}{2^{2x-1}}\,B(x, \tfrac{1}{2})$$

i.e. $$\frac{\{\Gamma(x)\}^2}{\Gamma(2x)} = \frac{1}{2^{2x-1}}\,\frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})} \text{ or } 2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2x).$$

Examples. (i) Express $\Gamma(\frac{1}{6})$ in terms of $\Gamma(\frac{1}{3})$.

$$2^{-\frac{2}{3}}\Gamma(\tfrac{1}{6})\Gamma(\tfrac{2}{3}) = \sqrt{\pi}\,\Gamma(\tfrac{1}{3}); \text{ but } \Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) = 2\pi/\sqrt{3}.$$

Therefore $\Gamma(\frac{1}{6}) = 2^{-\frac{1}{2}}3^{\frac{1}{2}}\pi^{-\frac{1}{2}}\{\Gamma(\frac{1}{3})\}^2$.

(ii) Let $I = \int_0^\infty e^{-ax^\alpha}x^\beta\,dx$ $(\alpha > 0,\ a > 0,\ \beta > -1)$

Take $t = ax^\alpha$, then $I = \dfrac{1}{\alpha a^{(\beta+1)/\alpha}}\displaystyle\int_0^\infty e^{-t}t^{\{(\beta+1)/\alpha\}-1}\,dt = \dfrac{1}{\alpha a^{(\beta+1)/\alpha}}\Gamma\{(\beta+1)/\alpha\}$.

Similarly (by taking $t = ax^{-\alpha}$)

$$\int_0^\infty e^{-a/x^\alpha}x^{-\beta}\,dx = \frac{1}{\alpha a^{(\beta-1)/\alpha}}\Gamma\{(\beta-1)/\alpha\}, \quad (a > 0,\ \alpha > 0,\ \beta > 1).$$

In particular

$$\int_0^\infty e^{-ax^2}x^\beta\,dx = \frac{1}{2a^{\frac{1}{2}(\beta+1)}}\Gamma\{\tfrac{1}{2}(\beta+1)\} \quad (\beta > -1,\ a > 0).$$

$$\int_0^\infty e^{-ax^2}x^{2m}\,dx = \frac{1.3.5\ldots(2m-1)}{2^{m+1}}\,\frac{\sqrt{\pi}}{a^{\frac{1}{2}(2m+1)}} = \frac{(2m)!}{2^{2m+1}(m!)}\cdot\frac{\sqrt{\pi}}{a^{m+\frac{1}{2}}} \quad (a > 0).$$

$$\int_0^\infty e^{-ax^2}x^{2m+1}\,dx = \frac{m!}{2a^{m+1}} \quad (a > 0).$$

(m in the last two integrals being zero or a positive integer).

$$\int_0^\infty e^{-ax^\alpha}\,dx = \frac{1}{\alpha a^{1/\alpha}}\Gamma\left(\frac{1}{\alpha}\right) \quad (\alpha > 0,\ a > 0).$$

(iii) If $I = \int_0^1 \left(\log\frac{1}{x}\right)^\alpha x^\beta\,dx$ $(\alpha > -1,\ \beta > -1)$, we find by taking $x = e^{-t}$, that

$$I = \frac{1}{(1+\beta)^{1+\alpha}}\Gamma(\alpha+1).$$

Similarly $\displaystyle\int_1^\infty (\log x)^\alpha x^{-\beta}\,dx = \frac{1}{(\beta-1)^{\alpha+1}}\Gamma(\alpha+1)$ $(\alpha > -1,\ \beta > 1)$.

Thus $\displaystyle\int_0^1 (\log x)^m\,dx = \pm(m!)$ when m is an integer.

(iv) $\int_a^b (x-a)^{p-1}(b-x)^{q-1}\,dx = (b-a)^{p+q-1}B(p, q).$ $\left(\text{Take } u = \frac{x-a}{b-a}.\right)$

$$\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(x+k)^{p+q}}\,dx = k^{-q}(1+k)^{-p}B(p, q)\ (k > 0)\quad \left(\text{Take } u = \frac{x(1+k)}{x+k}.\right)$$

(v) $\int_0^{\frac{1}{2}\pi} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\{\frac{1}{2}(p+1)\}\Gamma\{\frac{1}{2}(q+1)\}}{2\Gamma\{\frac{1}{2}(p+q+2)\}}\ (p, q > -1).$

This follows at once by the substitution $t = \sin^2\theta$.

In particular,

$$\int_0^{\frac{1}{2}\pi} \tan^p\theta\,d\theta = \frac{\pi}{2\cos\frac{1}{2}p\pi}\ (|p| < 1);\quad \int_0^{\pi/2}\sqrt{(\tan\theta)}d\theta = \int_0^{\pi/2}\sqrt{(\cot\theta)}d\theta = \frac{\pi}{\sqrt{2}}.$$

(vi) $\int_0^1 x^\alpha(1-x^\beta)^\gamma\,dx = \frac{1}{\beta}\frac{\Gamma\left(\frac{\alpha+1}{\beta}\right)\Gamma(\gamma+1)}{\Gamma\left(\frac{\alpha+1}{\beta}+\gamma+1\right)}$ $(\alpha > -1, \beta > 0, \gamma > -1),$
(taking $t = x^\beta$).

In particular

$$\int_0^1 \frac{x^\alpha\,dx}{\sqrt{(1-x^\beta)}} = \frac{\sqrt{\pi}}{\beta}\cdot\frac{\Gamma\left(\frac{\alpha+1}{\beta}\right)}{\Gamma\left(\frac{\alpha+1}{\beta}+\frac{1}{2}\right)} = \frac{2^{\frac{\alpha+1}{\beta}-1}}{\beta}\cdot\frac{\left\{\Gamma\left(\frac{\alpha+1}{\beta}\right)\right\}^2}{\Gamma\left(2\frac{\alpha+1}{\beta}\right)}$$

?

(by the Reduplication Formula).

$$\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = 2^{\frac{2}{n}-1}\frac{\left\{\Gamma\left(\frac{1}{n}\right)\right\}^2}{\Gamma\left(\frac{2}{n}\right)}\quad (n > 0)$$

$2\uparrow\left\{\frac{\alpha+1}{\beta}-1\right\}$

(vii) $\int_0^\infty \frac{x^\alpha\,dx}{1+x^\beta} = \frac{2}{\beta}\int_0^{\frac{\pi}{2}}(\tan\theta)^{\{2(\alpha+1)/\beta\}-1}\,d\theta$ (if $x = (\tan\theta)^{2/\beta}$)

$$= \frac{\pi}{\beta\sin\{(\alpha+1)\pi/\beta\}}\ (0 < \alpha+1 < \beta)$$

In particular $\int_0^\infty \frac{dx}{1+x^{2n}} = \frac{\pi}{2n\sin(\pi/2n)}\ (n > \frac{1}{2}).$

12.3. The Gamma Function. Weierstrass's Definition. The infinite product $\prod_1^\infty\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$ has been proved uniformly and absolutely convergent for all finite z (real or complex) (§ *11.24* (v)). Its value is zero when z is a negative integer (i.e. it converges to zero).

The Gamma Function may be defined by the relation

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_1^\infty\left(1+\frac{z}{n}\right)e^{-z/n}$$

where $\gamma\left(= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right)\right)$ is Euler's Constant.

This equation therefore defines $\Gamma(z)$ as an analytic function of z for all finite values of z except $z = 0, -1, -2, -3, \ldots$ where there are simple poles.

It will be shown in § *12.34* that this is equivalent to Euler's definition when $\mathbf{R}(z) > 0$.

12.31. Euler's Product. From the above definition

$$\frac{1}{\Gamma(z)} = z \lim_{n \to \infty} e^{z\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)} n^{-z} \prod_1^n \left(\frac{n+z}{n}\right) e^{-z/n}$$

i.e.
$$\Gamma(z) = \lim_{n \to \infty} \frac{n!\,n^z}{z(z+1)(z+2)\dots(z+n)}.$$

Also $\Gamma(z) = \lim\limits_{n \to \infty} \dfrac{(n-1)!\,n^z}{z(z+1)\dots(z+n-1)}$ since $\left(\dfrac{n-1}{n}\right)^z \to 1$

and therefore $\Gamma(1+z) = \lim\limits_{n \to \infty} \dfrac{n!\,n^z}{(z+1)(z+2)\dots(z+n)}$.

The limit on the right was denoted by $\Pi(z)$ in Gauss's notation, so that $\Pi(z) = \Gamma(1+z)$.

12.32. The Relation $\Gamma(1+z) = z\Gamma(z)$. From the last result in § *12.31*, we have $\dfrac{\Gamma(z+1)}{\Gamma(z)} = z$ so that $\Gamma(z+1) = z\Gamma(z)$ when z is not zero nor a negative integer.

Thus $\Gamma(1+z) = \Pi(z) = (z)!$ when z is a positive integer.

12.33. The Relation $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$.

$$\Gamma(z)\Gamma(1-z) = \lim \frac{n!\,n^z}{z(z+1)\dots(z+n)} \lim \frac{n!\,n^{1-z}}{(1-z)(2-z)\dots(n+1-z)}.$$

Therefore

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \lim z\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\dots\left(1-\frac{z^2}{n^2}\right)\left(\frac{n+1-z}{n}\right)$$
$$= \frac{\sin \pi z}{\pi} \quad (\S\ 11.34)$$

Corollary. $\lim\limits_{z \to -m} (z+m)\Gamma(z) = \dfrac{\pi}{m!} \lim\limits_{z \to -m} \dfrac{z+m}{\sin \pi z} = (-1)^m \dfrac{1}{m!}$ (m being a positive integer or zero),

i.e. $(-1)^m \dfrac{1}{m!}$ is the residue of $\Gamma(z)$ at $z = -m$.

12.34. The Equivalence of the Definitions of Euler and Weierstrass. Let $\Gamma_1(z)$, $\Gamma_2(z)$ be respectively the Euler and the Weierstrassian Gamma Function.

$$\int_0^1 (1-t)^n t^{z-1}\,dt = B(n+1,\ z) \quad (\mathbf{R}(z) > 0,\ n > -1)$$
$$= \frac{n!\,\Gamma_1(z)}{\Gamma_1(z+n+1)} \quad (n \text{ being a positive integer})$$
$$= \frac{n!}{z(z+1)\dots(z+n)}$$

Therefore $\Gamma_2(z) = \lim\limits_{n\to\infty} n^z\int_0^1 (1-t)^n t^{z-1}\,dt = \lim \int_0^n (1-u/n)^n\, u^{z-1}\,du,$
(taking $t = u/n$)

$$= \int_0^\infty e^{-u}u^{z-1}\,du \ (\S\ 11.57) = \Gamma_1(z).$$

12.35. The Infinite Product $\prod\limits_1^\infty R(n)$, *where* $R(n)$ *is Rational.* It is *necessary* for convergence that R_n should tend to 1 when n tends to infinity.

Assume therefore that $R(n) = \dfrac{(n+a_1)(n+a_2)\ldots(n+a_m)}{(n+b_1)(n+b_2)\ldots(n+b_m)}$ where the a's and b's, whilst not necessarily distinct, are not negative integers. Also no a is equal to a b. When n is large

$$R(n) = 1 + \frac{\Sigma a - \Sigma b}{n} + O\left(\frac{1}{n^2}\right)$$

and therefore it is necessary and sufficient that $\Sigma a = \Sigma b$.

$$\prod_{r=1}^{r=m}\frac{1}{\Gamma(1+a_r)} = e^{\gamma\Sigma a_r}\prod_{n=1}^{\infty}\prod_{r=1}^{m}\left(1+\frac{a_r}{n}\right)e^{-\frac{a_r}{n}}$$

$$= e^{\gamma\Sigma a_r}\prod_{n=1}^{\infty}e^{-\frac{1}{n}\Sigma a_r}\prod_{r=1}^{m}\left(\frac{n+a_r}{n}\right).$$

Similarly $\sum\limits_{r=1}^{m}\dfrac{1}{\Gamma(1+b_r)} = e^{\gamma\Sigma b_r}\prod\limits_{n=1}^{\infty}e^{-\frac{1}{n}\Sigma b_r}\prod\limits_{r=1}^{m}\left(\dfrac{n+b_r}{n}\right)$

i.e. (since $\Sigma a_r = \Sigma b_r$), $\prod\limits_1^\infty\dfrac{(n+a_1)\ldots(n+a_m)}{(n+b_1)\ldots(n+b_m)} = \prod\limits_{r=1}^{m}\dfrac{\Gamma(1+b_r)}{\Gamma(1+a_r)}.$

Examples. (i) Prove that

$$\frac{1-\frac{1}{5^2}}{1-\frac{1}{6^2}}\cdot\frac{1-\frac{1}{9^2}}{1-\frac{1}{10^2}}\cdot\frac{1-\frac{1}{13^2}}{1-\frac{1}{14^2}}\cdot\ \cdot\ \cdot = \frac{3\sqrt{2}}{32\sqrt{\pi}}\{\Gamma(\tfrac14)\}^2.$$

The product is $\prod\limits_1^\infty\dfrac{\{(4n+1)^2-1\}(4n+2)^2}{\{(4n+2)^2-1\}(4n+1)^2} = \prod\limits_1^\infty\dfrac{n(n+\frac12)^3}{(n+\frac14)^3(n+\frac34)}$

$$= \frac{\{\Gamma(\frac54)\}^3\Gamma(\frac74)}{\{\Gamma(\frac32)\}^3} = \frac{3}{32}\,\frac{\{\Gamma(\frac14)\}^3\Gamma(\frac34)}{\{\Gamma(\frac12)\}^3} = \frac{3\sqrt{2}}{32\sqrt{\pi}}\{\Gamma(\tfrac14)\}^2$$

since $\Gamma(\frac14)\Gamma(\frac34) = \pi\sqrt2$; $\Gamma(\frac12) = \sqrt\pi$.

(ii) Find the value of the derangement of the product

$$(1+\tfrac12)(1-\tfrac13)(1+\tfrac14)(1-\tfrac15)\ldots$$

(which equals 1) obtained by taking in order three terms > 1 followed by two terms < 1.

Denote the product of the first n factors of the deranged product by P_n. The derangement converges to the same value as $\lim P_{5n}$ when n tends to infinity.

Now $P_{5n} = \prod\limits_1^n\left(1+\dfrac{1}{6n-4}\right)\left(1+\dfrac{1}{6n-2}\right)\left(1+\dfrac{1}{6n}\right)\left(1-\dfrac{1}{4n-1}\right)\left(1-\dfrac{1}{4n+1}\right)$

$$= \prod_1^n\frac{(n-\frac12)^2(n-\frac16)(n+\frac16)}{(n-\frac23)(n-\frac13)(n-\frac14)(n+\frac14)} = \frac{\Gamma(\frac13)\Gamma(\frac23)\Gamma(\frac34)\Gamma(\frac54)}{\{\Gamma(\frac12)\}^2\Gamma(\frac56)\Gamma(\frac76)} = \frac{\sqrt3}{\sqrt2}$$?

since $\Gamma(\frac13)\Gamma(\frac23) = 2\pi/\sqrt3$; $\Gamma(\frac34)\Gamma(\frac54) = \frac14\pi\sqrt2$; $\Gamma(\frac56)\Gamma(\frac76) = \frac13\pi$.

12.4. Binet's Formulae and the Asymptotic Expansion of $\log \Gamma(z)$. Binet's First Formula for $\log \Gamma(z)$ is

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\frac{e^{-tz}}{t}\,dt$$

when $\mathbf{R}(z) > 0$.

Frullani's Integral (§ *11.56*, (iii)) gives

$$\log\left(\frac{z+n}{n}\right) = \int_0^\infty \frac{e^{-nt}}{t}(1 - e^{-zt})\,dt \quad (n = 1, 2, 3, \ldots), \quad (\mathbf{R}(z) > -1)$$

$$\log n = \int_0^\infty \frac{e^{-t} - e^{-nt}}{t}\,dt \quad (n > 0)$$

Thus $\log\left\{\dfrac{n^z n!}{(z+1)(z+2)\ldots(z+n)}\right\}$

$$= z\log n - \sum_{m=1}^{n}\log\left(\frac{z+m}{m}\right)$$

$$= z\int_0^\infty \frac{e^{-t} - e^{-nt}}{t}\,dt - \int_0^\infty (1 - e^{-zt})\left(\frac{1 - e^{-nt}}{e^t - 1}\right)\frac{dt}{t}$$

since $e^{-t} + e^{-2t} + \ldots + e^{-nt} = \dfrac{1 - e^{-nt}}{e^t - 1}$.

Thus

$$\log \Gamma(1+z) = \int_0^\infty \left(ze^{-t} - \frac{1 - e^{-zt}}{e^t - 1}\right)\frac{dt}{t} - \lim_{n\to\infty}\int_0^\infty e^{-nt}\left(z - \frac{1 - e^{-zt}}{e^t - 1}\right)\frac{dt}{t}.$$

When $0 < t < 2\pi$, $\dfrac{1}{t}\left(z - \dfrac{1 - e^{-zt}}{e^t - 1}\right)$ can be expanded in a power series in t which tends to $\frac{1}{2}(z + z^2)$ when t tends to zero; for

$$\frac{1 - e^{-zt}}{e^t - 1} = \{z - \tfrac{1}{2}z^2t + O(t^2)\}\{1 - \tfrac{1}{2}t + O(t^2)\} = z - \tfrac{1}{2}(z + z^2)t + O(t^2)$$

Thus $\dfrac{1}{t}\left(z - \dfrac{1 - e^{-zt}}{e^t - 1}\right)$ has an upper bound (all z) for $t < 1$ (at least).

When $t > 1$, $|e^{-zt}| = e^{-xt} < e^t$ (where $x = \mathbf{R}(z) > -1$).

$$\text{Therefore } \left|\frac{1}{t}\left(z - \frac{1 - e^{-zt}}{e^t - 1}\right)\right| < |z| + \frac{e+1}{e-1}$$

i.e. $$\left|\int_0^\infty e^{-nt}\left(z - \frac{1 - e^{-zt}}{e^t - 1}\right)\frac{dt}{t}\right| < K\int_0^\infty e^{-nt}\,dt = \frac{K}{n}$$

where K is the upper bound of $\dfrac{1}{t}\left(z - \dfrac{1 - e^{-zt}}{e^t - 1}\right)$ in the interval $(0, \infty)$ of t, and has been shown to finite for all finite z (for which $\mathbf{R}(z) > -1$), be

i.e. $$\log \Gamma(1+z) = \int_0^\infty \left(ze^{-t} - \frac{1 - e^{-zt}}{e^t - 1}\right)\frac{dt}{t}.$$

Now $\dfrac{1}{e^t - 1} - \dfrac{1}{t} + \dfrac{1}{2} = \sum_1^\infty \dfrac{2t}{t^2 + 4n^2\pi^2}$ (all real t) (§ *11.33*)

$$< \frac{t}{2\pi^2} \sum_1^\infty \frac{1}{n^2} \ (0 < t)$$

$$< \tfrac{1}{12} t.$$

So that $\left| \int_0^\infty \left(\dfrac{1}{e^t - 1} - \dfrac{1}{t} + \dfrac{1}{2} \right) \dfrac{e^{-tz}}{t} \, dt \right| < \int_0^\infty \dfrac{e^{-tx}}{12} \, dt = \dfrac{1}{12x}$ $(x > 0)$.

Now $\log \Gamma(1 + z) = F(z) + G(z)$ where

$$F(z) = \int_0^\infty \left\{ ze^{-t} - \frac{1}{e^t - 1} + \left(\frac{1}{t} - \frac{1}{2} \right) e^{-zt} \right\} \frac{dt}{t}$$

$$G(z) = \int_0^\infty \left\{ \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right\} \frac{e^{-zt}}{t} \, dt \text{ and } |G(z)| < \frac{1}{12x} \, (x > 0).$$

It follows therefore that $\lim_{x \to \infty} \{\log \Gamma(1 + z) - F(z)\} = 0$. The integral obtained by differentiating $F(z)$ with respect to z under the integral sign is

$$\int_0^\infty \{e^{-t} - (1 - \tfrac{1}{2}t)e^{-zt}\} \frac{dt}{t} = \int_0^\infty \frac{e^{-t} - e^{-zt}}{t} \, dt + \int_0^\infty \tfrac{1}{2} e^{-zt} \, dt$$

$$= \log z + \frac{1}{2z}.$$

It is uniformly convergent for $\mathbf{R}(z) \geqslant z > 0$.

Integrating, we obtain $F(z) = z \log z - z + \frac{1}{2} \log z + A$, where A is a constant.

Thus $\log \Gamma(1 + z) = (z + \frac{1}{2}) \log z - z + A + G(z)$ $(\mathbf{R}(z) > 0)$

$$\log \Gamma(z) = \log \Gamma(1 + z) - \log z$$
$$= (z - \tfrac{1}{2}) \log z - z + A + G(z)$$

where $G(z) \to 0$ when $\mathbf{R}(z) \to \infty$.

From the reduplication formula, we have

$$\log \Gamma(2z) - \log \Gamma(z) - \log \Gamma(z + \tfrac{1}{2}) = (2z - 1) \log 2 - \tfrac{1}{2} \log \pi$$

i.e. $(2z - \frac{1}{2}) \log 2z - (z - \frac{1}{2}) \log z - z \log (z + \frac{1}{2}) + \frac{1}{2} - A$
$+ G(2z) - G(z) - G(z + \frac{1}{2}) = (2z - 1) \log 2 - \frac{1}{2} \log \pi$

or $A = \frac{1}{2} \log (2\pi) - z \log \left(1 + \dfrac{1}{2z} \right) + \frac{1}{2} + G(2z) - G(z) - G(z + \frac{1}{2})$.

Let $\mathbf{R}(z) \to \infty$ and we find that $A = \frac{1}{2} \log (2\pi)$.

We have therefore proved the required result

$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log (2\pi)$

$$+ \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{e^{-zt}}{t} \, dt \ (\mathbf{R}(z) > 0).$$

12.41. Gauss's Formula for $\psi(z) = \Gamma'(z)/\Gamma(z)$. In the last paragraph we have shown that

$$\log \Gamma(1 + z) = \int_0^\infty \left\{ ze^{-t} - \frac{1 - e^{-zt}}{e^t - 1} \right\} \frac{dt}{t}.$$

The integral obtained by differentiation with respect to z is

$$\int_0^\infty \left(e^{-t} - \frac{te^{-zt}}{e^t - 1}\right)\frac{dt}{t}$$

and is uniformly convergent for $\mathbf{R}(z) > \varepsilon > -1$.

Thus
$$\frac{\Gamma'(1+z)}{\Gamma(1+z)} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{e^t - 1}\right)dt$$

so that
$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}}\right)dt \quad (\mathbf{R}(z) > 0)$$

Corollary. Since $\dfrac{\Gamma'(1)}{\Gamma(1)} = -\gamma$ (Euler's Constant), it follows that

$$\gamma = \int_0^\infty \left(\frac{1}{1 - e^{-t}} - \frac{1}{t}\right)e^{-t}\,dt = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{e^{-t}}{t}\right)dt.$$

12.42. The Asymptotic Expansion of $\log \Gamma(z)$. Consider

$$\int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\frac{e^{-zt}}{t}\,dt \quad (\mathbf{R}(z) > 0).$$

The integrand $H(z)$ is $\left\{\sum_1^\infty \dfrac{2t}{t^2 + 4n^2\pi^2}\right\}\dfrac{e^{-zt}}{t}$

$$= \sum_{n=1}^\infty \frac{1}{2n^2\pi^2}\left\{1 - \frac{t^2}{4n^2\pi^2} + \frac{t^4}{(4n^2\pi^2)^2} - \cdots + (-1)^r \frac{t^{2r}}{(2n\pi)^{2r}} + E_r\right\}e^{-zt}$$

where
$$E_r = (-1)^{r+1}\frac{t^{2r+2}}{(2n\pi)^{2r}(t^2 + 4n^2\pi^2)}.$$

Also $\displaystyle\int_0^\infty t^{2r}e^{-zt}\,dt = \frac{(2r)!}{z^{2r+1}}$ $(\mathbf{R}(z) > 0)$ and $\displaystyle\sum_1^\infty \frac{1}{n^{2r}} = \frac{(2\pi)^{2r}}{2(2r)!}B_r$.

Thus
$$\int_0^\infty H(z)\,dz = \sum_{r=0}^{r} (-1)^r \frac{(2r)!}{z^{2r+1}}\frac{B_{r+1}}{(2r+2)!} + \int_0^\infty \sum_1^\infty \frac{E_r e^{-zt}}{2n^2\pi^2}dt$$

$$= \frac{B_1}{1.2.z} - \frac{B_2}{3.4.z^3} + \cdots + (-1)^r \frac{B_{r+1}}{(2r+1)(2r+2)}\cdot\frac{1}{z^{2r+1}} + K(z)$$

where
$$|K(z)| < \int_0^\infty \sum_1^\infty \frac{2t^{2r+2}e^{-zt}\,dt}{(2n\pi)^{2r+2}(t^2 + 4n^2\pi^2)}$$

$$< \frac{2}{(2\pi)^{2r+4}}\int_0^\infty t^{2r+2}e^{-zt}\,dt \sum_1^\infty \frac{1}{n^{2r+4}}$$

i.e.
$$< \frac{B_{r+2}}{(2r+3)(2r+4)}\cdot\frac{1}{z^{2r+3}}.$$

Thus $\log \Gamma(z) \sim (z - \tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \dfrac{B_1}{1.2.z} - \dfrac{B_2}{3.4.z^3} + \dfrac{B_3}{5.6.z^5} - \cdots$ (*Stirling's Series*) the series being asymptotic.

By taking the exponential of this series we find that

$$\Gamma(z) \sim e^{-z} z^{z-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\left\{1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} \dots\right\}$$

In particular, when n is a positive integer

$$n! = n\Gamma(n) \sim e^{-n}n^{n+\frac{1}{2}}(2\pi)^{\frac{1}{2}}\left\{1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \dots\right\}$$

a result that has already been established otherwise.

Example. If $n = 20$, $e^{-n}n^{n+\frac{1}{2}}(2\pi)^{\frac{1}{2}} = 2422785 \times 10^{13}$ correct to six significant figures. The first correction gives $10{,}095 \times 10^{13}$, the second 21×10^{13} and the third $-0{\cdot}8 \times 10^{13}$. Thus $(20)! = 243290 \times 10^{13}$ correct to six significant figures. Its actual value is $243290\,200817664 \times 10^{13}$

12.43. Binet's Second Formula for $\log \Gamma(z)$. $(\mathbf{R}(z) > 0.)$ Differentiation of the infinite product for $\Gamma(z)$ gives $\dfrac{d^2}{dz^2}\{\log \Gamma(z)\} = \sum\limits_{0}^{\infty} \dfrac{1}{(z+n)^2}$ when z is not a negative integer.

Now consider the contour integral

$$\int_C \frac{dz}{(z+\alpha i)^2(e^{2\pi z}-1)}$$

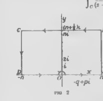

FIG. 2

where $\alpha = p + iq$ and $p > 0$, and C is the rectangle of corners $\pm n$, $\pm n + (n + \frac{1}{2})i$ indented by the upper half of the circle γ $(|z| = \varepsilon)$, n being a positive integer. (*Fig. 2.*)

The poles inside are $i, 2i, \dots, ni$. The pole $-\alpha i$ is outside since $p > 0$. Along the side $z = n + it$, t varies from 0 to $n + \frac{1}{2}$ and the corresponding part of the integral is I_{AB}, where

$$I_{AB} = \int_0^{n+\frac{1}{2}} \frac{i\,dt}{(n + it + pi - q)^2(e^{2\pi i(n+it)} - 1)}.$$

Thus $|I_{AB}| < \displaystyle\int_0^{n+\frac{1}{2}} \frac{dt}{(n - |\alpha|)^2(e^{2\pi n} - 1)}$, n large

$$= \frac{(2n+1)}{2(n - |\alpha|)^2(e^{2\pi n} - 1)} \text{ which} \to 0 \text{ as } n \to \infty.$$

Along CD where $z = -n + it$ and t varies from $n + \frac{1}{2}$ to 0, we have

$$|I_{CD}| < \int_0^{n+\frac{1}{2}} \frac{dt}{(n - |\alpha|)^2(1 - e^{-2\pi n})} < \frac{2n+1}{2(n - |\alpha|)^2(1 - e^{-2\pi n})}$$

which $\to 0$ as $n \to \infty$.

Along BC where $z = t + i(n + \frac{1}{2})$ and t varies from n to $-n$, we have

$$I_{BC} = \int_{-n}^{n} \frac{dt}{(z + \alpha i)^2(e^{2\pi t} + 1)}$$

so that $\quad |I_{BC}| < \int_{-n}^{n} \frac{4dt}{(2n + 1 + 2p)^2(e^{2\pi t} + 1)} < \frac{8n}{(2n + 1 + 2p)^2}$

which $\rightarrow 0$ as $n \rightarrow \infty$.

If $f(z)$ denotes the integrand, $zf(z)$ is continuous at $z = 0$ and tends to the value $\dfrac{1}{2\pi(\alpha i)^2}$ as $z \rightarrow 0$,

i.e. $$I_\gamma \rightarrow \frac{\pi i}{2\pi(\alpha i)^2} = -\frac{i}{2\alpha^2} \text{ when } e \rightarrow 0.$$

Thus $P\displaystyle\int_{-\infty}^{\infty} \frac{dt}{(t + \alpha i)^2(e^{2\pi t} - 1)}$ exists and is equal to $-\dfrac{i}{2\alpha^2} + 2\pi i \lim\limits_{n \rightarrow \infty} \sum\limits_{1}^{n} R_s$

if the latter limit exists, where R_s is the residue of $f(z)$ at $z = si$.

Now $\sum\limits_{1}^{n} R_s = \sum\limits_{1}^{n} \left(-\dfrac{1}{(\alpha + s)^2}\dfrac{1}{2\pi}\right)$ which tends to $-\dfrac{1}{2\pi}\left[\dfrac{d^2}{d\alpha^2}\{\log \Gamma(\alpha)\} - \dfrac{1}{\alpha^2}\right]$

i.e. $\displaystyle\int_0^{\infty} \left\{\frac{1}{(t+\alpha i)^2}\frac{1}{e^{2\pi t} - 1} + \frac{1}{(t - \alpha i)^2}\frac{1}{e^{-2\pi t} - 1}\right\}dt = \frac{i}{2\alpha^2} - i\frac{d^2}{d\alpha^2}\{\log \Gamma(\alpha)\}.$

Thus $$\frac{d^2}{d\alpha^2}\{\log \Gamma(\alpha)\} = \frac{1}{2\alpha^2} + i\int_0^{\infty} \left\{\frac{1}{(t+\alpha i)^2} - \frac{1}{(t-\alpha i)^2}\right\}\frac{dt}{e^{2\pi t} - 1} - i\int_0^{\infty} \frac{dt}{(t - \alpha i)^2}$$

$$= \frac{1}{2\alpha^2} + \frac{1}{\alpha} + \int_0^{\infty} \frac{4\alpha t}{(t^2 + \alpha^2)^2}\frac{dt}{e^{2\pi t} - 1}$$

or changing α to z, we have

$$\frac{d^2}{dz^2}\{\log \Gamma(z)\} = \frac{1}{2z^2} + \frac{1}{z} + \int_0^{\infty} \frac{4tz\,dt}{(t^2 + z^2)^2(e^{2\pi t} - 1)}.$$

Consider the infinite integral I in this equation:

$$\left|\frac{z}{(t^2 + z^2)^2}\right| < \frac{1}{|z|^3} < \frac{1}{\delta^3} \text{ where } \mathbf{R}(z) = p > \delta > 0$$

and therefore since $\displaystyle\int_0^{\infty} \frac{t\,dt}{e^{2\pi t} - 1}$ converges, I converges uniformly in $0 < \delta \leqslant \mathbf{R}(z)$ and tends to zero when $|z| \rightarrow \infty$.

Integration gives

$$\frac{d}{dz}\{\log \Gamma(z)\} = -\frac{1}{2z} + \log z + C - 2\int_0^{\infty} \frac{t\,dt}{(t^2 + z^2)(e^{2\pi t} - 1)}.$$

Denote the infinite integral in this last equation by J. Since

$$\left|\frac{1}{t^2 + z^2}\right| < \frac{1}{|z|^2}$$

J is uniformly convergent in $0 < \delta \leqslant \mathbf{R}(z)$ and converges to zero when $|z| \rightarrow \infty$ in this domain.

A further integration gives

$$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z + (C - 1)z + C' + 2\int_0^\infty \frac{\operatorname{arc\,tan}(t/z)}{e^{2\pi t} - 1}\, dt$$

where arc tan ζ is defined to be $\int_0^\zeta \frac{dz}{1 + z^2}$ and the path of integration (for definiteness) is the straight line joining 0 to ζ.

Suppose that z $(= x)$ is real (> 0); then $0 < \operatorname{arc\,tan}(t/x) < (t/x)$ when $0 < t < \infty$,

i.e. $$\int_0^\infty \frac{\operatorname{arc\,tan} t/x}{e^{2\pi t} - 1}\, dx < \frac{1}{x}\int_0^\infty \frac{t\, dt}{e^{2\pi t} - 1} = \frac{B_1}{4x} \text{ i.e. } = \frac{1}{24x}$$

so that $$|\log \Gamma(x) - (x - \tfrac{1}{2}) \log x - (C - 1)x - C'| < \frac{1}{12x}.$$

But $$\log \Gamma(1 + x) = \log x + \log \Gamma(x)$$

Therefore

$$(x + \tfrac{1}{2}) \log (x + 1) + (C - 1)(x + 1) + C' + \frac{\theta'}{12(x + 1)}$$
$$= \log x + (x - \tfrac{1}{2}) \log x + (C - 1)x + C' + \frac{\theta}{12x}$$

where θ, θ' are certain numbers (functions of x) that satisfy the inequalities $|\theta'| < 1$, $|\theta| < 1$,

i.e. $C - 1 = -1 + O(1/x)$ when x is large or $C = 0$.

Also by using the reduplication formula as in the first Binet formula, we may obtain $C' = \tfrac{1}{2} \log (2\pi)$.

Thus

$$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log 2\pi + 2\int_0^\infty \frac{\operatorname{arc\,tan}(t/z)}{e^{2\pi t} - 1}\, dt \;(\mathbf{R}(z) > 0).$$

Corollary. By taking $z = 1$ in the formula for $\Gamma'(z)/\Gamma(z)$ we find

$$\gamma = \tfrac{1}{2} + 2\int_0^\infty \frac{t\, dt}{(t^2 + 1)(e^{2\pi t} - 1)}.$$

Notes. (i) By comparing the two formulae for $\log \Gamma(z)$ we deduce that

$$\int_0^\infty \frac{e^{-tz}}{t}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) dt = 2\int_0^\infty \frac{\operatorname{arc\,tan}(t/z)\, dt}{e^{2\pi t} - 1};$$

$$\int_0^\infty e^{-tz}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) dt = 2\int_0^\infty \frac{t\, dt}{(t^2 + z^2)(e^{2\pi t} - 1)}.$$

These results may also be proved by evaluating the repeated integrals associated with the absolutely convergent double integral

$$\int_0^\infty \int_0^\infty \frac{e^{-tz} \sin tu}{e^{2\pi u} - 1}\, dt\, du \quad \text{(Bromwich, \textit{Infinite Series}, 177.)}$$

(ii) By expanding arc tan (t/z) in the form

$$\text{arc tan}\left(\frac{t}{z}\right) = \frac{t}{z} - \frac{1}{3}\frac{t^3}{z^3} + \dots + \frac{(-1)^{n-1}}{2n-1}\frac{t^{2n-1}}{z^{2n-1}} + \frac{(-1)^n}{z^{2n-1}}\int_0^t \frac{u^{2n}\,du}{u^2+z^2}$$

and using the result $\displaystyle\int_0^\infty \frac{t^{2n-1}\,dt}{e^{2\pi t}-1} = \frac{B_n}{4n}$, we may obtain Stirling's Series. (*Whittaker and Watson, Modern Analysis, XII.*) Similarly

$$\frac{\Gamma'(z)}{\Gamma(z)} \sim \log z - \frac{1}{2z} - \frac{B_1}{2z^2} + \frac{B_2}{4z^4} - \frac{B_3}{6z^6} + \dots$$

12.5. Gauss's Multiplication Formula.

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\dots\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}-nz}\,\Gamma(nz),$$

where n is a positive integer.

$$\Gamma\left(z+\frac{r}{n}\right) = \lim_{m\to\infty}\frac{(m-1)!\,m^{z+\frac{r}{n}}}{z(z+1)\dots\left(z+\frac{r}{n}+m-1\right)}.$$

Therefore

$$\prod_{r=0}^{n-1}\Gamma\left(z+\frac{r}{n}\right) = \lim_{m\to\infty}\frac{\{(m-1)!\}^n m^{nz+\frac{1}{2}(n-1)}}{z\left(z+\frac{1}{n}\right)\dots\left(z+\frac{n-1}{n}\right)(z+1)\dots\left(z+\frac{nm-1}{n}\right)}$$

$$= \lim_{m\to\infty}\frac{\{(m-1)!\}^n m^{nz+\frac{1}{2}(n-1)}n^{nm}}{nz(nz+1)\dots(nz+nm-1)}.$$

But $$\Gamma(nz) = \lim_{m\to\infty}\frac{(nm-1)!\,(nm)^{nz}}{nz(nz+1)\dots(nz+nm-1)}$$

i.e. $$\frac{n^{nz}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\dots\Gamma\left(z+\frac{n-1}{n}\right)}{\Gamma(nz)} = F(n)$$

(independent of z).

Using the asymptotic formula for $\Gamma(z)$, we find that the left-hand side is

$$\frac{n^{nz}\left\{\prod_{r=0}^{n-1} e^{-z-\frac{r}{n}}\left(z+\frac{r}{n}\right)^{z+\frac{r}{n}-\frac{1}{2}}\right\}(2\pi)^{\frac{n}{2}}}{e^{-nz}(nz)^{nz-\frac{1}{2}}(2\pi)^{\frac{1}{2}}}\left\{1+O\left(\frac{1}{|z|}\right)\right\}$$

$$= n^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(n-1)}e^{\frac{1}{2}(1-n)}\prod_1^{n-1}\left(1+\frac{r}{nz}\right)^{z+\frac{r}{n}-\frac{1}{2}}\left\{1+O\left(\frac{1}{|z|}\right)\right\}.$$

But when $|z|\to\infty$, $\displaystyle\prod_1^{n-1}\left(1+\frac{r}{nz}\right)^{z+\frac{r}{n}-\frac{1}{2}} \to e^{\frac{1}{2}(n-1)}$

i.e. $$F(n) = n^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(n-1)}.$$

Note. $F(n)$ is equal to $n\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)$.

and therefore $F^2 = \dfrac{n^2\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}} = n\pi^{n-1}2^{n-1}$ (§ *10.05.*)

i.e. $\qquad F = n^{\frac12}(2\pi)^{\frac12(n-1)}$ since $F > 0$.

Example. Show that $\Gamma(\tfrac{1}{12}) = \dfrac{3^{\frac38}}{\pi^{\frac12}2^{\frac14}}\Gamma(\tfrac13)\Gamma(\tfrac14)(\sqrt{3}+1)^{\frac12}$.

$$\Gamma(\tfrac{1}{12})\Gamma(\tfrac{7}{12}) = (2\pi)^{\frac12}2^{\frac13}\Gamma(\tfrac16)\,;\quad \Gamma(\tfrac{1}{12})\Gamma(\tfrac{5}{12})\Gamma(\tfrac{9}{12}) = 2\pi.3^{\frac14}\Gamma(\tfrac14)$$

$$\Gamma(\tfrac16)\Gamma(\tfrac23) = (2\pi)^{\frac12}2^{\frac16}\Gamma(\tfrac13)$$

Therefore $\qquad \{\Gamma(\tfrac{1}{12})\}^2\dfrac{\pi}{\sin\frac{5\pi}{12}}\Gamma(\tfrac34)\Gamma(\tfrac23) = (2\pi)^2 2^{\frac12}3^{\frac14}\Gamma(\tfrac14)\Gamma(\tfrac13)$

i.e. $\qquad \{\Gamma(\tfrac{1}{12})\}^2 = 2^{\frac12}3^{\frac14}\pi\sin\left(\dfrac{5\pi}{12}\right)\{\Gamma(\tfrac14)\}^2\{\Gamma(\tfrac13)\}^2\dfrac{\sin\frac{\pi}{4}\sin\frac{\pi}{3}}{\pi.\pi}$

$$= \frac{3^{\frac34}}{\pi.2^{\frac12}}\{\Gamma(\tfrac13)\}^2\{\Gamma(\tfrac14)\}^2(\sqrt{3}+1)$$

or $\qquad \Gamma(\tfrac{1}{12}) = \dfrac{3^{\frac38}(3^{\frac12}+1)^{\frac12}}{2^{\frac14}\pi^{\frac12}}\Gamma(\tfrac13)\Gamma(\tfrac14)$, since $\Gamma(\tfrac{1}{12}) > 0$

12.6. Dirichlet's Integral.

$$I = \iint\ldots\int f(t_1+t_2+\ldots t_n)t_1^{\alpha_1-1}t_2^{\alpha_2-1}\ldots t_n^{\alpha_n-1}\,dt_1\,dt_2\ldots dt_n$$

$$= \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\ldots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\alpha_2+\ldots+\alpha_n)}\int_0^1 f(\theta)\theta^{\alpha_1+\alpha_2+\ldots+\alpha_n-1}\,d\theta$$

where $\alpha_r > 0$, the integration extends over all positive and zero values of $t_1, t_2, \ldots, t_n$ that satisfy $0 < t_1 + t_2 + \ldots + t_n \leq 1$, and the integral on the right converges.

The method of proof is sufficiently indicated by taking $n = 3$. For simplicity also let us assume that $f(\theta)$ is continuous in $0 < \theta \leq 1$.

Let $t_1 + t_2 + t_3 = \theta$; $t_1 + t_2 = \theta\theta_1$; $t_1 = \theta\theta_1\theta_2$, so that $t_1 = \theta\theta_1\theta_2$, $t_2 = \theta\theta_1(1-\theta_2)$; $t_3 = \theta(1-\theta_1)$ and the transformation is $1-1$. The given region which is bounded by $t_1 = 0$, $t_2 = 0$, $t_3 = 0$, $t_1 + t_2 + t_3 = 1$ corresponds to the unit cube $\theta = 0, 1$; $\theta_1 = 0, 1$; $\theta_2 = 0, 1$. The only discontinuity of the integrand, if any, occurs when $t_1 = 0$, $t_2 = 0$ or $t_3 = 0$, i.e. on $\theta = 0$; $\theta_1 = 0, 1$; $\theta_2 = 0, 1$. The given integral is the limit (if it exists) of the integral throughout the region determined by $t_1 = \varepsilon$; $t_2 = \varepsilon$; $t_3 = \varepsilon$; $t_1 + t_2 + t_3 = 1$ $(\varepsilon > 0)$ where $\varepsilon \to 0$. Let δ be any preassigned positive number, however small. Then ε can be chosen sufficiently small to ensure that at every point of the surface $\theta\theta_1\theta_2 = \varepsilon$ (within the cube), one of the variables $\theta, \theta_1, \theta_2$ is $< \delta$. Similarly on the surface $\theta\theta_1(1-\theta_2) = \varepsilon$ either $\theta < \delta$ or $\theta_1 < \delta$ or $\theta_2 > 1-\delta$; and on $\theta(1-\theta_1) = \varepsilon$, $\theta < \delta$ or $\theta_1 > 1-\delta$, i.e. ε can be chosen so that the boundary of the transformed region lies between the surface of the unit cube and the rectangular space determined by $\theta = \delta, 1$; $\theta_1 = \delta, \theta_1 = 1-\delta$; $\theta_2 = \delta$, $\theta_2 = 1-\delta$. The convergence of the transformed integral therefore implies that of the original and the two integrals have the same value

If $X_1 = \theta = t_1 + t_2 + t_3$; $X_2 = \theta\theta_1 = t_2 + t_3$; $X_3 = \theta\theta_1\theta_2 = t_3$, then

$$\left|\frac{\partial(t_1t_2t_3)}{\partial(\theta\theta_1\theta_2)}\right| = \left|\frac{\partial(X_1X_2X_3)}{\partial(\theta\theta_1\theta_2)}\right| \div \left|\frac{\partial(X_1X_2X_3)}{\partial(t_1t_2t_3)}\right|$$

$$= \left|\left\{\begin{vmatrix}1 & 0 & 0\\ \theta_1 & \theta & 0\\ \theta_1\theta_2 & \theta\theta_2 & \theta\theta_1\end{vmatrix} \div \begin{vmatrix}1&1&1\\1&1&0\\1&0&0\end{vmatrix}\right\}\right| = \theta^2\theta_1.$$

Thus I

$$= \iiint f(\theta)(\theta\theta_1\theta_2)^{\alpha_3-1}(\theta\theta_1)^{\alpha_2-1}(1-\theta_2)^{\alpha_2-1}\theta^{\alpha_1-1}(1-\theta_1)^{\alpha_1-1}\theta^2\theta_1\,d\theta\,d\theta_1\,d\theta_2$$

$$= \int_0^1 \theta_2^{\alpha_3-1}(1-\theta_2)^{\alpha_2-1}\,d\theta_2\int_0^1 \theta_1^{\alpha_2+\alpha_3-1}(1-\theta_1)^{\alpha_1-1}\,d\theta_1\int_0^1 f(\theta)\theta^{\alpha_1+\alpha_2+\alpha_3-1}\,d\theta$$

$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}\int_0^1 f(\theta)\theta^{\alpha_1+\alpha_2+\alpha_3-1}\,d\theta \quad (\alpha_r > 0).$$

Notes. (i) For n variables, with the corresponding change of variables $t_1 + t_2 + \ldots + t_n = \theta$, $t_2 + t_3 + \ldots + t_n = \theta\theta_1, \ldots, t_n = \theta\theta_1\theta_2\ldots\theta_{n-1}$

we have $\left|\dfrac{\partial(t_1, t_2, \ldots, t_n)}{\partial(\theta, \theta_1, \ldots, \theta_{n-1})}\right| = \theta^{n-1}\theta_1^{n-2}\ldots\theta_{n-2}$.

(ii) $\iint\ldots\int f(a_1x_1^{m_1} + a_2x_2^{m_2} + \ldots + a_nx_n^{m_n})$

$$x_1^{\alpha_1-1}x_2^{\alpha_2-1}\ldots x_n^{\alpha_n-1}\,dx_1\,dx_2\ldots dx_n$$

where the integration extends over all positive and zero values of $x_1, x_2, \ldots, x_n$ for which $0 \leq \sum_1^n a_rx_r^{m_r} \leq 1$ is equal to

$$\frac{1}{\Gamma(S_n)}\left\{\prod_{r=1}^{n}\frac{\Gamma(\alpha_r/m_r)}{m_r a_r^{\alpha_r/m_r}}\right\}\int_0^1 f(\theta)\theta^{S_n-1}\,d\theta$$

where $$S_n = \sum_1^n\frac{\alpha_r}{m_r}\ \left(\frac{\alpha_r}{m_r} > 0\right),\ a_r > 0.$$

Examples. (i) $\iiint\log(x+y+z)dx\,dy\,dz$ for the region determined by $0 \leq x+y+z \leq 1$ is $\frac{1}{2}\int_0^1\theta^2\log\theta\,d\theta = -\frac{1}{18}$.

(ii) The volume V in the first octant determined by the surface

$$x^n + y^n + z^n = a^n \ (n > 0)$$

and the co-ordinate planes is $\iiint dx\,dy\,dz$ for $0 \leq x^n + y^n + z^n \leq a^n$.

Take $X = \left(\dfrac{x}{a}\right)^n$, $Y = \left(\dfrac{y}{a}\right)^n$, $Z = \left(\dfrac{z}{a}\right)^n$.

Then $V = \dfrac{a^3}{n^3}\iiint(XYZ)^{\frac{1}{n}-1}dX\,dY\,dZ$ for $0 \leq X + Y + Z \leq 1$

$$= \frac{a^3}{n^3}\frac{\left\{\Gamma\left(\frac{1}{n}\right)\right\}^3}{\Gamma\left(\frac{3}{n}\right)}\int_0^1\theta^{\frac{3}{n}-1}d\theta = \frac{\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^3}{\Gamma\left(1+\frac{3}{n}\right)}a^3.$$

12.7 *The Integrals* $\displaystyle\int_0^\infty x^{k-1}e^{-\lambda x\cos\alpha}\,{\cos \atop \sin}(\lambda x\sin\alpha)dx$. $(\lambda > 0)$ If $|\alpha| < \frac{1}{2}\pi$ both integrals converge when $k > 0$, if $|\alpha| = \dfrac{\pi}{2}$, the cosine

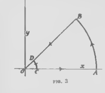

FIG. 3

integral converges for $0 < k < 1$ and the sine integral for $|k| < 1$. To determine their values, assume that $0 < \alpha < \frac{1}{2}\pi$ and $k > 0$. Consider the contour integral

$$I = \int_C z^{k-1} e^{-z}\, dz$$

where C is the boundary of the sector OAB of the circle $z = Re^{i\theta}$ determined by $\theta = 0$, $\theta = \alpha$ indented (when $k < 1$) by the arc CD of the small circle $|z| = \varepsilon$. (*Fig. 3.*)

Along the arc AB, $z = Re^{i\theta}$ and $I_{AB} = i\int_0^\alpha R^k e^{ik\theta} e^{-R(\cos\theta + i\sin\theta)}\, d\theta$.

Therefore
$$|I_{AB}| < R^k \int_0^\alpha e^{-R\cos\theta}\, d\theta$$
$$< \frac{\pi R^{k-1}}{2}\left\{e^{-\frac{2R}{\pi}\left(\frac{\pi}{2}-\alpha\right)} - e^{-R}\right\}$$

since $\cos\theta > \dfrac{2}{\pi}\left\{\dfrac{\pi}{2} - \theta\right\}$ for $0 < \theta < \dfrac{\pi}{2}$ i.e. $I_{AB} \to 0$ when $\alpha < \frac{1}{2}\pi$, all k, and $I_{AB} \to 0$ when $\alpha = \frac{1}{2}\pi$, $k < 1$.

Along the small arc CD where $z = \varepsilon e^{i\theta}$, $I_{CD} = i\int_0^\alpha \varepsilon^k e^{ik\theta} e^{-\varepsilon e^{i\theta}}\, d\theta$.

But $e^{-\varepsilon e^{i\theta}} = 1 + E$ where $E \to 0$ uniformly when $\varepsilon \to 0$ so that since $|I_{CD}| < \int_0^\alpha \varepsilon^k (1 + E)\, d\theta$, $I_{CD} \to 0$ when $k > 0$. Thus, under the conditions stated

$$\lim_{R\to\infty} \int_{OA} e^{-z} z^{k-1}\, dz = \lim_{R\to\infty} \int_{OB} e^{-z} z^{k-1}\, dz$$

since the integrand is analytic within and on C. On OA take $z = t$ and on OB take $z = te^{i\alpha}$; then

$$\int_0^\infty e^{-t(\cos\alpha + i\sin\alpha)} t^{k-1} e^{ik\alpha}\, dt = \int_0^\infty e^{-t} t^{k-1}\, dt = \Gamma(k)$$

i.e. $\displaystyle\int_0^\infty e^{-t\cos\alpha} t^{k-1} \frac{\cos}{\sin}(t\sin\alpha)\, dt = \Gamma(k) \frac{\cos}{\sin}(k\alpha)$ (all $k > 0$, $0 \leq \alpha < \frac{1}{2}\pi$)

and
$$\int_0^\infty t^{k-1} \frac{\cos}{\sin}(t)\, dt = \Gamma(k) \frac{\cos}{\sin}\left(\frac{k\pi}{2}\right) \quad (0 < k < 1).$$

By changing the sign of α, we see that the first two results are true also for $-\frac{1}{2}\pi < \alpha$ (all k).

The integral $\int_0^\infty t^{k-1} \cos t\, dt$ is not convergent for $k = 0$ but the integral

$\int_0^\infty t^{k-1}\sin t\,dt$ is convergent for $-1 < k < 1$ and the formula is correct for this increased interval.

For let $k = -1 + k'$ where $0 < k' < 1$.

Then $\int_0^\infty t^{k-1}\sin t\,dt = \left(\frac{t^k}{k}\sin t\right)_0^\infty - \frac{1}{k}\int_0^\infty t^{k'-1}\cos t\,dt = -\frac{\Gamma(k')}{k}\cos\frac{\pi k'}{2}$

since $t^k\sin t \to 0$ when $t \to 0$ and $t \to \infty$ when $-1 < k < 0$,

$$= -\frac{\Gamma(1+k)}{k}\cos\frac{\pi}{2}(1+k) = \Gamma(k)\sin\frac{k\pi}{2}.$$

The formula is also true when $k \to 0$ since $\frac{\pi}{2} = \int_0^\infty \frac{\sin t}{t}\,dt = \lim\limits_{k\to 0}\Gamma(k)\sin\frac{k\pi}{2}$

(using the relation $\Gamma(k)\Gamma(1-k) = \pi\,\mathrm{cosec}\,\pi k$).

Writing $t = \lambda x$ $(\lambda > 0)$, we obtain

$$\int_0^\infty x^{k-1}e^{-\lambda x\cos\alpha}\,{\cos \atop \sin}(\lambda x\sin\alpha)dx = \frac{\Gamma(k)}{\lambda^k}\,{\cos \atop \sin}\,(k\alpha)$$

when the integrals converge and $|\alpha| < \frac{\pi}{2}$.

Corollary. $\int_0^\infty x^{k-1}e^{-ax}\,{\cos \atop \sin}\,(bx)dx$

$$= \frac{\Gamma(k)}{(a^2+b^2)^{\frac{1}{2}k}}\,{\cos \atop \sin}\left(k\arctan\frac{b}{a}\right)(a,\ k > 0)$$

$$\int_0^\infty x^{k-1}\cos bx\,dx = \frac{\Gamma(k)}{b^k}\cos\frac{\pi k}{2}\ (0 < k < 1,\ b > 0)$$

$$\int_0^\infty x^{k-1}\sin bx\,dx = \frac{\Gamma(k)}{b^k}\sin\frac{\pi k}{2}\ (|k| < 1,\ b > 0)$$

$$\int_0^\infty \frac{\cos bx}{x^\mu}\,dx = \frac{\pi}{2b^{1-\mu}\Gamma(\mu)\cos\frac{\pi\mu}{2}}\ (b > 0,\ 0 < \mu < 1)$$

$$\int_0^\infty \frac{\sin bx}{x^\mu}\,dx = \frac{\pi}{2b^{1-\mu}\Gamma(\mu)\sin\frac{\pi\mu}{2}}\ (b > 0,\ 0 < \mu < 2)$$

Example. Find $\int_0^\infty \sin(x^n)dx$ and $\int_0^\infty \cos(x^n)dx$ $(n > 1)$.

Let $x = X^{\frac{1}{n}}$; then $\int_0^\infty \sin(x^n)dx = \frac{1}{n}\int_0^\infty X^{\frac{1}{n}-1}\sin X\,dX = \frac{\Gamma\left(\frac{1}{n}\right)\sin\frac{\pi}{2n}}{n}$ and

similarly $$\int_0^\infty \cos(x^n)dx = \frac{\Gamma\left(\frac{1}{n}\right)\cos\frac{\pi}{2n}}{n}$$

In particular $\int_0^\infty \sin(x^3)\,dx = \int_0^\infty \cos(x^3)\,dx = \dfrac{\sqrt{\pi}}{2\sqrt{2}}$;

$\int_0^\infty \sin(x^3)\,dx = \tfrac{1}{6}\Gamma(\tfrac{1}{3})$; $\int_0^\infty \cos(x^3)\,dx = \tfrac{1}{6}\sqrt{3}\,\Gamma(\tfrac{1}{3})$. $\int_0^\infty \begin{matrix}\sin\\ \cos\end{matrix}(x^4)\,dx = \tfrac{1}{8}\Gamma(\tfrac{1}{4})(2 \mp \sqrt{2})^{\frac{1}{2}}$.

12.3. Some Properties of the Function $\psi(z) = \Gamma'(z)/\Gamma(z)$.

1. By differentiating the infinite product for $\Gamma(x)$ we obtain

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_1^\infty \left(\frac{1}{n} - \frac{1}{n+z}\right)$$

and therefore $\psi(z) - \psi(u) = \sum_0^\infty \left(\dfrac{1}{n+u} - \dfrac{1}{n+z}\right)$.

2. By differentiating the relations

$$\Gamma(1+z) = z\,\Gamma(z),\quad \Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z$$

we find that

$$\psi(1+z) - \psi(z) = \frac{1}{z};\quad \psi(1-z) - \psi(z) = \pi \cot \pi z.$$

3. By differentiating Gauss's Multiplication formula, we obtain

$$\psi(z) + \psi\left(z + \frac{1}{n}\right) + \ldots + \psi\left(z + \frac{n-1}{n}\right) = n\psi(nz) - n \log n.$$

In particular $\psi(z) + \psi(z + \tfrac{1}{2}) = 2\psi(2z) - 2 \log 2$.

4. It has been shown that

$$\psi(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right)dt\ (\mathbf{R}(z) > 0);\ \gamma = \int_0^\infty \left(\frac{e^{-t}}{1-e^{-t}} - \frac{e^{-t}}{t}\right)dt$$

(§ *12.41*).

Therefore $\psi(z) + \gamma = \int_0^\infty \dfrac{e^{-t} - e^{-zt}}{1-e^{-t}}\,dt = \int_0^1 \dfrac{1-u^{z-1}}{1-u}\,du$ (where $u = e^{-t}$).

5. From the results $\psi(1+z) - \psi(z) = \dfrac{1}{z}$, $\psi(1) = -\gamma$ we obtain

$\psi(2) = 1 - \gamma$; $\psi(3) = \tfrac{3}{2} - \gamma$; $\psi(4) = \tfrac{11}{6} - \gamma$; . . . ;

$$\psi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} - \gamma$$

where n is a positive integer.

6. Since $\psi\left(\dfrac{1}{2}\right) + \gamma = \int_0^1 \dfrac{1 - u^{-\frac{1}{2}}}{1-u}\,du$ (from 4 above)

$$= -2\int_0^1 \frac{dv}{1+v} = -2 \log 2,$$

we have

$\psi(\tfrac{1}{2}) = -\gamma - 2 \log 2$; $\psi(\tfrac{3}{2}) = -\gamma - 2 \log 2 + 2$, . . . ,

$$\psi\left(n + \frac{1}{2}\right) = -\gamma - 2 \log 2 + 2\sum_1^n \frac{1}{2r-1}$$

using the relation $\psi(1+z) - \psi(z) = \dfrac{1}{z}$ (n being a positive integer)

Examples. (i) Find $\psi(\tfrac14)$, $\psi(\tfrac34)$.

$$\psi(\tfrac14) = -\gamma + \int_0^1 \frac{1-u^{-3/4}}{1-u}\,du = -\gamma - 2\int_0^1 \left(\frac{v+1}{v^2+1} + \frac{1}{v+1}\right)dv = -\gamma - 3\log 2 - \tfrac12\pi.$$

$$\psi(\tfrac34) - \psi(\tfrac14) = \pi \text{ and therefore } \psi(\tfrac34) + \gamma = -3\log 2 + \tfrac12\pi.$$

(ii) The integral obtained by differentiating $\int_0^{\frac12\pi} \sin^{2\alpha-1}x\,dx = \dfrac{\sqrt{\pi}}{2}\dfrac{\Gamma(\alpha)}{\Gamma(\alpha+\frac12)}$ is $2\int_0^{\frac12\pi} \sin^{2\alpha-1}x \log\sin x\,dx$, which is uniformly convergent for $\alpha \geq \alpha_0 > 0$ since $\dfrac{\sin x}{x}$ is bounded and near $x = +0$, $x^{2\alpha} \leq x^{2\alpha_0}$; for we may apply the M test taking $M(x) = \sin^{2\alpha_0-1}x \log\sin x$.

Thus $\displaystyle\int_0^{\frac12\pi} \sin^{2\alpha-1}x \log\sin x\,dx = \frac{\sqrt{\pi}}{4}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac12)}\{\psi(\alpha) - \psi(\alpha+\tfrac12)\}.$

Similarly, by a further differentiation

$$\int_0^{\frac12\pi} \sin^{2\alpha-1}x(\log\sin x)^2\,dx = \frac{\sqrt{\pi}}{8}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac12)}[\{\psi(\alpha) - \psi(\alpha+\tfrac12)\}^2 + \psi'(\alpha) - \psi'(\alpha+\tfrac12)]$$

(iii) Find $\int_0^{\frac12\pi} \sin^{\frac13}x \log\sin x\,dx$.

Put $\alpha = \tfrac23$ in *Example* (ii) above and obtain

$$\int_0^{\frac12\pi} \sin^{\frac13}x \log\sin x\,dx = \frac{\sqrt{\pi}\,\Gamma(\frac23)}{4\,\Gamma(\frac76)}\{\psi(\tfrac23) - \psi(\tfrac76)\}.$$

Now $\psi(\tfrac16) + \psi(\tfrac23) = 2\psi(\tfrac13) - 2\log 2.$

$$\psi(\tfrac76) - \psi(\tfrac16) = 6, \qquad \psi(\tfrac23) - \psi(\tfrac13) = \frac{\pi}{\sqrt3}.$$

Also $\psi(\tfrac13) + \psi(\tfrac23) = \gamma - 3\gamma - 3\log 3.$
From these we may determine $\psi(\tfrac13)$, $\psi(\tfrac16)$ and we find that

$$\int_0^{\frac12\pi} \sin^{\frac13}x \log\sin x\,dx = -\frac{2^{4/3}\pi^2}{(\Gamma(\frac13))^3}\left\{\frac{\pi}{\sqrt3} + \log 2 - 3\right\}$$

(iv) Find $\int_0^{\frac12\pi} (\log\sin x)^2\,dx$.

Put $\alpha = \tfrac12$ in the second formula of *Example* (ii) above, then

$$\int_0^{\frac12\pi} (\log\sin x)^2\,dx = \frac{\sqrt{\pi}}{8}\Gamma(\tfrac12)\{(\psi(\tfrac12) - \psi(1))^2 + \psi'(\tfrac12) - \psi'(1)\}.$$

Now $\qquad \psi'(z) = \sum_0^\infty \dfrac{1}{(z+n)^2}$ and therefore $\psi'(1) = \tfrac16\pi^2$

and $\qquad \psi'(\tfrac12) = 4\sum_0^\infty \dfrac{1}{(2n+1)^2} = 3\sum_1^\infty \dfrac{1}{n^2} = \dfrac{\pi^2}{2}.$

Also $\qquad \psi(\tfrac12) = -\gamma - 2\log 2;\quad \psi(1) = -\gamma.$

Therefore $\displaystyle\int_0^{\pi/2} (\log\sin x)^2\,dx = \tfrac12\pi\{(\log 2)^2 + \tfrac1{12}\pi^2\}.$

Examples XII

Establish the expansions given in *Examples 1–8*.

1. $z\cot z = 1 - \frac{1}{3}z^2 - \frac{1}{45}z^4 - \dots - \frac{(2z)^{2k}}{(2k)!}B_k - \dots$

2. $\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \dots + \frac{2^{2k}(2^{2k}-1)}{(2k)!}z^{2k-1}B_k + \dots$

3. $z\operatorname{cosec} z = 1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \dots + \frac{(2^{2k}-2)z^{2k}}{(2k)!}B_k + \dots$

4. $z\coth z = 1 + \frac{1}{3}z^2 - \frac{1}{45}z^4 + \dots + (-1)^{k-1}\frac{(2z)^{2k}}{(2k)!}B_k + \dots$

5. $z^2\operatorname{cosec}^2 z = 1 + \frac{1}{3}z^2 + \frac{1}{15}z^4 + \dots + \frac{2^{2k}(2k-1)z^{2k}}{(2k)!}B_k + \dots$

6. $\frac{1}{e^z+1} = \frac{1}{2} - B_1(2^2-1)\frac{z}{2!} + \dots + (-1)^k(2^{2k}-1)\frac{z^{2k-1}}{(2k)!}B_k + \dots$

7. $\log(z\operatorname{cosec} z) = \sum_{k=1}^{\infty}\frac{B_k(2z)^{2k}}{2k.(2k)!}$

8. $\log\left(\frac{\cosh z - \cos z}{z^2}\right) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{4k+1}}{(4k).(4k)!}B_{2k}z^{4k}$

9. If the numbers E_r are defined by the equation

$$\sec z = 1 + \frac{E_1z^2}{2!} + \frac{E_2z^4}{4!} + \dots + \frac{E_n}{(2n)!}z^{2n} + \dots$$

prove that

(i) $E_1 = 1,\ E_2 = 5,\ E_3 = 61,\ E_4 = 1385$.

(ii) $E_n - {}^{2n}C_2E_{n-1} + {}^{2n}C_4E_{n-2} - \dots + (-1)^{n-1}\,{}^{2n}C_2E_1 + (-1)^n = 0$

(iii) $E_n = \frac{(2n)!2^{2n+2}}{\pi^{2n+1}}\left(1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \dots\right)$

Prove the results given in *Examples 10–23*.

10. $1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32} = 0{\cdot}968946$

11. $1 - \frac{1}{3^5} + \frac{1}{5^5} - \dots = \frac{5\pi^5}{1536} = 0{\cdot}996158$

12. $1 - \frac{1}{3^7} + \frac{1}{5^7} - \dots = \frac{61\pi^7}{184320} = 0{\cdot}999555$

13. $\frac{B_1}{2!}\pi^2 + \frac{B_2}{4!}\pi^4 + \frac{B_3}{6!}\pi^6 + \dots = 1$

14. $\frac{B_1}{2!}\pi^2 + \frac{B_2}{2^2.4!}\pi^4 + \frac{B_3}{2^4.6!}\pi^6 + \dots = 4 - \pi$

15. ${}^{2m+1}C_2B_1 - {}^{2m+1}C_4B_2 + \dots + (-1)^{m-1}\,{}^{2m+1}C_{2m}B_m = \frac{1}{2}$

16. ${}^{2m}C_2B_1 - {}^{2m}C_4B_2 + \dots + (-1)^{m-2}\,{}^{2m}C_2B_{m-1}$
$$+ (-1)^{m-1}(2m+1)B_m = \tfrac{1}{2}$$

17. ${}^{2m+1}C_1B_m - {}^{2m+1}C_3B_{m-1} + \dots + (-1)^{m-1}\,{}^{2m+1}C_3B_1$
$$= (-1)^{m-1}\left(\frac{2m-1}{2}\right)$$

18. $\frac{2^{2n}B_n}{(2n)!} - \frac{2^{2n-2}B_{n-1}}{(2n-2)!3!} + \frac{2^{2n-4}B_{n-2}}{(2n-4)!5!} - \dots + (-1)^{n-1}\frac{2^2B_1}{2!(2n-1)!}$
$$= (-1)^{n-1}\frac{2n}{(2n+1)!}$$

19. $$\frac{2^{2n}B_{2n}}{(4n)!\,2!} - \frac{2^{2n-2}B_{2n-2}}{(4n-4)!\,6!} + \dots + (-1)^{n-1}\frac{2^2B_1}{4!(4n-2)!} + \frac{(-1)^n 2n}{(4n+2)!} = 0$$

20. $$\frac{2^{2n-1}B_{2n-1}}{(4n-2)!\,2!} - \frac{2^{2n-3}B_{2n-3}}{(4n-6)!\,6!} + \dots + (-1)^{n-1}\frac{2B_1}{2!(4n-2)!} + \frac{(-1)^n}{2(4n-1)!} = 0$$

21. $$(2^{2n}-1)B_n = (2^{2n-2}-1)\,{}^{2n}C_2B_{n-1} - (2^{2n-4}-1)\,{}^{2n}C_4B_{n-2} + \dots + (-1)^n\,{}^{2n}C_{2n-2}B_1 + \tfrac{1}{2}(-1)^{n+1}$$

22. $$(2n-1)E_{n-1} - {}^{2n-1}C_3E_{n-2} + \dots + (-1)^{n-2}\,{}^{2n-1}C_3E_1 + (-1)^{n-1} = \frac{2^{2n}(2^{2n}-1)}{2n}B_n$$

23. $$\frac{E_n}{(2n)!} = \frac{2^{2n+2}(2^{2n+2}-1)}{(2n+2)!}B_{n+1} + \sum_{s=0}^{n-1}\frac{2^{2s+2}(2^{2n-2s}-2)(2^{2s+2}-1)}{(2n-2s)!(2s+2)!}B_{s+1}B_{n-s}.$$

24. Show that
$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \dots + \frac{1}{(2n+1)^3} \sim C - \frac{1}{4(2n+1)^2} + \frac{1}{2(2n+1)^3} + \sum_1^\infty (-1)^m \frac{(2m+1)2^{2m-2}B_m}{(2n+1)^{2m+2}}$$
and deduce (taking $n = 5$) that
$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \dots = 1{\cdot}051800 \text{ (correct to six decimal places)}$$

25. Show that
$$\frac{1}{3^3} + \frac{1}{7^3} + \dots + \frac{1}{(4n-1)^3} \sim C - \frac{1}{8(4n-1)^2} + \frac{1}{2(4n-1)^3} - \sum_1^\infty (-1)^{m-1}\frac{(2m+1)2^{4m-3}B_m}{(4n-1)^{2m+2}}$$
and deduce (taking $n = 4$) that

(i) $\frac{1}{3^3} + \frac{1}{7^3} + \frac{1}{11^3} + \dots = 0{\cdot}041427$

(ii) $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = 0{\cdot}968946$ (*see Example 24*)

26. Show that
$$\frac{1}{5^5} + \frac{1}{9^5} + \dots + \frac{1}{(4n+1)^5} \sim C - \frac{1}{16(4n+1)^4} + \frac{1}{2(4n+1)^5} - \frac{5}{3(4n+1)^6}$$
with an error less than 10^{-6} if $n > 2$. Hence show that $\frac{1}{5^5} + \frac{1}{9^5} + \dots = 0{\cdot}000341$ approximately.

27. Show that
$$\frac{1}{3^5} + \frac{1}{7^5} + \dots + \frac{1}{(4n-1)^5} \sim C - \frac{1}{16(4n-1)^4} + \frac{1}{2(4n-1)^5} - \frac{5}{3(4n-1)^6}$$
with an error less than 10^{-6} if $n > 3$. Hence show that $\frac{1}{3^5} + \frac{1}{7^5} + \dots = 0{\cdot}004183$ approximately.

28. Deduce from *Examples 26, 27* that the approximate values of

(i) $1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \dots$ is $1{\cdot}004524$

(ii) $1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \dots$ is $0{\cdot}996158$.

By using the summation formula obtain the approximate results given in *Examples 29–33*.

29. $1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots = 1{\cdot}017344$

30. $\frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \ldots = 0{\cdot}158867$

31. $\frac{1}{3^4} + \frac{1}{7^4} + \frac{1}{11^4} + \ldots = 0{\cdot}012867$

32. $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots = 0{\cdot}915967$

33. $1 - \frac{1}{3^4} + \frac{1}{5^4} - \frac{1}{7^4} + \ldots = 0{\cdot}988937$

34. Show that

$$\sum_1^\infty \frac{1}{1+n^2} \sim \sum_1^n \frac{1}{1+n^2} + \arctan\left(\frac{1}{n}\right) - \frac{1}{2(1+n^2)} - \sum_1^\infty (-1)^{m-1}\frac{B_m}{2m}\sin^{2m}\theta \sin 2m\theta$$

where $\tan\theta = \frac{1}{n}$ and deduce that $\sum_1^\infty \frac{1}{1+n^2} = 1{\cdot}0767$.

35. Show that

$$\frac{1}{B_n} = \frac{(2\pi)^{2n}}{2(2n)!}\prod_{r=1}^{\infty}\left(1 - \frac{1}{p_r^{2n}}\right)$$

where $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... (the prime numbers $\neq 1$).

36. Prove that the integral part of B_8 is 54 and deduce that $B_8 = 43867/798$.

37. Prove that if $f(x)$ is a polynomial

$$f(1) - f(2) + f(3) - \ldots$$
$$+ (-1)^{n-1}f(n) = (-1)^{n-1}\left\{\frac{1}{2}f(n) + \frac{2^2-1}{2!}B_1f'(n) - \frac{2^4-1}{4!}B_2f'''(n) + \ldots\right\} + C$$

Deduce expressions for (i) $1^3 - 2^3 + 3^3 - \ldots + (-1)^{n-1}n^3$

(ii) $1^4 - 2^4 + 3^4 - \ldots + (-1)^{n-1}n^4$

Prove the results given in *Examples 38–100*.

38. $2^{\frac13}\pi^{\frac12}\Gamma(\frac16) = 3^{\frac12}\{\Gamma(\frac13)\}^2$

39. $3^{\frac12}\Gamma(\frac56)\{\Gamma(\frac13)\}^2 = 2^{\frac43}\pi^{\frac32}$

40. $2^{\frac{7}{10}}\pi^{\frac12}\Gamma(\frac{1}{10}) = (5+5^{\frac12})^{\frac12}\Gamma(\frac15)\Gamma(\frac25)$

41. $2^{\frac35}\Gamma(\frac{3}{10})\Gamma(\frac45) = \pi^{\frac12}(5^{\frac12}-1)\Gamma(\frac25)$

42. $\Gamma(\frac{1}{15})\Gamma(\frac{1}{3}) = \pi^{\frac12}2^{\frac12}\Gamma(\frac{2}{5})$

43. $5^{\frac14}\Gamma(\frac{1}{10})\Gamma(\frac{1}{5})\Gamma(\frac25) = 2\pi(5^{\frac12}+1)^{\frac12}$

44. $2^{\frac14}\pi^{\frac12}\Gamma(\frac{1}{12}) = 3^{\frac38}(3^{\frac12}+1)^{\frac12}\Gamma(\frac13)\Gamma(\frac14)$

45. $3^{\frac14}\Gamma(\frac{5}{12})\Gamma(\frac14) = 2^{\frac14}(3^{\frac12}-1)^{\frac12}\pi^{\frac12}\Gamma(\frac13)$

46. $\Gamma(\frac{7}{12})\Gamma(\frac14) = 2^{\frac14}3^{\frac18}(3^{\frac12}-1)^{\frac12}\pi^{\frac12}\Gamma(\frac23)$

47. $3^{\frac38}\Gamma(\frac{11}{12})\Gamma(\frac13)\Gamma(\frac14) = 2^{\frac14}(3^{\frac12}+1)^{\frac12}\pi^{\frac32}$

48. $2^{\frac19}\pi^{\frac12}\Gamma\left(\frac{1}{18}\right)\sin\frac{\pi}{9} = 3^{\frac12}\Gamma\left(\frac13\right)\Gamma\left(\frac29\right)$

49. $\frac{\Gamma(\frac{5}{24})\Gamma(\frac38)}{\Gamma(\frac{1}{24})} = \frac{\{\pi(\sqrt3-1)(\sqrt2-1)\}^{\frac12}}{2^{\frac14}3^{\frac18}}$

50. $\frac{4}{3}\cdot\frac{16}{15}\cdot\frac{36}{35}\cdot\frac{64}{63}\cdot\ldots = \frac{\pi}{2}$

51. The minimum value of $\Gamma(x)$ is $0{\cdot}886\ldots$, when $x = 1{\cdot}46\ldots$.

52.
$$\frac{\left(1 + \frac{1}{5^2}\right)\left(1 + \frac{1}{9^2}\right)\left(1 + \frac{1}{13^2}\right)\ldots}{\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\ldots} = \frac{\{\Gamma(\frac14)\}^4}{16\pi^3}$$

53. $$\frac{\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{10^2}\right)\left(1-\frac{1}{14^2}\right)\cdots}{\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\cdots}=\frac{\sqrt{2}\,\{\Gamma(\frac14)\}^2}{3\pi^{3/2}}$$

54. $$\frac{\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{8^2}\right)\left(1-\frac{1}{12^2}\right)\cdots}{\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\cdots}=\frac{\{\Gamma(\frac14)\}^2}{\pi^{5/2}}$$

55. $$\frac{\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{17^2}\right)\cdots}{\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{13^2}\right)\left(1-\frac{1}{19^2}\right)\cdots}=\frac{5\sqrt[3]{\pi^2}}{3^{\frac12}\{\Gamma(\frac13)\}^3}$$

56. $(1-\frac12)(1+\frac13)(1+\frac14)(1-\frac15)(1-\frac16)(1+\frac17)\ldots=\sqrt{2}$

57. $(1-\frac12)(1+\frac13)(1+\frac14)(1+\frac15)(1-\frac16)(1+\frac17)(1+\frac18)(1+\frac1{10})(1-\frac1{11})\ldots=\sqrt{3}$

58. $$\prod_1^\infty\left(1-\frac{1}{2n+1}\right)\left\{1+\frac{1}{2m(n-1)+2}\right\}\left\{1+\frac{1}{2m(n-1)+4}\right\}\cdots\left(1+\frac{1}{2mn}\right)=\sqrt{m}\ (m\text{ positive integer})$$

59. $$\prod_1^\infty\frac{(4n+1)(8n-3)(8n-1)}{64n(2n-1)(2n+1)}=\frac{2\cdot\pi^{\frac12}}{\{\Gamma(\frac14)\}^2}$$

60. $$\prod_1^\infty\frac{(n-\alpha)(n-1-\beta-\gamma)}{(n-\beta)(n-\gamma)}\left(1+\frac{\alpha-1}{n}\right)=\frac{\sin\alpha\pi}{\pi}\cdot\frac{\Gamma(1-\beta)\Gamma(1-\gamma)}{\Gamma(2-\beta-\gamma)}$$

61. $$\Gamma\left(\frac1n\right)\Gamma\left(\frac2n\right)\ldots\Gamma\left(\frac{3n-1}{n}\right)=\frac{n!(2n)!}{n^{3n+\frac12}}2^{\frac{3n-5}{2}}\pi^{\frac32(n-1)}$$

62. $$\prod_1^\infty\frac{(3n-1)(6n+1)^2}{27n(4n^2-1)}=\frac{3\sqrt[3]{2}\,\pi}{\{\Gamma(\frac13)\}^3}$$

63. $$\prod_1^\infty\left(1+\frac{x}{2n-1}\right)\left(1-\frac{x}{2n}\right)\left(1-\frac{2x}{2n+1}\right)=\frac{2^{1-x}\cos\pi x}{x(1-2x)}\frac{\Gamma(2x)}{\{\Gamma(x)\}^2}\ (2x\neq1)$$

and equals $\sqrt{2}$ $(2x=1)$.

64. $$\prod_0^\infty\left(1-\frac{1}{c+2n}\right)e^{\frac{1}{2n+2}}=\frac{e^{\frac12\gamma}}{2^{1-c}\sqrt{\pi}}\frac{\{\Gamma(c/2)\}^2}{\Gamma(c)}$$

65. $$\prod_0^\infty\left\{1+\frac{y^2}{(x+n)^2}\right\}=\frac{\{\Gamma(x)\}^2}{\Gamma(x+iy)\Gamma(x-iy)}$$

66. $|\Gamma(1+ic)|=\sqrt{(\pi c\operatorname{cosech}\pi c)}$ (c real).

67. $|\Gamma(\frac12+ic)|=\sqrt{(\pi\operatorname{sech}\pi c)}$ (c real).

68. $\Gamma(\omega)\Gamma(\omega^2)=\Gamma(-\omega)\Gamma(-\omega^2)=\pi\operatorname{sech}(\frac12\pi\sqrt{3})$, where ω, ω^2 are the imaginary cube roots of unity

69. $$\lim_{n\to\infty}\frac{a(a+1)(a+2)\ldots(a+2n-1)}{1.3.5\ldots(2n-1)2a(2a+2)\ldots(2a+2n-2)}=2^{a-1}$$

70. $$\lim_{n\to\infty}\frac{(2n)!\{(mn)!\}^2 2^{2n(m-1)}}{(2mn)!(n!)^2}=\sqrt{m}\ (m\text{ being a positive integer})$$

71. $$\lim_{n\to\infty}\frac{(2n)!(3n)!2^{2n}}{(n!)^2\,1^2.5^2\ldots(4n-3)^2.3^{3n}}=\frac{3^{\frac12}}{2\pi^{3/2}}\{\Gamma(\tfrac14)\}^2$$

72. $$\frac{3.5.7\ldots(2n+1)}{2.4.6\ldots2n}\sim2\sqrt{\left(\frac{n}{\pi}\right)}\left(1+\frac{3}{8n}\right)$$

73. $B(p, p + \frac{1}{2}) = 2^{2p}B(2p, 2p)$

74. $B(p, p).B\left(p + \frac{1}{2}, p + \frac{1}{2}\right) = \frac{\pi}{2^{4p-1}p}$

75. $B(p, q).B(p + q, r) = B(q, r).B(q + r, p)$

76. $B(p, q).B(p + q, r).B(p + q + r, s) = B(p, r).B(p + r, s).B(p + r + s, q)$

77. $B(p, p + \frac{1}{3}).B(p, p + \frac{2}{3}) = 3^{3p}B(p, 2p).B(3p, 3p) = 3^{3p}B(p, 3p).B(2p, 4p)$
$= 3^{3p}B(2p, 3p).B(p, 5p)$

78. $(6p + 1)2^{6p-2}B(p, p).B(p + \frac{1}{3}, p + \frac{1}{3}).B(p + \frac{2}{3}, p + \frac{2}{3}) = 3^{\frac{1}{2}}\pi B(2p, \frac{1}{3})$

79. $$\int_0^\infty \frac{\cosh 2x}{\cosh^{2m} x}dx = \frac{4^{m-1}m\{\Gamma(m)\}^2}{(m-1)\Gamma(2m)} \quad (m > 1)$$

80. $$\int_0^1 \frac{x^{\alpha-1} + x^{\beta-1}}{(1+x)^{\alpha+\beta}}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\alpha, \beta > 0)$$

81. $$\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{\sqrt{2}}{8\sqrt{\pi}}\{\Gamma(\tfrac{1}{4})\}^2$$

82. $$\int_0^1 \frac{dx}{\sqrt{(1-x^6)}} = \frac{\sqrt{3}}{2}\int_0^1 \frac{dx}{\sqrt{(1-x^3)}} = \frac{1}{2^{\frac{7}{3}}\pi}\{\Gamma(\tfrac{1}{3})\}^3$$

83. $$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(a\cos^4\theta + b\sin^4\theta)}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4(ab)^{\frac{1}{4}}\pi^{\frac{1}{2}}} \quad (a, b > 0)$$

84. $$\int_{-\frac{1}{4}\pi}^{\frac{3}{4}\pi}(\cos\theta + \sin\theta)^{\frac{1}{2}}d\theta = \frac{2\sqrt{2\pi^3}}{\{\Gamma(\frac{1}{4})\}^2}$$

85. $$\int_{-1}^1 \frac{(1+x)^{2m-1}(1-x)^{2n-1}dx}{(1+x^2)^{m+n}} = 2^{m+n-2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (m, n > 0)$$

86. $$\int_0^1 \frac{x^{2n}dx}{(x-x^3)^{1/3}} = \frac{\pi}{\sqrt{3}}\cdot\frac{1\cdot 4\cdot 7\ldots(3n-2)}{3^n n!}$$ when n is an integer > 0 and equals $\frac{\pi}{\sqrt{3}}$ if $n = 0$.

87. $$\int_{-1}^{+1}\frac{(1+x)^{2m-1}}{(1+x^2)^{m+\frac{1}{2}}}dx = 2^{3m-\frac{5}{2}}\frac{\{\Gamma(m)\}^2}{\Gamma(2m)} \quad (m > 0)$$

88. $$\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{(1+\sin^2\theta)}} = \frac{\sqrt{2}}{8\sqrt{\pi}}\{\Gamma(\tfrac{1}{4})\}^2$$

89. $$\int_0^1 \frac{x^{a-1}dx}{(x+c)(1-x)^a} = \frac{\pi c^{a-1}}{(1+c)^a\sin a\pi} \quad (0 < a < 1,\ c > 0)$$

90. $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)^\lambda d\theta = \frac{\pi}{2}\sec\frac{\pi\lambda}{2} \quad (|\lambda| < 1)$$

91. $$\int_0^\pi \frac{\sin^{a-1}\theta\, d\theta}{(a+b\cos\theta)^a} = \frac{2^{a-1}}{(a^2-b^2)^{a/2}}\cdot\frac{\{\Gamma(\frac{1}{2}a)\}^2}{\Gamma(a)} \quad (a^2 > b^2,\ a > 0)$$

92. $$\int_0^\pi \frac{d\theta}{(2+\cos\theta-2\cos^2\theta-\cos^3\theta)^{\frac{1}{3}}} = \left(\frac{243}{256}\right)^{\frac{1}{4}}\frac{\{\Gamma(\frac{1}{3})\}^3}{\pi}$$

93. $$\int_{-1}^{+1}\frac{(x+1)^{a-1}(1-x)^{b-1}dx}{(x+2)^{a+b}} = \frac{2^{a+b-1}}{3^a}\cdot\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (a, b > 0)$$

94. $2^{\frac12}\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = 3^{\frac14}\int_0^1 \frac{x\,dx}{(1-x^6)^{2/3}}$.

95. $2^{\frac12}\int_0^1 \frac{x^3\,dx}{(1-x^{10})^{3/5}} = (\sqrt5 - 1)\int_0^1 \frac{x\,dx}{\sqrt{(1-x^{10})}}$.

96. $\int_0^1 \frac{dx}{\sqrt{(1-x^{12})}} = \frac{(\sqrt3+1)^{\frac12}}{3^{5/8}2^{1/4}}\int_0^1 \frac{dx}{(1-x^4)^{2/3}}$.

97. $\frac{1}{\sqrt5-1}\int_0^1 \frac{x\,dx}{(1-x^5)^{7/10}} = \frac{1}{2^{9/5}}\int_0^1 \frac{dx}{\sqrt{(1-x^5)}} = \frac{1}{2^{6/5}}\int_0^1 \frac{dx}{(1-x^5)^{4/5}}$.

98. $\int_0^\infty \frac{\sin x}{x^{\frac14}}\,dx = \frac{\pi}{2\Gamma(\frac14)\sin\frac{\pi}{8}}$.

99. $\int_0^\infty \frac{\sin x}{\sqrt x}\,dx = \sqrt{\left(\frac{\pi}{2}\right)}$.

100. $\int_0^\infty \frac{\sin x^4}{x}\,dx = \frac{\pi}{8}$.

101. Show that $\iiiint \frac{dx_1\,dx_2\,dx_3\,dx_4}{\sqrt{(1-x_1^2-x_2^2-x_3^2-x_4^2)}} = \frac{\pi^2}{12}$ where the integration extends to all positive and zero values of x_1, x_2, x_3, x_4 for which

$$0 \leqslant x_1^2 + x_2^2 + x_3^2 + x_4^2 < 1.$$

102. Show that the total perimeter of the curve $r^4 = a^4\cos 4\theta$ is $a\,\frac{\{\Gamma(\frac14)\}^2}{2^{3/4}\Gamma(\frac12)}$.

103. Prove that the perimeter of the curve $r^3 = a^3\cos 3\theta$ is $3a.\frac{\{\Gamma(\frac13)\}^2}{2^{4/3}\pi}$.

104. Find the area of a loop of the curve $r^4 = a^4\sin^2\theta\cos\theta$.

105. Show that (assuming $a, b, c, \alpha, \beta, \gamma, \delta > 0$) the curve $ax^\alpha + by^\beta = cx^\gamma y^\delta$ has a loop in the first quadrant if $\alpha\beta > \beta\gamma + \alpha\delta$; and that the area of the loop is

$$\frac{c^{\lambda+\mu}}{a^\lambda b^\mu}\,\frac{\Gamma(\lambda)\Gamma(\mu)}{(\alpha+\beta)\Gamma(\lambda+\mu)} \text{ where } \lambda = \frac{\beta+\gamma-\delta}{\alpha\beta-\beta\gamma-\alpha\delta},\ \mu = \frac{\alpha+\delta-\gamma}{\alpha\beta-\beta\gamma-\alpha\delta}.$$

Prove the following results for the areas of the loops in the first quadrant determined by the curves given in *Examples 106–11*.

106. $x^\alpha + y^\alpha = \alpha a^{\alpha-2}xy$; $\frac{a^2}{2\alpha^{\frac{\alpha-4}{\alpha-2}}}\,\frac{\left\{\Gamma\left(\frac{1}{\alpha-2}\right)\right\}^2}{\Gamma\left(\frac{2}{\alpha-2}\right)}$ $(\alpha > 2)$

107. $x^\alpha + y^\alpha = cx^\beta y^\beta$; $\frac{c^{\frac{2}{\alpha-2\beta}}}{2\alpha}\,\frac{\left\{\Gamma\left(\frac{1}{\alpha-2\beta}\right)\right\}^2}{\Gamma\left(\frac{2}{\alpha-2\beta}\right)}$ $(\alpha > 2\beta)$

108. $x^5 + y^5 = 5ax^2y^2$; $\frac52 a^2$

109. $x^{2m+1} + y^{2m+1} = (2m+1)ax^my^m$; $(2m+1)\frac{a^2}{2}$

110. $x^{2m} + y^{2m} = ma^2x^{m-1}y^{m-1}$; $\frac{\pi}{4}a^2$

111. $x^{12} + y^6 = x^2y^6$; $\frac{2}{21}\,\frac{\pi^3}{\{\Gamma(\frac13)\}^3}$

112. Show that (assuming $a, b, c, \alpha, \beta, \gamma > 0$) if $\alpha > \gamma$, the area in the first quadrant determined by $y = 0$ and the curve $ax^\alpha + by^\beta = cx^\gamma$ is

$$\frac{c^{\lambda+\mu}}{a^\lambda b^\mu}\,\frac{\Gamma(\lambda)\Gamma(\mu)}{(\alpha+\beta)\Gamma(\lambda+\mu)} \text{ where } \lambda = \frac{\beta+\gamma}{\beta(\alpha-\gamma)},\ \mu = \frac{1}{\beta}.$$

Prove the following results for the areas in the first quadrant bounded by $y = 0$ and the curves given in *Examples 113–19*.

113. $x^m + y^m = cx$; $\quad c^{\frac{2}{m-1}} \dfrac{\Gamma\left(\frac{m+1}{m(m-1)}\right)\Gamma\left(\frac{1}{m}\right)}{2m\Gamma\left(\frac{2}{m-1}\right)} \quad (m > 1)$

114. $x^2 + y^2 = 2ax$; $\quad \frac{1}{2}\pi a^2$ **115.** $x^3 + y^3 = a^2x$; $\quad \dfrac{\pi}{3\sqrt{3}}a^2$

116. $x^m + y^m = cx^2$; $\quad c^{\frac{2}{m-2}} \dfrac{\Gamma\left(\frac{m+2}{m(m-2)}\right)\Gamma\left(\frac{1}{m}\right)}{2m\Gamma\left(\frac{2}{m-2}\right)} \quad (m > 2)$

117. $x^3 + y^3 = ax^2$; $\quad \dfrac{2\pi}{9\sqrt{3}}a^2$ **118.** $x^4 + y^4 = a^2x^2$; $\quad \dfrac{\pi\sqrt{2}}{8}a^2$

119. $x^m + y^m = ax^{m-1}$; $\quad \dfrac{\pi a^2(m-1)}{2m^2 \sin\frac{\pi}{m}}$

120. Show that (assuming $a, b, c, \alpha, \beta, \gamma > 0$) the area in the first quadrant bounded by the curve $ax^\alpha + by^\beta = c$ and the co-ordinate axes are

$$\frac{c^{\frac{1}{\alpha}+\frac{1}{\beta}}}{a^{\frac{1}{\alpha}}b^{\frac{1}{\beta}}} \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1+\frac{1}{\beta}\right)}{\Gamma\left(1+\frac{1}{\alpha}+\frac{1}{\beta}\right)}.$$

Prove the following results for the areas determined in the first quadrant by the co-ordinate axes and the curves given in *Examples 121–7*.

121. $x^m + y^m = a^m$; $\quad \dfrac{\left\{\Gamma\left(\frac{1}{m}\right)\right\}^2}{2m\Gamma\left(\frac{2}{m}\right)}a^2$

122. $x^3 + y^3 = a^3$; $\quad \dfrac{\sqrt{3}a^2}{12\pi}\left\{\Gamma\left(\frac{1}{3}\right)\right\}^3$

123. $x^4 + y^4 = a^4$; $\quad \dfrac{a^2}{8\sqrt{\pi}}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2$

124. $x^6 + y^6 = a^6$; $\quad \dfrac{a^2 2^{\frac{1}{3}}}{8\pi}\left\{\Gamma\left(\frac{1}{3}\right)\right\}^3$

125. $a^mx^m + y^{2m} = a^{2m}$; $\quad \dfrac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{1}{2m}\right)}{3m\Gamma\left(\frac{3}{2m}\right)}a^2$

126. $a^3x^3 + y^6 = a^6$; $\quad \dfrac{\sqrt{3}a^2}{9\pi 2^{\frac{1}{3}}}\left\{\Gamma\left(\frac{1}{3}\right)\right\}^3$

127. $a^2x^2 + y^4 = a^4$; $\quad \dfrac{a^2}{6\sqrt{(2\pi)}}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2$

128. Show that the volume in the first octant $(x, y, z > 0)$, bounded by the surface $x^\alpha + y^\beta + z^\gamma = 1$, is

$$\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{1}{\gamma}\right)}{\Gamma\left(1+\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)} \quad (\alpha, \beta, \gamma > 0).$$

Prove the following results for the approximate volumes in the first octant determined by the surfaces given in *Examples 129–37*.

129. $x^2+y^2+z^4=1$; 0.628

130. $x^2+y^2+z^6=1$; 0.673 **131.** $x^2+y^4+z^4=1$; 0.728

132. $x^2+y^4+z^6=1$; 0.770 **133.** $x^2+y^6+z^6=1$; 0.811

134. $x^4+y^4+z^4=1$; 0.810 **135.** $x^4+y^4+z^6=1$; 0.844

136. $x^4+y^6+z^6=1$; 0.875 **137.** $x^6+y^6+z^6=1$; 0.901

138. Show that the volume enclosed by the surface $a^4x^2+a^2y^4+z^6=a^6$ is

$$\frac{3\cdot(3^{\frac12}-1)^{\frac12}\{\Gamma(\frac13)\}^3\{\Gamma(\frac14)\}^2a^3}{2^{\frac13}\cdot 11\pi^{\frac12}}$$

139. Prove that the distance from $x=0$ of the centroid of the volume determined in the first octant by the surface $ax^\alpha+by^\beta+cz^\gamma=d$ is

$$\frac12\left(\frac da\right)^{\frac1\alpha}\frac{\Gamma\left(1+\frac2\alpha\right)\Gamma\left(1+\frac1\alpha+\frac1\beta+\frac1\gamma\right)}{\Gamma\left(1+\frac1\alpha\right)\Gamma\left(1+\frac2\alpha+\frac1\beta+\frac1\gamma\right)}\quad(\alpha,\beta,\gamma,a,b,c,d>0).$$

140. Show that the distance from $x=0$ of the centroid of the volume determined in the first octant by the surface $\left(\frac xa\right)^m+\left(\frac yb\right)^m+\left(\frac zc\right)^m=1$ is

$$\frac{3a}{4}\frac{B\left(\frac2m,\frac3m\right)}{B\left(\frac1m,\frac4m\right)}\quad(m>0).$$

141. Show that the square of the radius of gyration about the z-axis of the volume in the first octant bounded by $ax^\alpha+by^\beta+cz^\gamma=d$ is

$$\frac13\left(\frac da\right)^{\frac2\alpha}\frac{\Gamma\left(1+\frac3\alpha\right)\Gamma\left(1+\frac1\alpha+\frac1\beta+\frac1\gamma\right)}{\Gamma\left(1+\frac1\alpha\right)\Gamma\left(1+\frac3\alpha+\frac1\beta+\frac1\gamma\right)}$$

$$+\frac13\left(\frac db\right)^{\frac2\beta}\frac{\Gamma\left(1+\frac3\beta\right)\Gamma\left(1+\frac1\alpha+\frac1\beta+\frac1\gamma\right)}{\Gamma\left(1+\frac1\beta\right)\Gamma\left(1+\frac1\alpha+\frac3\beta+\frac1\gamma\right)}\quad(a,b,c,d,\alpha,\beta,\gamma>0).$$

142. Prove that the square of the radius of gyration about the z-axis of the solid determined by $\left(\frac xa\right)^m+\left(\frac yb\right)^m+\left(\frac zc\right)^m=1$ is

$$\frac35(a^2+b^2)\frac{B\left(\frac3m,\frac3m\right)}{B\left(\frac1m,\frac5m\right)}.$$

Find for the solids in the first octant determined by the surfaces given in *Examples 143–5*, (i) the volume, (ii) the distance from $x=0$ of the centroid, (iii) the square of the radius of gyration about the z-axis.

143. $x^{\frac23}+y^{\frac23}+z^{\frac23}=a^{\frac23}$ **144.** $\sqrt x+\sqrt y+\sqrt z=\sqrt a$

145. $x^4+y^4+z^4=a^4$

146. Show that the volume in the first octant determined by the surface $\left(\frac xa\right)^m+\left(\frac yb\right)^m+\left(\frac zc\right)^m=\left(\frac{xyz}{abc}\right)^n$ where $m>3n$ is

$$\frac{abc}{3m^2}\frac{\left\{\Gamma\left(\frac{1}{m-3n}\right)\right\}^3}{\Gamma\left(\frac{3}{m-3n}\right)}.$$

147. Prove the volume of the solid bounded by $x^4 + y^4 + z^4 = axyz$ is $\frac{1}{12}a^3$.

Prove the results given in *Examples 148–60.*

148. $\psi(\frac{1}{3}) = -\gamma - \frac{3}{2}\log 3 - \frac{1}{6}\pi\sqrt{3}$

149. $\psi(\frac{2}{3}) = -\gamma - \frac{3}{2}\log 3 + \frac{1}{6}\pi\sqrt{3}$

150. $\psi(\frac{4}{3}) = 3 - \gamma - \frac{3}{2}\log 3 - \frac{1}{6}\pi\sqrt{3}$

151. $\psi(\frac{1}{6}) = -\gamma - \frac{1}{2}\pi\sqrt{3} - \frac{3}{2}\log 3 - 2\log 2$

152. $\psi(\frac{5}{6}) = -\gamma + \frac{1}{2}\pi\sqrt{3} - \frac{3}{2}\log 3 - 2\log 2$

153. $\psi\left(\frac{1}{12}\right) = -\gamma - (3+\sqrt{3})\log 2 - \frac{3}{2}\log 3 + 2\sqrt{3}\log(\sqrt{3}-1) - \frac{2+\sqrt{3}}{2}\pi$

154. $\psi\left(\frac{1}{n}\right) = -\gamma - \log n - \frac{1}{2}\pi\cot\frac{\pi}{n} + \sum_{r=1}^{n-1}\cos\frac{2\pi r}{n}\log\left(2\sin\frac{r\pi}{n}\right)$

155. $\int_0^{\frac{1}{2}\pi} \sin x.(\log\sin x)^2\,dx = (\log 2 - 1)^2 + 1 - \frac{1}{12}\pi^2$

156. $\int_0^{\frac{1}{2}\pi} \sin^{-1}x\log\sin x\,dx = -\frac{\sqrt{3}}{2^{7/3}}\left\{\Gamma\left(\frac{1}{3}\right)\right\}^3\left(\frac{\sqrt{3}}{\pi}\log 2 + 1\right)$

157. $\int_0^{\frac{1}{2}\pi} \sin^{2\alpha-1}x\cos^{2\beta-1}x\log\sin x\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{4\Gamma(\alpha+\beta)}\{\psi(\alpha) - \psi(\alpha+\beta)\}$ $(\alpha, \beta > 0)$

158. $\int_0^{\frac{1}{2}\pi} \sin^{2\alpha-1}x\cos^{2\beta-1}x.\log\sin x.\log\cos x\,dx$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{8\Gamma(\alpha+\beta)}[\{\psi(\alpha) - \psi(\alpha+\beta)\}\{\psi(\beta) - \psi(\alpha+\beta)\} - \psi'(\alpha+\beta)] \quad (\alpha, \beta > 0)$$

159. $\int_0^{\frac{1}{2}\pi} \log\sin x.\log\cos x\,dx = \frac{\pi}{2}(\log 2)^2 - \frac{\pi^3}{48}$

160. $\int_0^{\frac{1}{2}\pi} \frac{\log\sin x\log\cos x}{\sin^{\frac{1}{2}}x\cos^{\frac{1}{2}}x}\,dx$

$$= \frac{\sqrt{2}\{\Gamma(\frac{1}{4})\}^2}{2\sqrt{\pi}}\left\{\frac{\pi^2}{4} + \pi\sqrt{3}\log\left(\frac{2}{3}\right) + (3\log 3)(\log 2) - \frac{9}{4}(\log 3)^2\right\}$$

$\log\Gamma(x)$

	0	1	2	3	4	5	6	7	8	9
1·0	0000	$\bar{1}$·9975	9951	9928	9905	9883	9862	9841	9821	9802
1·1	$\bar{1}$·9783	9765	9748	9731	9715	9699	9684	9669	9655	9642
1·2	9629	9617	9605	9594	9583	9573	9564	9554	9546	9538
1·3	9530	9522	9516	9510	9505	9500	9495	9491	9487	9483
1·4	9481	9478	9476	9475	9473	9473	9472	9473	9473	9474
1·5	9475	9477	9479	9482	9485	9488	9492	9496	9501	9506
1·6	9511	9517	9523	9529	9536	9543	9550	9558	9566	9575
1·7	9584	9593	9603	9613	9623	9633	9644	9655	9667	9679
1·8	9691	9704	9717	9730	9743	9757	9771	9786	9800	9815
1·9	9831	9846	9862	9879	9895	9912	9929	9946	9964	9982

Solutions

2. $\tan z = \cot z - 2\cot 2z$ 3. $z \operatorname{cosec} z = z\cot z - \ldots \cot \frac{1}{2}z$

5. $z^2 \operatorname{cosec}^2 z = -z^2 \frac{d}{dz}(\cot z)$ 6. $(e^z + 1)^{-1} = (e^z - 1)^{-1} - 2(e^{2z} - 1)^{-1}$

7. Integrate $\frac{1}{z} - \cot z = \sum_1^\infty \frac{2^{2k} z^{2k-1}}{(2k)!} B_k$.

8. Find $\log\left(\frac{\sin \alpha z}{\alpha z}\right) + \log\left(\frac{\sin \beta z}{\beta z}\right)$ from Example 7, where $\alpha = \frac{1}{2}(1 + i)$, $\beta = \frac{1}{2}(1 - i)$.

9. (ii) The coefficient of z^{2n} in the product $\left\{\sum_0^\infty (-1)^r \frac{z^{2r}}{(2r)!}\right\}\left\{1 + \sum_1^\infty \frac{E_r z^{2r}}{(2r)!}\right\}$ is unity (iii) Use the series $\sec z = \sum_0^\infty \frac{(-1)^n 4(2n+1)\pi}{\{(2n+1)^2\pi^2 - 4z^2\}}$.

13. Series for $\cot z$ when $z = \frac{1}{2}\pi$

14. Series for $\cot z$ when $z = \frac{1}{2}\pi$.

15. $(\cos z - 1)\frac{d}{dz}(\frac{1}{2}z \cot \frac{1}{2}z) = \frac{1}{2}(z - \sin z)$

16. $\frac{1}{2}(1 + \cos z) - \frac{1}{2}z \cot \frac{1}{2}z = \sin z \frac{d}{dz}(\frac{1}{2}z \cot \frac{1}{2}z)$

17. $z(1 + \cos z) = (\sin z)(z \cot \frac{1}{2}z)$ 18. $z \cos z = (\sin z)(z \cot z)$

19, 20. If $A = \sinh \frac{z}{\sqrt{2}} \sin \frac{z}{\sqrt{2}} = \frac{z^2}{2!} - \frac{z^6}{6!} + \ldots$;

$$B = \cosh \frac{z}{\sqrt{2}} \sin \frac{z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(z + \frac{z^3}{3!} - \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \frac{z^{11}}{11!} \ldots\right)$$

then $A \coth \frac{z}{\sqrt{2}} = B$

21. $z \cot z = (\cos z)(z \operatorname{cosec} z)$ 22. $\sin z \sec z = \tan z$

23. $(\tan z)(z \operatorname{cosec} z) = z \sec z$

35. If $S = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots$, $S\left(1 - \frac{1}{2^{2n}}\right) = 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \ldots = S_1$

$S_1\left(1 - \frac{1}{3^{2n}}\right) = 1 + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \ldots$ etc., $S \to 1$ when $n \to \infty$.

37. If n is even, $\sum_1^n f(r) = \int^n f(x)dx + \frac{1}{2}f(n) + \sum_1 (-1)^{s-1} \frac{f^{(2s-1)}(n)}{(2s)!} B_s$ + constant and $\sum_1^{\frac{1}{2}n} F(r) = \int^{\frac{1}{2}n} F(x)dx + \frac{1}{2}F(\frac{1}{2}n) + \sum_1 (-1)^{s-1} \frac{F^{(2s-1)}(\frac{1}{2}n)}{(2s)!} B_s$ + constant where $F(x) = f(2x)$, i.e. $\sum_1^{\frac{1}{2}n} f(2r) = \frac{1}{2}\int^n f(x)dx + \frac{1}{2}f(n) + \sum_1 (-1)^{s-1} \frac{2^{2s-1} f^{(2s-1)}(n)}{(2s)!} B_s$ + constant,

i.e. $f(1) - f(2) + f(3) \ldots - f(n)$

$$= -\left\{\frac{1}{2}f(n) + \sum_1 (-1)^{s-1} \frac{(2^{2s} - 1)}{(2s)!} B_s f^{(2s-1)}(n)\right\} + \text{constant}.$$

If n is odd, $f(2) - f(3) + \ldots - f(n) = F(1) - F(2) + F(3) \ldots - F(n-1)$ where $F(x) = f(x + 1)$. (i) $-\frac{1}{4}n^2(2n + 3)$, n even; $\frac{1}{4}(n + 1)^2(2n - 1)$, n odd. (ii) $(-1)^{n-1}\frac{1}{2}n(n + 1)(n^2 + n - 1)$

38–49. Use the Multiplication Formula and the relation

$$\Gamma(x)\Gamma(1 - x) = \pi \operatorname{cosec} \pi x$$

49. $2^{-1}3^{-\frac{1}{4}}\pi^{\frac{1}{2}}\left(\sin\frac{\pi}{4}\right)^{\frac{1}{4}}\left(\sin\frac{3\pi}{8}\right)^{-1}\left(\sin\frac{\pi}{3}\right)^{-\frac{1}{4}}\left(\sin\frac{5\pi}{12}\right)^{-1}$

51. The value is $x = 1.5 - h$ where h is approximately $-\psi(1.5)/\psi'(1.5)$, i.e. $(4 - 2\gamma - 4\log 2)/(\pi^2 - 8)$.

52. $\prod_1^\infty \dfrac{(4n-1)^2(4n+2)}{(4n+1)^2(4n-2)}$

56. $\prod_1^\infty \left(1 - \dfrac{1}{2n+1}\right)\left(1 + \dfrac{1}{4n-2}\right)\left(1 + \dfrac{1}{4n}\right)$

69. $\prod_1^\infty \dfrac{(a+2n-2)(a+2n-1)}{(2n-1)(2a+2n-2)}$; $\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}a) = 2^{1-a}\sqrt{\pi}\,\Gamma(a)$

70. Use the Asymptotic Formula for $n!$, $(2n)!$, $(mn)!$, $(2mn)!$.

71. $\prod_1^\infty \dfrac{(2n-1)2n(3n-2)(3n-1)3n.4}{n^2(4n-3)^2.27}$

72. Expression is $\dfrac{(2n-1)(2n)!}{2^{2n}(n!)^2}$. Use the Asymptotic formula for $(2n)!$ and $(n!)$

79. Take $t = \tanh^2 x$.

80. $\displaystyle\int_0^1 \frac{x^{\alpha-1}\,dx}{(1+x)^{\alpha+\beta}} = \int_0^{\frac{1}{2}} t^{\alpha-1}(1-t)^{\beta-1}\,dt$ where $t = \dfrac{x}{x+1}$.

$$\int_0^1 \frac{x^{\beta-1}\,dx}{(1+x)^{\alpha+\beta}} = \int_{\frac{1}{2}}^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \text{ where } t = \frac{1}{x+1}.$$

83. Take $\dfrac{1}{y} = 1 + \dfrac{b}{a}\tan^4\theta$.

84. Take $\phi = \theta + \frac{1}{4}\pi$.

85. Take $t = \dfrac{(1+x)^2}{2(1+x^2)}$ in the integral $\displaystyle\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$

88. Take $t = \sin^4\theta$.

89. Take $y = \dfrac{x(1+c)}{x+c}$.

90. Take $\phi = \theta + \frac{1}{4}\pi$.

91. If $t = \dfrac{(a-b)\sin^2\frac{\theta}{2}}{a+b\cos\theta}$, $1 - t = \dfrac{(a+b)\cos^2\frac{\theta}{2}}{a+b\cos\theta}$, $\dfrac{dt}{d\theta} = \dfrac{(a^2-b^2)\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{(a+b\cos\theta)^2}$

92. Put $a = 2$, $b = 1$ and $x = \frac{1}{2}$ in Example 91.

93. Take $y = \dfrac{3(x+1)}{2(x+2)}$.

94. Each equals $\dfrac{\{\Gamma(\frac{1}{4})\}^2}{4\pi}$.

95. $\dfrac{\sqrt{5}-1}{5.2^{6/5}}\cdot\dfrac{\{\Gamma(\frac{1}{5})\}^2}{\Gamma(\frac{2}{5})}$

96. $\dfrac{\sqrt{3}-1}{8.3^{3/4}}\cdot\dfrac{\{\Gamma(\frac{1}{3})\}^3}{\sqrt{\pi}}$

97. $\dfrac{1}{2^{6/5}}\cdot\dfrac{\{\Gamma(\frac{1}{5})\}^2}{5\Gamma(\frac{2}{5})}$

104. $\dfrac{7\sqrt{2}}{16}a^2$

105. Take $u = \dfrac{ax^{\alpha-\gamma}}{cy^\delta}$, $v = \dfrac{by^{\beta-\delta}}{cx^\gamma}$.

112. Take $u = \dfrac{a}{c}x^{\alpha-\gamma}$, $v = \dfrac{by^\beta}{cx^\gamma}$.

120. Take $u = \dfrac{a}{c}x^\alpha$, $v = \dfrac{b}{c}y^\beta$.

138. $\dfrac{2\sqrt{\pi}\,\Gamma(\frac{1}{3})\Gamma(\frac{1}{4})}{11\Gamma(\frac{11}{12})}a^2$ and use *Examples 38, 47.*

143. (i) $\frac{\pi a^3}{70}$; (ii) $\frac{21a}{128}$; (iii) $\frac{14a^2}{143}$. **144.** (i) [illegible]; (ii) $\frac{3a}{28}$; (iii) [illegible]

145. (i) $\frac{\{\Gamma(\frac{1}{4})\}^2 a^2}{\Gamma(\frac{3}{4})}$ [illegible] $\frac{a^2\sqrt{2}\,\{\Gamma(\frac{1}{4})\}^2}{96\pi}$ [illegible] $0{\cdot}8[illegible]0a^2$ approx.

(ii) $\frac{3a\pi^{\frac{1}{2}}}{2^{\frac{1}{4}}\{\Gamma(\frac{1}{4})\}^2} = 0{\cdot}449a$ approx.

(iii) $\frac{48\pi^2 a^2}{5\{\Gamma(\frac{1}{4})\}^4} = 0{\cdot}548a^2$ approx.

146. Take $u = \left(\frac{x}{a}\right)^m$ [illegible] $\left(\frac{yz}{bc}\right)$ [illegible], $v = \left(\frac{y}{b}\right)^n$ [illegible] $\left(\frac{zx}{ca}\right)$ [illegible] $\left(\frac{xy}{ab}\right)$ [illegible]

148. $\psi(\frac{1}{2})$ [illegible] γ [illegible] $3\int_0^1 \frac{1+u}{1+u}$ [illegible] $u^2\,du$ **149.** $\psi(\frac{1}{3})$ [illegible] $\sqrt{3}$

151. $\psi(\frac{1}{4})$ [illegible] $\frac{1}{2}\{\psi(\frac{1}{8}) + \psi(\frac{5}{8})\}$ [illegible] $\log 2$

153, 154 *Bromwich, Infinite Series, Appendix III, Example 43*

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